

Light Nuclei as Quantized Skyrmions: Energy Spectra and Form Factors

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Abstract

We review the $SU(2)$ Skyrme model and describe its topological soliton solutions, which are called Skyrmions. Skyrmions provide a model of nuclei in which the conserved topological charge is identified with the baryon number of a nucleus. The semiclassical quantum theory of Skyrmions, in which they are treated as rigid bodies spinning in space and isospace, is described. We derive the energy spectra corresponding to various light nuclei, and predict a few new states. We also calculate the electromagnetic form factors describing the structure of the α -particle and lithium-6. Our recent reparametrization of the model gives results that are in reasonable quantitative agreement with experiment.

1 Introduction

The Skyrme model, a nonlinear classical field theory of pions, was introduced in the early 1960s as a tentative description of the strongly interacting elementary particles [43]. The model provides a low energy effective theory of quantum chromodynamics (becoming exact as the number of quark colours becomes large) [51, 52]. QCD has an important broken symmetry: the approximate chiral $SU(2) \times SU(2)$ symmetry of strong interactions. By considering the conserved vector and axial-vector currents of QCD, it can be deduced that if this symmetry is exact and unbroken, parity doubling would be seen in the hadron spectrum. However, no such phenomenon is observed. So chiral symmetry must be spontaneously broken to its isospin $SU(2)$ subgroup. A spontaneously broken approximate chiral symmetry entails the existence of approximately massless Goldstone bosons. The three pion particles π^+ , π^0 and π^- behave as these approximate Goldstone bosons [50]. The Skyrme model captures this broken chiral symmetry.

Interestingly, the Skyrme model admits topological soliton solutions, *Skyrmions*, with an integer-valued conserved topological charge. Skyrmions provide a model of atomic nuclei in which one interprets a quantized charge B Skyrmion as a nucleus with baryon number B . We describe a semiclassical quantum theory of Skyrmions and our recent progress in describing light nuclei within this framework.

2 The Skyrme Model

The Skyrme model is a nonlinear theory of pions defined in terms of four fields: σ , π_1 , π_2 and π_3 , subject to the constraint $\sigma^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = 1$ [35]. The Skyrme field is an $SU(2)$ matrix defined as

$$U = \sigma 1_2 + i\boldsymbol{\pi} \cdot \boldsymbol{\tau} = \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi_1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix}, \quad (1)$$

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where $\boldsymbol{\tau}$ denotes the triplet of Pauli matrices.

The Lagrangian density is given by

$$\mathcal{L} = \frac{F_\pi^2}{16} \text{Tr} \partial_\mu U \partial^\mu U^\dagger + \frac{1}{32e^2} \text{Tr} [\partial_\mu U U^\dagger, \partial_\nu U U^\dagger] [\partial^\mu U U^\dagger, \partial^\nu U U^\dagger] + \frac{1}{8} m_\pi^2 F_\pi^2 \text{Tr} (U - 1_2), \quad (2)$$

where F_π is the pion decay constant, e is a dimensionless parameter and m_π is the pion mass. One imposes the boundary condition $U(\mathbf{x}) \rightarrow 1_2$ as $|\mathbf{x}| \rightarrow \infty$. The vacuum, the unique field of minimal energy, is then $U(\mathbf{x}) = 1_2$ for all \mathbf{x} . In the absence of the term involving the pion mass, the Lagrangian would be symmetric under the $SU(2) \times SU(2)$ chiral symmetry $U \mapsto A_1 U A_2^\dagger$, where A_1 and A_2 are constant elements of $SU(2)$. The vacuum $U = 1_2$ spontaneously breaks this symmetry down to the isospin $SU(2)$ subgroup $U \mapsto A U A^\dagger$, where $A \in SU(2)$. In addition, with the pion mass term present there is a small explicit breaking of chiral symmetry. The Skyrme model therefore captures the most fundamental property of QCD, that of spontaneous chiral symmetry breaking.

Using energy and length units of $F_\pi/4e$ and $2/eF_\pi$ respectively, the Skyrme Lagrangian can be rewritten as

$$L = \int \left\{ -\frac{1}{2} \text{Tr} (R_\mu R^\mu) + \frac{1}{16} \text{Tr} ([R_\mu, R_\nu] [R^\mu, R^\nu]) + m^2 \text{Tr} (U - 1_2) \right\} d^3x, \quad (3)$$

where $R_\mu = (\partial_\mu U) U^\dagger$, and the dimensionless pion mass $m = 2m_\pi/eF_\pi$.

At a fixed time, U is a map from \mathbb{R}^3 to S^3 , the group manifold of $SU(2)$. The boundary condition $U \rightarrow 1_2$ implies a one-point compactification of space, so that topologically U can be regarded as a map from S^3 to S^3 . As $\pi_3(S^3) = \mathbb{Z}$, Skyrme field configurations fall into homotopy classes labelled by an integer B , the baryon number, which is equal to the integral over space of the baryon density $B_0(\mathbf{x})$:

$$B = \int B_0(\mathbf{x}) d^3x, \quad (4)$$

where

$$B_\mu(\mathbf{x}) = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{Tr} \partial^\nu U U^\dagger \partial^\alpha U U^\dagger \partial^\beta U U^\dagger \quad (5)$$

is the baryon current.

Restricting to static fields $U(\mathbf{x})$, the Skyrme energy functional derived from the Lagrangian is

$$E = \int \left\{ -\frac{1}{2} \text{Tr} (R_i R_i) - \frac{1}{16} \text{Tr} ([R_i, R_j] [R_i, R_j]) - m^2 \text{Tr} (U - 1_2) \right\} d^3x. \quad (6)$$

For a given baryon number B , we denote the minimized energy by \mathcal{M}_B , and we call the field that minimizes E a Skyrmion. \mathcal{M}_B can be identified with the static Skyrmion mass. Occasionally, we also refer to some non-global minima of the energy and some low-lying saddle point solutions as Skyrmions too. The Skyrme energy functional has a nine-dimensional symmetry group, consisting of translations and rotations in \mathbb{R}^3 , together with isospin transformations. Consequently, Skyrmions lie on orbits of this symmetry group. Generically, this orbit is nine-dimensional, although for especially symmetric Skyrmions it is of lower dimension.

3 Symmetric Skyrmions

The minimal energy Skyrmion in the $B = 1$ sector is spherically symmetric and takes the form

$$U(\mathbf{x}) = \exp(i f(r) \hat{\mathbf{x}} \cdot \boldsymbol{\tau}) = \cos f(r) 1_2 + i \sin f(r) \hat{\mathbf{x}} \cdot \boldsymbol{\tau}, \quad (7)$$

where f is a radial profile function obeying an ordinary differential equation with the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$. Skyrmions with $B > 1$ all have interesting shapes; they

are not spherical like the $B = 1$ Skyrmion. Figure 1 shows surfaces of constant baryon density for Skyrmions with $1 \leq B \leq 8$, with the dimensionless pion mass parameter $m = 0$ [35]. The surfaces of constant energy density are qualitatively rather similar. The $B = 2$ Skyrmion, for example, has axial symmetry and its baryon density has a toroidal structure [29, 34, 48]. The Skyrmions presented in Fig. 1 have only discrete symmetries for $B > 2$. The $B = 3$ and $B = 4$ Skyrmions have tetrahedral and octahedral symmetry respectively. The $B = 5, 6$ and 8 Skyrmions have extended dihedral symmetries, and the $B = 7$ Skyrmion has icosahedral symmetry.

Some of these symmetric Skyrmions can be formed instantaneously during the collision of well separated $B = 1$ Skyrmions. For example, three Skyrmions initially placed on the vertices of a large contracting equilateral triangle scatter through the tetrahedral $B = 3$ Skyrmion, which then splits into a single Skyrmion and a $B = 2$ torus [8]. The dynamics is remarkably similar to the scattering of three $SU(2)$ Bogomolny-Prasad-Sommerfield monopoles.

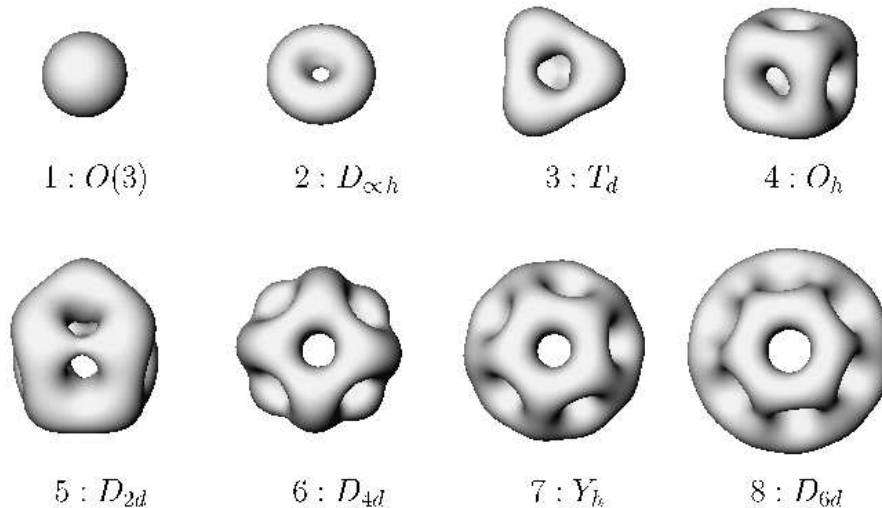


Figure 1: Skyrmions for $1 \leq B \leq 8$, with $m = 0$. A surface of constant baryon density is shown, together with the baryon number and symmetry.

In order to apply the Skyrme model to nuclear physics, a suitable parameter set must be chosen. A key consideration is the value of the dimensionless pion mass parameter m . When $m = 0$, the Skyrmions with $B \geq 3$, up to $B = 22$ [9, 10, 11] and beyond [5], are hollow polyhedra. The baryon density is concentrated in a shell of roughly constant thickness, surrounding a region in which the energy and baryon density are very small. Such a hollow structure is acceptable for small B , but clearly disagrees with the rather uniform baryon density observed in the interior of larger real nuclei. However, in the interior region of these Skyrmions the value of U is close to -1_2 , and for positive values of m , this is the value of U with highest potential energy. Not surprisingly, therefore, it is found that for baryon numbers $B \geq 8$ the hollow polyhedral Skyrmions do not remain stable when the pion mass parameter m is of order 1 [12, 13].

New stable Skyrmion solutions with baryon number a multiple of four have recently been found [7]. These solutions are clusters of cubic $B = 4$ Skyrmions, and so make contact with the α -particle model of nuclei [17]. They are of more uniform density than the old solutions. For example, when $m = 0$ the $B = 8$ Skyrmion is a hollow polyhedron with D_{6d} symmetry. However, when $m = 1$ the stable solution is found to be a bound configuration of two $B = 4$ cubic Skyrmions, with D_{4h} symmetry (see Fig. 2). This matches the known physics that beryllium-8 is an almost bound state of two α -particles. For $B = 12$, the new solution is

an equilateral triangle of three $B = 4$ cubes. The lowest energy solution has C_3 symmetry and is shown in Fig. 3, but there is a solution of very slightly higher energy with a larger D_{3h} symmetry. Rearranged solutions are analogous to the rearrangements of the α -particles which model excited states of nuclei. An example is the Skyrme model analogue of the three α -particles in a chain configuration for an excited state of carbon-12 [21, 37], which is displayed in Fig. 4.

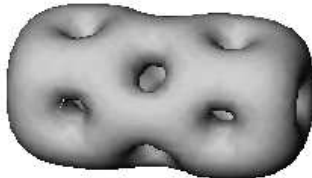


Figure 2: Baryon density isosurface for the $B = 8$ Skyrmion with $m = 1$, resembling two touching $B = 4$ Skyrmons.

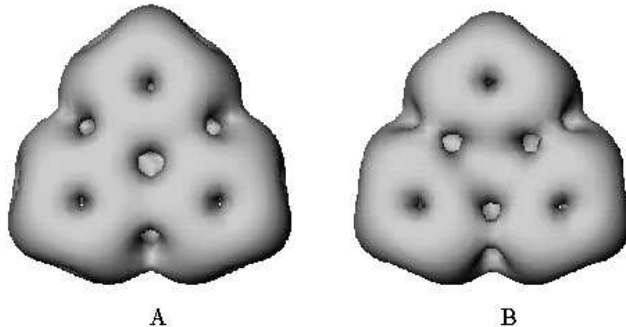


Figure 3: Top and bottom views of the $B = 12$ Skyrmion with triangular symmetry.

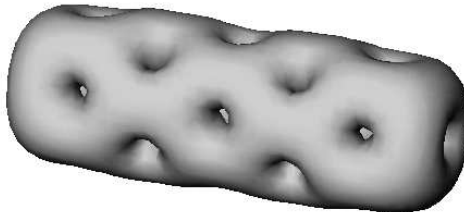


Figure 4: $B = 12$ Skyrmion formed from three cubes in a line.

4 The Rational Map Ansatz

Skyrmion solutions are known for several values of B , but they can only be obtained numerically. Motivated by similarities between lumps, monopoles and Skyrmons, an underlying connection in terms of rational maps between Riemann spheres was investigated by Houghton, Manton and Sutcliffe [26], and this leads to a method of constructing good approximations to several

known Skyrmions for $m = 0$, and also for non-zero m . The rational maps have exactly the same symmetries as the Skyrmions in almost all cases.

A rational map is a map from S^2 to S^2 , which can be expressed as

$$R(z) = \frac{p(z)}{q(z)}, \quad (8)$$

where p and q are polynomials in z . Via stereographic projection, a point on S^2 having complex coordinate z is identified as having conventional spherical polars given by

$$z = \tan \frac{\theta}{2} e^{i\phi}, \quad (9)$$

or equivalently as being the unit Cartesian vector

$$\frac{1}{1 + |z|^2} (2 \operatorname{Re}(z), 2 \operatorname{Im}(z), 1 - |z|^2). \quad (10)$$

We denote a point in \mathbb{R}^3 by its coordinates (r, z) , where r is the radial distance from the origin. The rational map ansatz for the Skyrme field is constructed from a rational map $R(z)$ and a profile function $f(r)$ as

$$U(r, z) = \begin{pmatrix} \cos f + i \sin f \frac{1 - |R|^2}{1 + |R|^2} & i \sin f \frac{2\bar{R}}{1 + |R|^2} \\ i \sin f \frac{2R}{1 + |R|^2} & \cos f - i \sin f \frac{1 - |R|^2}{1 + |R|^2} \end{pmatrix}. \quad (11)$$

For this to be well-defined at the origin and infinity, one imposes $f(0) = \pi$ and $f(\infty) = 0$.

The rational map ansatz leads to some simplifications. The baryon number is given by

$$B = \int \frac{-f'}{2\pi^2} \left(\frac{\sin f}{r} \right)^2 \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^2 \frac{2i dz d\bar{z}}{(1 + |z|^2)^2} r^2 dr, \quad (12)$$

and this reduces to the topological degree of the rational map, which is the greater of the algebraic degrees of p and q . The energy is given by

$$E = 4\pi \int_0^\infty \left(r^2 f'^2 + 2B \sin^2 f (f'^2 + 1) + \mathcal{I} \frac{\sin^4 f}{r^2} + 2m^2 r^2 (1 - \cos f) \right) dr, \quad (13)$$

in which \mathcal{I} denotes the angular integral

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i dz d\bar{z}}{(1 + |z|^2)^2}. \quad (14)$$

Minimal energy solutions within this ansatz are found by first minimizing \mathcal{I} over all maps of degree B . The profile function f is then found by solving the second order ordinary differential equation that is the Euler-Lagrange equation obtained from (13) with B , m and \mathcal{I} as parameters. Table 1 lists the energy-minimizing rational maps, their symmetries and the total Skyrme energy for $1 \leq B \leq 8$ and $m = 0$, and in Fig. 5 we present a graph of their corresponding profile functions. The Skyrmions increase in size with increasing B .

B	$R(z)$	$E/12\pi^2$	Symmetry
1	z	1.23	$O(3)$
2	z^2	2.42	$D_{\infty h}$
3	$\frac{\sqrt{3}iz^2-1}{z(z^2-\sqrt{3}i)}$	3.55	T_d
4	$\frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$	4.55	O_h
5	$\frac{z(z^4+bz^2+a)}{az^4-bz^2+1}$	5.74	D_{2d}
6	$\frac{z^4+ic}{z^2(icz^4+1)}$	6.82	D_{4d}
7	$\frac{7z^5+1}{z^2(z^5-7)}$	7.75	Y_h
8	$\frac{z^6-id}{z^2(idz^6-1)}$	8.94	D_{6d}

Table 1: Energy-minimizing rational maps. The parameters a , b , c and d are numerically determined as 3.07, 3.94, 0.16 and 0.14 respectively.

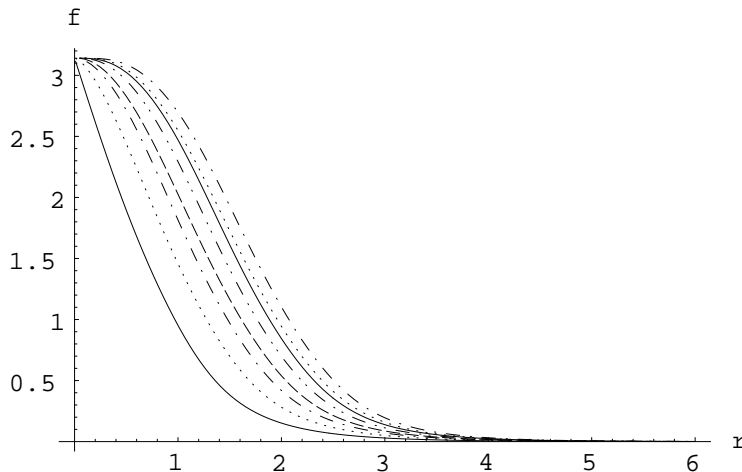


Figure 5: The profile functions $f(r)$ for $B = 1$ to 8. B increases from left to right.

5 Skymion Quantization

In the semiclassical method of Skymion quantization, one quantizes the spin and isospin rotational degrees of freedom while treating the Skymion as a rigid body. As mentioned previously, the symmetry group of the model is nine-dimensional. Given a generic static Skymion U_0 , there is a nine-parameter set of configurations, all degenerate in energy, obtained from U_0 by some combination of translation, rotation and isorotation:

$$U(\mathbf{x}) = A_1 U_0 (D(A_2)(\mathbf{x} - \mathbf{X})) A_1^\dagger, \quad (15)$$

where $A_1, A_2 \in SU(2)$ and $D(A_2)_{ij} = \frac{1}{2} \text{Tr}(\tau_i A_2 \tau_j A_2^\dagger) \in SO(3)$. The quantization procedure promotes the collective coordinates A_1, A_2, \mathbf{X} to dynamical variables, each depending on time [15]. In what follows, the translational degrees of freedom are ignored and the Skymions are quantized in their rest frames.

The ansatz for the dynamical Skyrme field is then given by

$$\hat{U}(\mathbf{x}, t) = A_1(t)U_0(D(A_2(t))\mathbf{x})A_1(t)^\dagger. \quad (16)$$

Inserting this into the Lagrangian (3), we obtain the kinetic energy

$$T = \frac{1}{2}a_i U_{ij} a_j - a_i W_{ij} b_j + \frac{1}{2}b_i V_{ij} b_j, \quad (17)$$

where b_i and a_i are the angular velocities in space and isospace respectively,

$$a_j = -i \text{Tr } \tau_j A_1^\dagger \dot{A}_1, \quad b_j = i \text{Tr } \tau_j \dot{A}_2 A_2^\dagger, \quad (18)$$

and the inertia tensors U_{ij} , W_{ij} and V_{ij} are functionals of the Skyrmion U_0 given by

$$U_{ij} = - \int \text{Tr} \left(T_i T_j + \frac{1}{4} [R_k, T_i] [R_k, T_j] \right) d^3 x, \quad (19)$$

$$W_{ij} = \int \epsilon_{jlm} x_l \text{Tr} \left(T_i R_m + \frac{1}{4} [R_k, T_i] [R_k, R_m] \right) d^3 x, \quad (20)$$

$$V_{ij} = - \int \epsilon_{ilm} \epsilon_{jnp} x_l x_n \text{Tr} \left(R_m R_p + \frac{1}{4} [R_k, R_m] [R_k, R_p] \right) d^3 x, \quad (21)$$

where $R_k = (\partial_k U_0) U_0^\dagger$ and $T_i = \frac{i}{2} [\tau_i, U_0] U_0^\dagger$.

The momenta corresponding to b_i and a_i are the body-fixed spin and isospin angular momenta L_i and K_i :

$$L_i = -W_{ij}^T a_j + V_{ij} b_j, \quad (22)$$

$$K_i = U_{ij} a_j - W_{ij} b_j. \quad (23)$$

The space-fixed spin and isospin angular momenta are denoted J_i and I_i respectively. One regards L_i , K_i , J_i and I_i as quantum operators, each set satisfying the $\mathfrak{su}(2)$ commutation relations. The quantum Hamiltonian is obtained by re-expressing (17) in terms of L_i and K_i . The symmetries of the inertia tensors are related to the symmetries of the Skyrmion U_0 . In several cases, these tensors are proportional to the identity matrix, in which case the Hamiltonian is that of a spherical top. If the matrices have two or three distinct eigenvalues, the Hamiltonian is that of a symmetric or asymmetric top, respectively.

Finkelstein and Rubinstein showed that it is possible to quantize a single Skyrmion as a fermion by defining wavefunctions on the covering space of the classical configuration space, which is a double cover for any value of B [20]. The wavefunction is defined in such a way that it has opposite signs on the two points of the covering space that cover one point in the configuration space. The basic Finkelstein-Rubinstein (FR) constraints on Skyrmion states for $B > 1$ implement the requirements of the Pauli exclusion principle (for nucleons). In particular they imply that for even B the spin and isospin are integral, and for odd B they are half-integral. Further FR constraints arise whenever the Skyrmion has special nontrivial symmetries. These constraint equations are now relatively easy to determine with the help of the rational map ansatz [30]. For example, the toroidal symmetry of the $B = 2$ Skyrmion leads to FR constraints which imply that the quantum ground state has spin 1 and isospin 0, in agreement with the quantum numbers of the deuteron [15]. Also, the FR constraints imply that the lowest quantum state of the double cube $B = 8$ Skyrmion has spin 0 and isospin 0, which is consistent with the quantum numbers of beryllium-8 [7].

A rational map $R(z)$, and hence the corresponding Skyrmion, has a rotational symmetry if it satisfies an equation of the form $R(M_2(z)) = M_1(R(z))$, where M_1 and M_2 are Möbius transformations. M_2 corresponds to a spatial rotation defined by an angle θ_2 and an axis \mathbf{n}_2 ; and M_1 corresponds to an isorotation defined by an angle θ_1 and an axis \mathbf{n}_1 . Such a symmetry

gives rise to a loop in configuration space (one thinks of this as a loop by letting the isorotation angle increase from 0 to θ_1 , while the rotation angle increases from 0 to θ_2). The symmetry leads to the following constraint on the wavefunction:

$$e^{i\theta_2 \mathbf{n}_2 \cdot \mathbf{L}} e^{i\theta_1 \mathbf{n}_1 \cdot \mathbf{K}} |\Psi\rangle = \chi_{\text{FR}} |\Psi\rangle, \quad (24)$$

where the FR sign χ_{FR} enforces the fermionic quantization condition:

$$\chi_{\text{FR}} = \begin{cases} +1 & \text{if the loop induced by the symmetry is contractible,} \\ -1 & \text{otherwise.} \end{cases} \quad (25)$$

Krusch showed that the FR sign depends only on the angles θ_1 and θ_2 (provided a crucial base point condition is satisfied for further details of which we refer the reader to Ref. [30]) through the formula

$$\chi_{\text{FR}} = (-1)^{\mathcal{N}}, \quad \text{where } \mathcal{N} = \frac{B}{2\pi}(B\theta_2 - \theta_1). \quad (26)$$

A convenient basis for the wavefunctions is given by the direct products $|J, J_3, L_3\rangle \otimes |I, I_3, K_3\rangle$, which is effectively shorthand for tensor products of Wigner D -functions parametrized by the rotational and isorotational Euler angles. In what follows, the arbitrary third components of the space and isospace angular momenta are omitted, so basis states are denoted $|J, L_3\rangle \otimes |I, K_3\rangle$. By considering the reflection symmetries of the rational maps, one can determine the parities of the quantum states, but we do not discuss this further here.

6 Reparametrizing the Model

Adkins, Nappi and Witten first quantized the $B = 1$ Skyrmion and showed that the lowest energy states may be identified with the proton/neutron isospin doublet [1, 2]. The masses of the nucleons and deltas were used to calibrate the model, and they obtained these values for the Skyrme parameters:

$$e = 4.84, \quad F_\pi = 108 \text{ MeV} \quad \text{and} \quad m_\pi = 138 \text{ MeV} \quad (\text{which implies } m = 0.528). \quad (27)$$

However, the delta is rotating at relativistic speeds and decays very rapidly, so this calibration is not very reliable [6].

In [36] we considered the lowest lying quantum state of the $B = 6$ Skyrmion, which has the quantum numbers of the lithium-6 nucleus in its ground state, i.e. spin 1 and isospin 0. We calculated a number of its static properties, dependent only on the Skyrme model parameters, and then chose e , F_π and m such that the predictions of the model agree precisely with the experimentally determined values. It appears that this new parameter choice more accurately describes properties of small nuclei than the traditional parameter set (27). By roughly doubling m to 1.125, we obtained a mean charge radius of the quantized $B = 6$ Skyrmion in close agreement with that of the lithium-6 nucleus. This sets a new Skyrme length scale. We then showed, provided a slight modification of the rational map is performed (while preserving its symmetry), that the quadrupole moment agrees with experiment. Finally, by equating the mass of the quantized $B = 6$ Skyrmion to the lithium-6 nucleus, we fitted the energy scale of the model. Keeping the pion mass fixed at its physical value, we have a new set of Skyrme parameters:

$$e = 3.26, \quad F_\pi = 75.2 \text{ MeV} \quad \text{and} \quad m_\pi = 138 \text{ MeV} \quad (\text{which implies } m = 1.125). \quad (28)$$

Reconsidering the α -particle and deuteron as quantized $B = 4$ and $B = 2$ Skyrmions gives further support for these new values. In what follows, this new parameter set is used throughout.

7 Energy Spectra of Light Nuclei

Here we review and slightly extend our work on the semiclassical quantization of Skyrmions with $B = 4, 6$ and 8 , as approximated by the rational map ansatz [33]. By exploiting the holomorphic character of the rational map one obtains useful general expressions for the elements of the inertia tensors (19,20,21) in terms of the approximating rational map. These formulae are slightly simpler than those obtained by Kopeliovich [28]. Using these formulae, one can apply techniques detailed in Ref. [31] to calculate the energy spectra of the quantized Skyrmions.

7.1 $B = 4$

The $B = 4$ Skyrmion has octahedral symmetry and a cubic shape, and is described by the rational map

$$R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \quad (29)$$

The generating symmetries of the rational map are

$$R(iz) = \frac{1}{R(z)}, \quad R\left(\frac{iz+1}{-iz+1}\right) = e^{i\frac{2\pi}{3}} R(z), \quad (30)$$

which lead to the FR constraints

$$e^{i\frac{\pi}{2}L_3} e^{i\pi K_1} |\Psi\rangle = |\Psi\rangle, \quad e^{i\frac{2\pi}{3\sqrt{3}}(L_1+L_2+L_3)} e^{i\frac{2\pi}{3}K_3} |\Psi\rangle = |\Psi\rangle. \quad (31)$$

Solving these in the basis described previously, one obtains the ground state $|0, 0\rangle \otimes |0, 0\rangle$ with spin 0 and isospin 0, which are the quantum numbers of the helium-4 nucleus, or α -particle. This is in agreement with earlier work of Walhout [49]. The next lowest lying states are a spin 2, isospin 1 state given by

$$\left(|2, 2\rangle + \sqrt{2}i|2, 0\rangle + |2, -2\rangle\right) \otimes |1, 1\rangle - \left(|2, 2\rangle - \sqrt{2}i|2, 0\rangle + |2, -2\rangle\right) \otimes |1, -1\rangle, \quad (32)$$

and a spin 4, isospin 0 state given by

$$\left(|4, 4\rangle + \sqrt{\frac{14}{5}}|4, 0\rangle + |4, -4\rangle\right) \otimes |0, 0\rangle. \quad (33)$$

The symmetries imply that the elements of the inertia tensors U_{ij} , V_{ij} and W_{ij} are all diagonal, with $U_{11} = U_{22}$, $V_{ij} = v\delta_{ij}$ and $W_{ij} = 0$. U_{11} , U_{33} and v are numerically determined. The quantum Hamiltonian is given by

$$T = \frac{1}{2v}\mathbf{J}^2 + \frac{1}{2U_{11}}\mathbf{I}^2 + \frac{1}{2}\left(\frac{1}{U_{33}} - \frac{1}{U_{11}}\right)K_3^2. \quad (34)$$

The eigenvalue of J^2 in states of spin J is $J(J+1)$, the standard result. Similarly I^2 has eigenvalues $I(I+1)$. The energy eigenvalues are calculated as:

$$E_{J=0, I=0} = \mathcal{M}_4 = 3679 \text{ MeV}, \quad (35)$$

$$E_{J=2, I=1} = \mathcal{M}_4 + 28.7 \text{ MeV} = 3708 \text{ MeV}, \quad (36)$$

$$E_{J=4, I=0} = \mathcal{M}_4 + 39.4 \text{ MeV} = 3718 \text{ MeV}. \quad (37)$$

For the ground state, the energy is simply the static mass of the Skyrmion, \mathcal{M}_4 . Comparing this to the mass of the α -particle, 3727 MeV, we see that our prediction comes to within 2% of the experimental value. For the spin 2, isospin 1 state, the excitation energy is 28.7 MeV. We note that hydrogen-4, helium-4 and lithium-4 form an isospin triplet, whose lowest energy

state has spin 2 and has an average excitation energy of 23.7 MeV relative to the ground state of helium-4, so here the Skyrmion picture works well [45]. Finally, we predict a spin 4, isospin 0 state with an excitation energy of 39.4 MeV. Such a state of helium-4 has not yet been seen experimentally. However, suggestions that such a state exists with an excitation energy of 24.6 MeV have been made previously [23, 24]. In Fig. 6 we present the energy level diagram for the quantized $B = 4$ Skyrmion and the corresponding experimentally observed states.

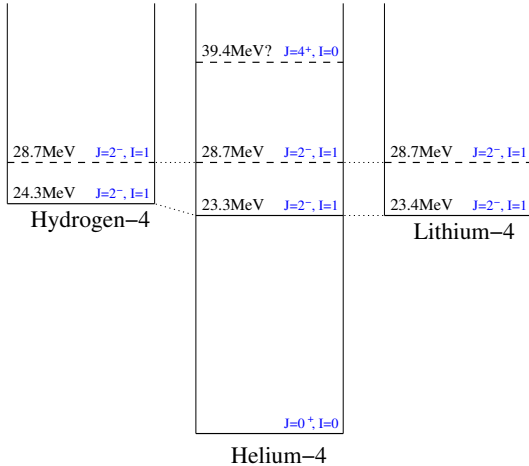


Figure 6: Energy level diagram for the quantized $B = 4$ Skyrmion. Solid lines indicate experimentally observed states, while dashed lines indicate our predictions.

7.2 $B = 6$

The quantization of the $B = 6$ Skyrmion was first considered by Irwin [27]. The Skyrmion has D_{4d} symmetry and can be approximated using the rational map

$$R(z) = \frac{z^4 + ia}{z^2(iaz^4 + 1)}, \quad a = 0.16. \quad (38)$$

Solving the FR constraints arising from the Skyrmion's symmetry one obtains its allowed quantum states [36]. The lowest three are $|1, 0\rangle \otimes |0, 0\rangle$, $|3, 0\rangle \otimes |0, 0\rangle$ and $|0, 0\rangle \otimes |1, 0\rangle$. The Hamiltonian is given by

$$T = \frac{1}{2V_{11}} [\mathbf{J}^2 - L_3^2] + \frac{1}{2U_{11}} [\mathbf{I}^2 - K_3^2] + \frac{1}{2(U_{33}V_{33} - W_{33}^2)} [U_{33}L_3^2 + V_{33}K_3^2 + 2W_{33}L_3K_3]. \quad (39)$$

The static Skyrmion mass, \mathcal{M}_6 , is set to be 5600 MeV, just below the mass of the lithium-6 nucleus, to allow for the spin energy which is of order 1 MeV¹. The energy eigenvalues corresponding to the lowest three states are then given by

$$E_{J=1, I=0} = \mathcal{M}_6 + \frac{1}{V_{11}} = \mathcal{M}_6 + 1.7 \text{ MeV} = 5601 \text{ MeV}, \quad (40)$$

$$E_{J=3, I=0} = \mathcal{M}_6 + \frac{6}{V_{11}} = \mathcal{M}_6 + 10.3 \text{ MeV} = 5610 \text{ MeV}, \quad (41)$$

$$E_{J=0, I=1} = \mathcal{M}_6 + \frac{1}{U_{11}} = \mathcal{M}_6 + 12.1 \text{ MeV} = 5612 \text{ MeV}. \quad (42)$$

These states, together with further allowed states, are displayed on the energy level diagram in Fig. 7. The experimental energy level diagram is displayed in Fig. 8 [46].

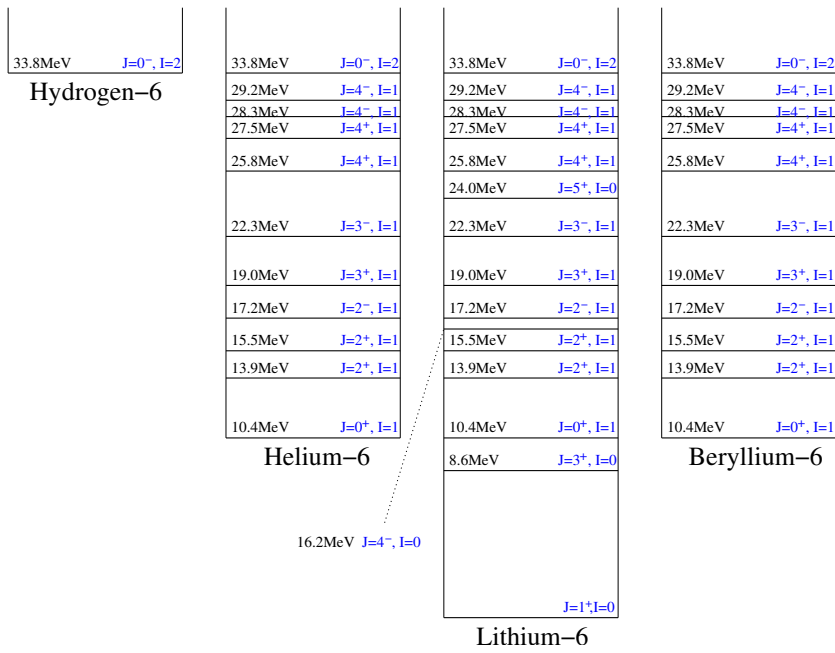


Figure 7: Energy level diagram for the quantized $B = 6$ Skyrmion. Energies are given relative to the spin 1, isospin 0 ground state.

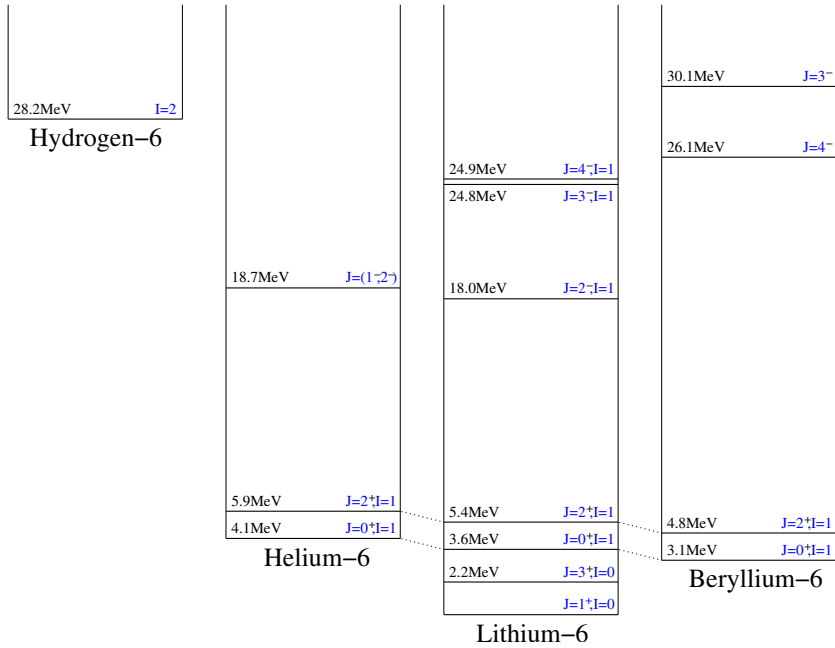


Figure 8: Energy level diagram for nuclei with $B = 6$.

The ground state of the lithium-6 nucleus has spin 1 and isospin 0, and there is an excited state with spin 3 with excitation energy 2.2 MeV. The lowest states of the isotriplet of helium-6, lithium-6 and beryllium-6 have spin 0 and spin 2. Further states of this isotriplet with spins 2, 3 and 4 are experimentally observed although the data are not complete. One state of an isospin 2 multiplet, the ground state of hydrogen-6, is also observed. The Skyrme model qualitatively

¹This updates Ref. [36] where we treated the spin energy as negligible

reproduces this spectrum and has some further states. The hydrogen-6 state is predicted to have spin 0. The only problem is for the lower spin states of lithium-6 as the energy splittings are too large by a factor of three to four. Note that we predict a number of states that have not yet been seen experimentally, for example spin 4 and spin 5 excited states of lithium-6 with isospin 0. The quantization of further degrees of freedom, such as vibrational modes, may lead to improvements of these predictions. For example, we have not considered the possibility of the nucleus separating into an α -particle and a deuteron.

7.3 $B = 8$

As described in section 3, the stable $B = 8$ Skyrmion when m is of order 1 resembles two touching $B = 4$ cubes (see Fig. 2). In this case, the rational map ansatz does not provide a quantitatively accurate approximation. However, a rational map that has the equivalent D_{4h} symmetry can be used to determine the FR constraints and quantum states of the Skyrmion. Such a rational map is

$$R(z) = \frac{z^8 + bz^6 - az^4 + bz^2 + 1}{z^8 - bz^6 - az^4 - bz^2 + 1}, \quad (43)$$

with a and b real. The FR constraints are given by

$$e^{i\frac{\pi}{2}L_3} e^{i\pi K_1} |\Psi\rangle = |\Psi\rangle, \quad e^{i\pi L_1} |\Psi\rangle = |\Psi\rangle. \quad (44)$$

These allow a ground state with spin 0, isospin 0, an excited state with spin 2, isospin 0, and many more excited states.

To estimate the moments of inertia one can work directly with the known inertia tensors of two $B = 4$ cubes and as a simplifying approximation use the parallel axis theorem [33]. The resulting inertia tensors are diagonal and satisfy the relations $U_{11} = U_{22}$, $V_{11} = V_{22}$ and $W_{ij} = 0$, and so the Hamiltonian is

$$T = \frac{1}{2V_{11}} [\mathbf{J}^2 - L_3^2] + \frac{1}{2U_{11}} [\mathbf{I}^2 - K_3^2] + \frac{L_3^2}{2V_{33}} + \frac{K_3^2}{2U_{33}}. \quad (45)$$

Figures 9 and 10 are energy level diagrams for the quantized $B = 8$ Skyrmion and for the $B = 8$ nuclei [47], respectively. Our predictions agree well with experiment. The predicted energy of the spin 2, isospin 0 state is 2.9 MeV, which is a very good match to the experimental value of 3 MeV. We obtain a spin 2 isotriplet with energy 13.3 MeV. Experimentally such a triplet exists with an average excitation energy of 16.5 MeV. We also obtain a spin 3 isotriplet with energy 16.2 MeV, which is experimentally seen with an average excitation energy of 19.0 MeV. We have also found quintets of $I = 2$ states. The lowest of these, with spin 0, has been detected experimentally with excitation energies very close to our prediction, and includes the helium-8 and carbon-8 ground states. We have recently started working with the exact values of the inertia tensors and have obtained similar results.

Of particular interest is the prediction from the Skyrme model of a spin 0 isotriplet of negative parity states, which if established experimentally could include new ground states of the lithium-8 and boron-8 nuclei. Low-lying spin 0, negative parity states could be difficult to observe, as experienced by the difficulty and the long time taken to observe the bottomonium and charmonium ground state mesons η_b and η_c [4, 39].

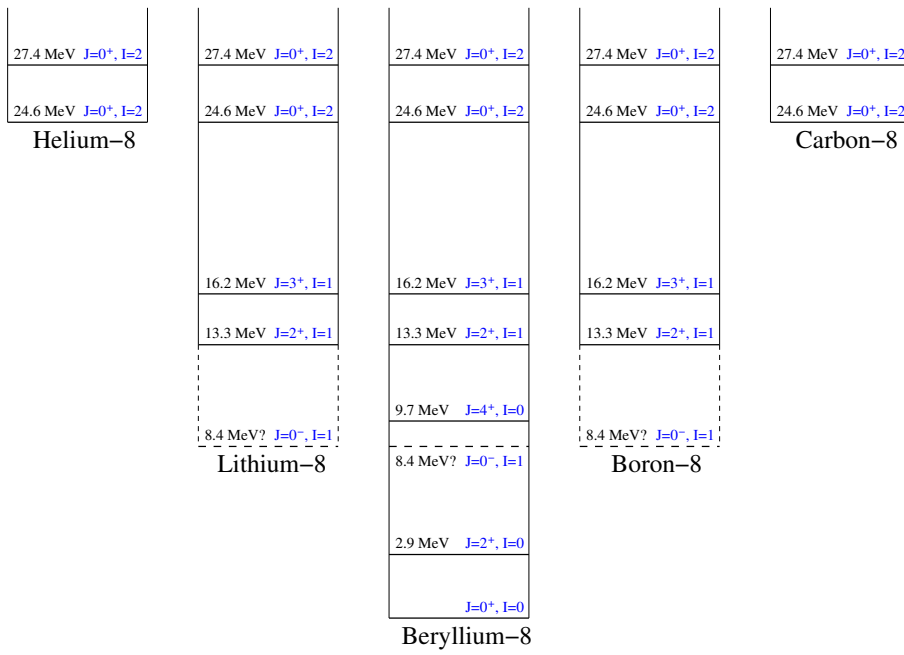


Figure 9: Energy level diagram for the quantized $B = 8$ Skyrmion. The putative $J = 0^-$ isotriplet is represented by dashed lines. Higher energy negative parity states are also predicted, but are omitted here.

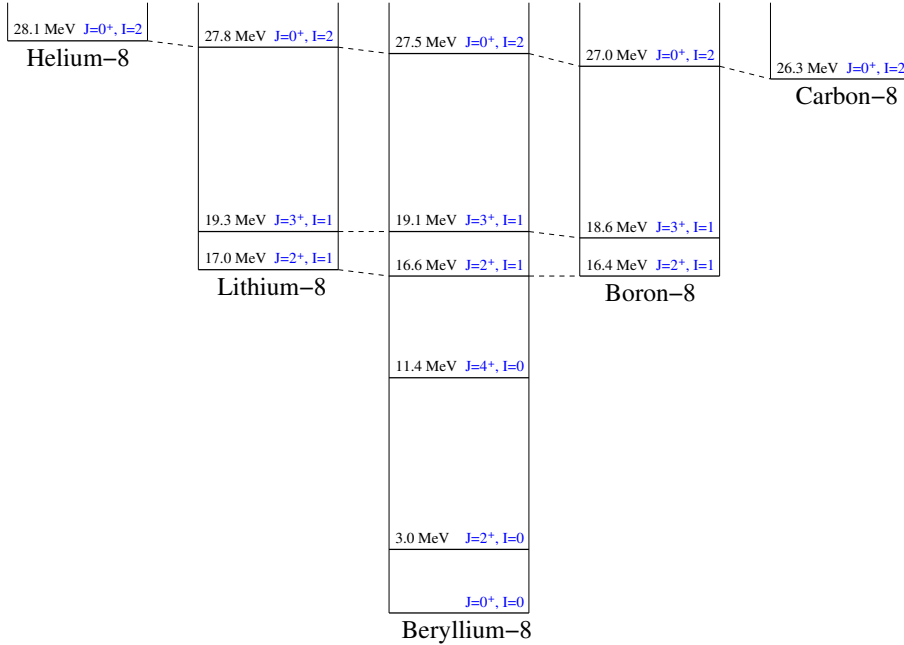


Figure 10: Energy level diagram for nuclei with $B = 8$ (selected levels).

8 Electromagnetic Form Factors of Quantized Skyrmions

The results described so far are encouraging for the Skyrme model of nuclei. Firstly, stable Skyrmions exist for all baryon numbers that we wish to consider. Secondly, they have quantum states with the same spin and isospin quantum numbers as their corresponding nuclei. And thirdly their energy spectra are in reasonable agreement with experiment. A criticism of the

model is that nuclei, bound states of protons and neutrons, bear little resemblance to highly symmetric classical Skyrmions, even taking into account that the classical Skyrmion occurs in all possible orientations in space and in isospace, with probability density determined by the collective coordinate wavefunction. The comparison of static properties and energy levels does not really address this criticism.

Now, the internal structure of nuclei can be investigated by electron scattering. The finite size of the nuclear charge distribution produces large deviations from the differential cross section for scattering from a point charge. A measure of this departure is provided by electromagnetic form factors. In this section we investigate whether the symmetries of the classical Skyrmions give rise to unusual behaviour in the form factors which would be incompatible with those of the nuclei they are supposed to model. The form factors of lithium-6 are calculated here, following the method developed by Braaten and Carson that was applied to the deuteron [16]. We will also consider the charge form factor of the α -particle.

We need to consider the full moduli space of Skyrmion collective coordinates, including translations, so we make the ansatz for the dynamical Skyrme field:

$$\hat{U}(\mathbf{x}, t) = A_1(t)U_0(D(A_2(t))(\mathbf{x} - \mathbf{X}(t)))A_1(t)^\dagger. \quad (46)$$

Form factors are calculated by inserting this ansatz into the expression for the electromagnetic current

$$\mathcal{J}_\mu = \frac{1}{2}B_\mu + I_\mu^3, \quad (47)$$

and determining matrix elements of the resulting operator between quantum states. I_μ^3 is the third component of the isospin current density, and B_μ the baryon current (5).

The ground state of the quantized $B = 6$ Skyrmion, which we recall has spin 1 and isospin 0, is

$$\Psi_{J_3}(\mathbf{p}) = \frac{\sqrt{3}}{8\pi^2} D_{0J_3}^1(\phi, \theta, \psi) D_{00}^0(\alpha, \beta, \gamma) e^{i\mathbf{p}\cdot\mathbf{X}} \quad (48)$$

in terms of Wigner D -functions parametrized by the rotational and isorotational Euler angles. J_3 is the third component of the space-fixed spin, and \mathbf{p} is the momentum. The charge and quadrupole form factors are defined in the Breit frame (i.e. the frame in which the sum of the initial and final momenta is zero) in terms of the matrix element of \mathcal{J}_0 between ground states:

$$\langle \Psi_{J'_3}(\mathbf{p}') | \mathcal{J}_0(\mathbf{x} = 0) | \Psi_{J_3}(\mathbf{p}) \rangle = G_C(q^2) \delta_{J'_3 J_3} + \frac{1}{6\mathcal{M}_6^2} G_Q(q^2) \Omega_{J'_3 a} (3q^a q^b - q^2 \delta^{ab}) \Omega_{b J_3}^\dagger, \quad (49)$$

where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ is the momentum transfer, $q^2 = \mathbf{q} \cdot \mathbf{q}$, \mathcal{M}_6 is the Skyrmion mass and $\Omega_{J_3 a}$ is the unitary matrix relating the Cartesian basis to the spin 1 angular momentum basis.

Braaten and Carson [16] obtained a general expression for the matrix element of \mathcal{J}_0 , which may be simplified to

$$\langle \Psi_{J'_3}(\mathbf{p}') | \mathcal{J}_0(0) | \Psi_{J_3}(\mathbf{p}) \rangle = \delta_{J'_3 J_3} \frac{1}{2} \int j_0(qr) B_0(\mathbf{x}) d^3x + \Omega_{J'_3 a} (3q^a q^b - q^2 \delta^{ab}) \Omega_{b J_3}^\dagger \frac{1}{4} \frac{1}{q^2} \int (1 - 3 \cos^2 \theta) j_2(qr) B_0(\mathbf{x}) d^3x, \quad (50)$$

where $j_n(qr)$ denote spherical Bessel functions and B_0 is the baryon density of the Skyrmion U_0 in its initial orientation. Therefore

$$G_C(q^2) = \frac{1}{2} \int j_0(qr) B_0(\mathbf{x}) d^3x, \quad (51)$$

$$\frac{1}{\mathcal{M}_6^2} G_Q(q^2) = \frac{3}{2} \frac{1}{q^2} \int (1 - 3 \cos^2 \theta) j_2(qr) B_0(\mathbf{x}) d^3x. \quad (52)$$

As $q^2 \rightarrow 0$, $G_C(q^2) \rightarrow 3 - \frac{1}{2}q^2 \langle r^2 \rangle$ and $G_Q(q^2) \rightarrow \mathcal{M}_6^2 Q$, where $\langle r^2 \rangle$ and Q are the squared mean charge radius and quadrupole moment. Note that $\langle r^2 \rangle$ is proportional to the derivative of $G_C(q^2)$ with respect to q^2 .

The structure of the lithium-6 nucleus is also described by a magnetic form factor. Before going into this, we will firstly describe the calculation of the magnetic dipole moment of the quantized $B = 6$ Skyrmion in the Skyrme model. The classical magnetic moment is defined by

$$\mu_a = \frac{1}{2} \int \epsilon_{abc} x_b \mathcal{J}_c d^3x. \quad (53)$$

As the ground state has isospin 0, the electromagnetic current density \mathcal{J}_c is equal to half the baryon current density (5), and so

$$\mu_a = \frac{1}{4} \int \epsilon_{abc} x_b B_c d^3x. \quad (54)$$

Inserting the expression for the rotated classical Skyrmion (16), we obtain

$$\hat{\mu}_a = D(A_2)_{a\alpha}^T (M_{\alpha k} a_k + N_{\alpha k} b_k), \quad (55)$$

where

$$M_{\alpha k} = \frac{1}{32\pi^2} \int x_\beta \text{Tr} (T_k [R_\alpha, R_\beta]) d^3x, \quad (56)$$

$$N_{\alpha k} = \frac{1}{32\pi^2} \int \epsilon_{krs} x_\beta x_r \text{Tr} (R_s [R_\alpha, R_\beta]) d^3x, \quad (57)$$

and these are calculated for the unrotated Skyrmion. These matrices are straightforward to calculate approximately using the rational map ansatz. The D_{4d} symmetry of the Skyrmion implies that $M_{\alpha k}$ and $N_{\alpha k}$ are diagonal and satisfy $M_{11} = M_{22} = 0$ and $N_{11} = N_{22}$. These relations imply that the terms of $\hat{\mu}_a$ involving N_{33} multiply the operator K_3 , which annihilates the quantum state of the Skyrmion. We ultimately obtain

$$\hat{\mu}_a = -\frac{N_{11}}{V_{11}} J_a + \text{terms proportional to } K_3, \quad (58)$$

where V_{11} is a component of the spatial inertia tensor (21). The magnetic dipole moment, μ , is defined to be the expectation value of $\hat{\mu}_3$ between quantum ground states with $J_3 = 1$, and so

$$\mu = -\frac{N_{11}}{V_{11}} = \frac{1}{8V_{11}} \int r^2 (1 + \cos^2 \theta) B_0(\mathbf{x}) d^3x. \quad (59)$$

μ is calculated to be 0.54 nm (in physical units), which is quite close to the experimental value for lithium-6 of 0.82 nm [40].

The magnetic form factor is defined in the Breit frame in terms of the matrix elements of the spatial components of the electromagnetic current density between quantum ground states of the $B = 6$ Skyrmion [16]:

$$\langle \Psi_{J'_3}(\mathbf{p}') | \mathcal{J}_i(\mathbf{x} = 0) | \Psi_{J_3}(\mathbf{p}) \rangle = \frac{1}{2\mathcal{M}_6} G_M(q^2) \Omega_{J'_3 a} (q^a \delta_i^b - \delta_i^a q^b) \Omega_{J_3 b}^\dagger. \quad (60)$$

This matrix element is given by

$$\langle \Psi_{J'_3}(\mathbf{p}') | \mathcal{J}_i(0) | \Psi_{J_3}(\mathbf{p}) \rangle = \Omega_{J'_3 a} (q^a \delta_i^b - \delta_i^a q^b) \Omega_{J_3 b}^\dagger \frac{3}{8V_{11}} \frac{1}{q} \int r (1 + \cos^2 \theta) j_1(qr) B_0(\mathbf{x}) d^3x, \quad (61)$$

from which the magnetic form factor is obtained as:

$$\frac{1}{2\mathcal{M}_6} G_M(q^2) = \frac{3}{8V_{11}} \frac{1}{q} \int r (1 + \cos^2 \theta) j_1(qr) B_0(\mathbf{x}) d^3x. \quad (62)$$

As $q^2 \rightarrow 0$, $G_M(q^2) \rightarrow 2\mathcal{M}_6\mu$. So the static electromagnetic properties are recovered from the electromagnetic form factors in the limit of zero momentum transfer.

In Fig. 11 we present a graph of the absolute normalized charge form factor for lithium-6 and for the quantized $B = 6$ Skyrmion. There is no method of experimentally separating $G_C(q^2)$ and $G_Q(q^2)$. Due to the smallness of $G_Q(q^2)$, we have made the comparison under the assumption that the observed electron scattering cross sections are due entirely to monopole charge scattering. This is consistent with other approaches [32]. We observe that the slopes of the theoretical and experimental form factors agree at $q^2 = 0$. This was, of course, to be expected as our new parameter set was chosen such that the mean charge radius is correctly predicted. The first cusp in the form factor is experimentally seen somewhere in the range $7 \text{ fm}^{-2} \leq q^2 \leq 8 \text{ fm}^{-2}$. We underpredict the location of this first cusp to be at roughly 2 fm^{-2} . We recall that the Skyrmion baryon density vanishes at the centre of the Skyrmion, which is unlike the conventional picture of a nucleus with its baryon density having at most a small dip in the centre. This may be the reason why the form factor cusps appear at too low values of the momentum transfer. Softening the Skyrmion by allowing it to vibrate may give a better fit. In comparison, in Fig. 12 we observe that our prediction for the absolute normalized magnetic form factor agrees rather well with experiment.

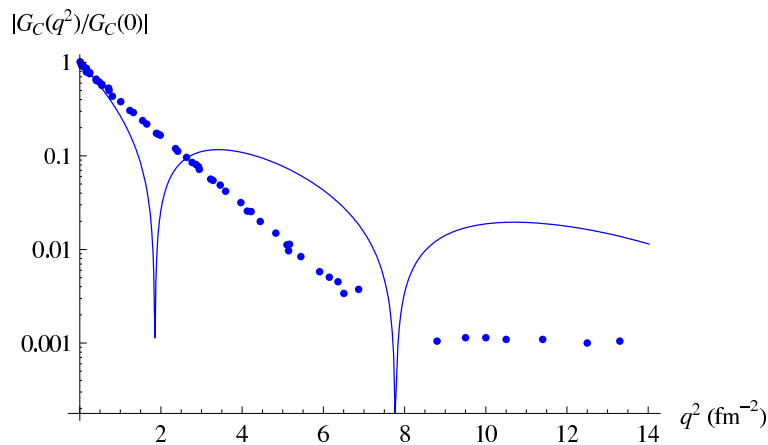


Figure 11: Absolute values of the charge form factor of the quantized $B = 6$ Skyrmion (solid) compared with experimental data for lithium-6 (dots) [18, 32, 44].

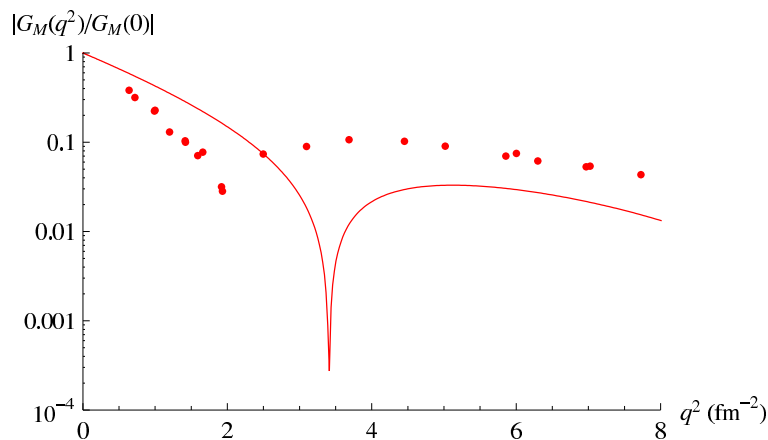


Figure 12: Absolute values of the magnetic form factor of the quantized $B = 6$ Skyrmion (solid) compared with experimental data for lithium-6 (dots) [14, 41]

We turn now to the α -particle. The cross section for the elastic scattering of electrons off a spin 0 nucleus, such as the α -particle, depends only on the charge form factor, which is given by

$$G_C(q^2) = \frac{1}{2} \int j_0(qr) B_0(\mathbf{x}) d^3x. \quad (63)$$

In Fig. 13 we have plotted the absolute normalized values of the charge form factor of the α -particle and the quantized $B = 4$ Skyrmion. Certainly, the two graphs have the same qualitative features, with the appearance of cusps in both cases. However, again our predicted cusps appear at smaller values of momentum transfer than the experimental cusps. Another thing to note is that we overpredict the magnitude of the slope of $G_C(q^2)$ at $q^2 = 0$, and correspondingly overpredict the mean charge radius of the nucleus. A further reparametrization of the Skyrme model, simultaneously leading to an accurate mean charge radius and to the correct location of the first cusp might be worth investigating. New data on the form factors of light nuclei are currently being collected at JLAB and preliminary results indicate a second cusp in the charge form factor of the α -particle [19].

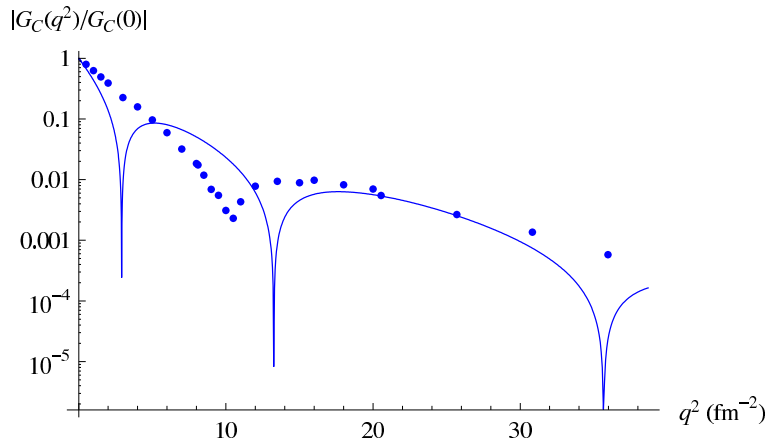


Figure 13: Absolute values of the charge form factor for the quantized $B = 4$ Skyrmion (solid) compared with experimental data for helium-4 (dots) [3, 22].

9 Conclusion

The work presented here has certainly provided support for Skyrme's field theoretical model for nuclei. In the Skyrme model, the individual $B = 1$ Skyrmions merge and lose their identities in the Skyrmion solutions with $B > 1$, which is unlike conventional nucleon potential models. This may capture an essential feature of nuclei in the configurations where nucleons are as close together as possible. It leads to a new geometry with which to describe nuclei. Our constraints on the allowed quantum numbers of states provide indirect support for this geometrical picture. The results which we have obtained for energy levels agree well with experiment, and the new parameter set which we proposed has enabled us to more accurately estimate properties of nuclei over a wider range than before. We have made predictions of a number of excited states of nuclei that have not been seen experimentally. This includes the prediction of a spin 4 state of helium-4, with an excitation energy higher than those of the experimentally established states. We have also predicted new spin 0, negative parity ground states of lithium-8 and boron-8. The qualitative behaviour of our form factors is in quite good agreement with experiment, and the symmetries of the classical Skyrmions do not lead to contradictions with experiment.

The approach here is the semiclassical method of quantization, which unifies the treatment of spin and isospin excitations. We have quantized the collective coordinates for translations,

rotations and isospin rotations, while ignoring further degrees of freedom, which are referred to as vibrational modes. Allowing the individual Skyrmions, or subclusters of Skyrmions, to move relative to each other, and performing a quantization of these degrees of freedom, would be a significant refinement. We have mainly used the rational map ansatz, which gives good approximations for small baryon numbers, but cannot easily be extended to higher B . With our collaborators Battye and Sutcliffe, we have recently started working with the numerically determined, exact Skyrmion solutions including $B = 10$ and $B = 12$, in order to model the corresponding nuclei, including boron-10 and carbon-12. Despite achieving considerable success in describing nuclei with even B , we would like to understand better the odd baryon number sectors of the model.

We have worked with the standard $SU(2)$ Skyrme model. The inclusion of strange quarks in the model would be a further refinement. The introduction of explicit vector meson fields leads to an improved description of the short-range structure of the nucleons [38, 53]. This could lead to further refinement of the modelling of nuclei, but not much is known about Skyrmions with $B > 1$ in these extended models.

Recent work by Sakai and Sugimoto and others has given further credence to the idea that at large N_c , baryons and nuclei are described by some variant of the Skyrme model [42]. Properties of baryons were predicted in this framework and it should be possible to predict properties of nuclei using this model [25].

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