Naturalness of Scalar Fields and a Solution to the Hierarchy Problem

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Abstract

We reconsider the issue of naturalness of scalar fields. Contrary to expectations based on leading order perturbation theory, our analysis demonstrates that resummation of perturbative series removes the naturalness problem and provides an explanation to the smallness of the physical mass of a scalar excitation with respect to the cutoff scale. This explanation suggests a dynamical solution to the hierarchy problem.

Since the late seventies of the past century, it is recognized that there is a qualitative difference between the mass renormalization of scalar fields and the one of the fields of higher spins involved in the Standard Model [1, 2, 3] (for a recent review, see, e.g., [4]). Putting the mass of a scalar field to zero does not increase the symmetry of the theory, and, therefore, one expects that radiative corrections to mass of a scalar should be of order of the cutoff. Indeed, one-loop mass correction for scalar field diverges as cutoff squared:

$$m^2 = m_0^2 + \Lambda^2 P(\lambda_0, g). \tag{1}$$

Here m^2 is the squared mass of a scalar particle, m_0^2 is the corresponding bare mass of the fundamental Lagrangian of the model defined at the fundamental scale Λ, which is also used as a cutoff in the Feynman integrals, $P(\lambda_0, g)$ is a polynomial of dimensionless bare scalar field self-coupling λ_0 and the rest of dimensionless bare couplings g of the model, and we neglected the corrections depending logarithmically on the cutoff. (For example, in the Standard Model, $P(\lambda_0, g) = 3(3g_2^2 + g_1^2 + 2\lambda_0 - 4y_t^2)/(32\pi^2)$, where g_1, g_2, and y_t are the gauge couplings of the gauge groups $SU(1)$, $SU(2)$, and top quark Yukawa coupling, respectively [5].)

Thus, to have the mass m much less than the cutoff, we are to fine tune the values of the bare couplings. The accuracy of the fine tuning should be enormous if we take the Plank scale as the cutoff Λ and the weak scale as the scalar mass. In this respect, scalar field is not a natural system, because it requires unnatural fine tunings of bare parameters. Besides, we are left without explanation to the magnitude of the weak scale (hierarchy problem), because the fine tuning does not explain anything. Rather, it should be explained itself.

This should be contrasted to the case of asymptotically free chromodynamics, where the hadronic mass scale naturally appears via dimensional transmutation:

$$m_h = \Lambda \exp\Big(-\frac{1}{2b\alpha(\Lambda)}\Big), \tag{2}$$

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where m_h is a hadronic scale, b is a computable positive number, and $\alpha(\Lambda)$ is the strong coupling normalized at the cutoff Λ. Due to asymptotic freedom, $\alpha(\Lambda)$ is small, and we have a natural explanation for the smallness of the hadronic mass scale with respect to the cutoff.

All this is a common lore. And we are going to reexamine it in the part related to the scalar field. Our motivation comes from comparing the above formulas for mass scales. There is a striking difference between the two formulas. The formula for hadronic mass scale is nonperturbative. It involves resummation of leading order logarithms in all orders of perturbation theory. At the same time, the formula for the scalar mass is a one-loop formula. What happens to it if we take into account higher order perturbative corrections?

We observe that expanding the scalar mass in powers of bare couplings generates powers of the cutoff. At one loop we have the second power, at higher orders, the higher powers. And the higher the order of perturbation theory, the higher is the power of the cutoff appearing in the expansion. Obviously, we should resum all leading powers of the cutoff in all orders of perturbation theory before making any conclusions on the relation between the scalar mass and the cutoff. The problem has been considered in Ref. [6]. Let us sketch that consideration.

First, it is convenient to inverse the problem of expressing physical parameters via the bare ones. Namely, let us consider the expressions of the bare parameters in terms of the physical ones. This is advantageous because, for renormalizable theories, dependence of the *bare couplings* on the cutoff is known if they are expressed in terms of the *physical couplings* [7]. Let us reiterate: for renormalizable theory, the *bare* mass squared of a scalar particle expressed as a series in powers of *physical* couplings with coefficients of the expansion depending on the cutoff, *physical* masses and renormalization scale grows not faster than the cutoff squared. Is this statement compatible with the appearance of higher powers of the cutoff in the right-hand-side of Eq. (1)? It is easy to check that there is no contradiction. Indeed, schematically, if we take the renormalization scale to be of the order of physical mass, the bare mass squared and the bare coupling are expressed as follows

$$m_0^2 = m^2 - \Lambda^2 P(\lambda, g), \tag{3}$$

$$\lambda_0 = \lambda + \log\left(\frac{\Lambda^2}{m^2}\right)\frac{\beta(\lambda, g)}{2}, \tag{4}$$

where $P(\lambda, g)$ is (in the leading order) the same polynomial as in Eq.(1), and $\beta(\lambda, g)$ is the leading order of the beta function governing the renormalization group evolution of coupling λ. If we use the above expressions as equations for m^2 and λ, we can determine the expansions of m^2 and λ in powers of λ_0. It is easy to check that both power series involve arbitrary high powers of the cutoff. The reason for the appearance of the high powers of Λ in the expansions is the presence of m^2 in the argument of the logarithm. (Logarithmic term is also present in the formula for bare mass, but we dropped it, because it is insignificant for further reasoning.)

Now we need to solve Eqs. (3), (4) for the physical parameters. On the way of doing this, let us compute the Jacobian of the mapping from the bare variables to the physical ones,

$$A = \begin{pmatrix} \frac{\partial\lambda}{\partial\lambda_0} & \frac{\partial\lambda}{\partial m_0^2} \\ \frac{\partial m^2}{\partial\lambda_0} & \frac{\partial m^2}{\partial m_0^2} \end{pmatrix}. \tag{5}$$

The inverse of the desired A can be computed with Eqs. (3) and (4):

$$A^{-1} \equiv B = \begin{pmatrix} \frac{\partial\lambda_0}{\partial\lambda} & \frac{\partial\lambda_0}{\partial m^2} \\ \frac{\partial m_0^2}{\partial\lambda} & \frac{\partial m_0^2}{\partial m^2} \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} 1 + \log(\frac{\Lambda^2}{m^2})\frac{\beta'(\lambda,g)}{2} & -\frac{\beta(\lambda,g)}{2m^2} \\ -\Lambda^2 P'(\lambda, g) & 1 \end{pmatrix}, \tag{7}$$

where the prime over β and P denotes the derivative with respect to λ. Thus, the desired A is

$$A = \frac{1}{\det(B)} \begin{pmatrix} 1 & \frac{\beta(\lambda,g)}{2m^2} \\ \Lambda^2 P'(\lambda,g) & 1+\log(\frac{\Lambda^2}{m^2})\frac{\beta'(\lambda,g)}{2} \end{pmatrix}, \quad (8)$$

where

$$\det(B) = -\frac{\Lambda^2}{m^2} P'(\lambda,g)\frac{\beta(\lambda,g)}{2} + \log(\frac{\Lambda^2}{m^2})\frac{\beta'(\lambda,g)}{2} + 1. \quad (9)$$

Finally, in the limit of infinite Λ, we have:

$$A = \begin{pmatrix} 0 & 0 \\ -\frac{2m^2}{\beta(\lambda,g)} & 0 \end{pmatrix}. \quad (10)$$

Let us comment on Eq. (10). As we see, physical parameters—the observable mass and coupling—are not oversensitive to the values of the bare parameters defined at a large (e.g., fundamental) scale Λ. The leading order relation, Eq. (1), is misleading in this respect. In other words: Derivative of observable mass in bare coupling has a finite limit expressible in terms of observable parameters when the cutoff is removed. (This is the worst sensitivity we have: the physical coupling exhibits *universality*, i.e., it becomes independent of bare parameters at infinite cutoff; the physical mass becomes independent of the bare mass at infinite cutoff.) We conclude that the fine tuning problem is the problem of the leading order perturbative approximation, Eq. (1).

Let us come back to the problem of expressing physical parameters in terms of the bare ones. Eq. (10) implies that physical coupling does not depend on the bare couplings (naturally, this is the case only in the limit of infinite cutoff). Physical mass is independent of the bare mass, but it does depend on the bare coupling:

$$\frac{\partial m^2}{\partial \lambda_0} = -\frac{2m^2}{\beta(\lambda,g)}. \quad (11)$$

This can be easily integrated:

$$m^2 = \Lambda^2 \exp\Big(-\frac{2(\lambda_0 - \tilde{\lambda}_0)}{\beta(\lambda,g)}\Big). \quad (12)$$

Here $\tilde{\lambda}_0$ is the value of bare coupling at which the physical mass equals the cutoff.

Let us discuss the expression for the mass squared in terms of the bare coupling, Eq. (12). We see that if the beta-function involved in the exponent is small, and the exponent is negative, the physical mass is much smaller than the cutoff. This is quite similar to what happens in the case of chromodynamics, Eq. (2). Indeed, the smallness of the physical mass scale is explained in both cases by a smallness of an exponential generated via summation of the leading terms of the perturbative expansion. There is also a difference between the two cases: in the case of the scalar field, the mass scale is determined in terms of both physical and bare couplings, while in the case of the asymptotic freedom only the bare coupling is involved. Thus, for the scalar field we need the smallness of the beta-function at the physical coupling, while in the case of chromodynamics we need the smallness of the bare coupling.

We also mention that the above picture imposes certain constraints on the model involving the scalar field. Apart of the smallness of the beta-function governing the evolution of the scalar self-coupling, we also should require that the determinant of the Jacobian, Eq. (9), would be nonzero at any cutoff. Otherwise, the mass and self-coupling of the scalar field may become infinite at a finite cutoff, and the limit of infinite cutoff employed in obtaining Eq. (12) is not justified. We postpone consideration of this requirement for future studies.

In conclusion, we demonstrated that the hierarchy problem is the problem of the leading order perturbation theory. Proper resummation of the higher orders of perturbation theory solves this problem. The weak mass scale is now more or less on par with the hadronic mass scale. Details of the dimensional transmutation for the weak scale differ the ones for the hadronic scale, but the basic feature persists: weak scale appears as the cutoff downsized with a very small exponential factor whose exponent is expressed via small couplings.

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