Recent Developments in the Skyrme Model

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Abstract

In this talk, we describe recent developments in the Skyrme model. Our main focus is on discussing various effects which need to be taken into account, when calculating the properties of light atomic nuclei in the Skyrme model. We argue that an important step is to understand "spinning Skyrmions" and discuss the theory of relative equilibria in this context.

1 Introduction

The Skyrme model is a classical field theory modelling the strong interaction between atomic nuclei [1]. It has to be quantized in order to compare it to nuclear physics. Since Adkins et al. [2, 3] quantized the translational and rotational zero-modes of the $B = 1$, a lot of progress has been made in understanding both the classical solutions of the Skyrme model and the quantization of Skyrmions. The rational map ansatz [4] gives a very successful approximation to Skyrmions when the pion mass is zero and helped finding all the Skyrmions up to baryon number $B = 22$, [5]. It was shown in [6] that solitons in scalar field theories can consistently be quantized as fermions provided that the fundamental group of configuration space has a $\mathbb{Z}_2$ subgroup generated by a loop in which two identical solitons are exchanged. Noting that such loops arise from the symmetries of classical Skyrme configurations lead to the calculation of the quantum ground states up to $B = 22$, [7, 8]. Currently, there is a lot of interest in understanding light atomic nuclei with the aid of the Skyrme model. Now, the focus has shifted to understanding excited states and making quantitative predictions about their excitation energies. There are a number of issues which need to be taken into account. The Skyrme model only has three parameters. Traditionally, the value of [2, 3] have been used, but there are now a number of separate arguments why different values of the Skyrme parameters should be used instead, [9, 10]. It was also discovered that the pion mass has a big influence on the shape of Skyrmions, [11, 12, 13], in particular for $B > 7$. Finally, it is important to take into account that Skyrmions change their shape when they rotate, see [9, 14] for a discussion of $B = 1$. In this talk, we describe how the theory of relative equilibria [15, 16, 17] may shed light on spinning Skyrmions.

The talk is organized as follows. Section 2 gives a brief introduction to the Skyrme model, the rational map ansatz and the quantization of Skyrmions. In Section 3, we discuss the theory of relative equilibria. The final section describes preliminary results of applying the theory of relative equilibria to the Skyrme model and gives an outlook to future work.

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2 The Skyrme Model

The Skyrme model is a nonlinear theory of pions which models the strong interaction between atomic nuclei. It is defined in terms of a field $U(t, \mathbf{x}) \in SU(2)$. The Skyrme Lagrangian is given by

$$L = \int \left(-\frac{1}{2} \operatorname{Tr} (R_\mu R^\mu) + \frac{1}{16} \operatorname{Tr} ([R_\mu, R_\nu][R^\mu, R^\nu]) + m^2 \operatorname{Tr} (U - 1_2) \right) d^3x \,, \tag{1}$$

where $R_\mu = (\partial_\mu U)U^\dagger$, and m is proportional to the pion mass, [11]. The Skyrme Lagrangian has localized finite energy solutions which behave like particles and are known as Skyrmions.

The Lagrangian is written in "geometric units" in which length is measured in units of $2/ef_\pi$ and energy in units of $f_\pi/4e$. The parameters f_π and e are known as the pion decay constant and the Skyrme constant, respectively, and will be discussed in more detail later.

In order to have finite energy, Skyrme fields have to take a constant value, $U(|\mathbf{x}| = \infty) = 1$, at infinity. Therefore, from a topological point of view the Skyrme field U at a fixed time can be regarded as a map $U : S^3 \to S^3$, and such maps are characterized by an integer-valued winding number. This topological charge is interpreted as the baryon number, which for our purposes can be thought of as the number of protons and neutrons.

The Skyrme Lagrangian is invariant under the Poincaré group of Lorentz transformations and translations in space, together with rotations in target space. There is also a discrete symmetry

$$\mathcal{P} : U(\mathbf{x}) \mapsto U^\dagger(-\mathbf{x}) \tag{2}$$

known as parity. In the following, we will describe static solutions. We will ignore translations and focus on the symmetry group $SO(3) \times SO(3)$: rotations in space and rotations in target space (known as isorotations). As usual, rotations are given by $\mathbf{x} \mapsto R\mathbf{x}$ where $R^T R = 1$, and isorotations by $U \mapsto AUA^\dagger$, where $A \in SU(2)$. These symmetries will play a very important role for the calculation of quantum states in the Skyrme model.

2.1 The Rational Map Ansatz

In this section, we describe the rational map ansatz [4] which is a very successful approximation to minimal energy Skyrme configurations. The most convenient way for obtaining the explicit formula is the geometric approach of Manton [18, 19].

The main idea is to use rational maps, i.e. holomorphic maps between two-spheres in space and target space. In complex stereographic coordinates,

$$S^2 \to \mathbb{C} \cup \infty : (\theta, \phi) \mapsto z = \tan\left(\frac{\theta}{2}\right) \mathrm{e}^{i\phi} \,,$$

rational maps can simply be written as ratios of polynomials $p(z)$ and $q(z)$,

$$R(z) = \frac{p(z)}{q(z)}.$$

The inverse map for the stereographic projection is

$$\mathbf{n}_z = \frac{1}{1 + |z|^2} \begin{pmatrix} z + \bar{z} \\ i(\bar{z} - z) \\ 1 - |z|^2 \end{pmatrix}.$$

The ansatz for the Skyrme field is then given by

$$U(r, z) = \exp\left(i f(r)\, \mathbf{n}_{R(z)} \cdot \boldsymbol{\tau} \right),$$

where $\boldsymbol{\tau}$ denotes the Pauli matrices. This approximation leads to various simplifications. Angular and radial integrals decouple, and the angular integrals only depend on a finite number of parameters. The baryon number is given by

$$B = -\frac{1}{2\pi^2}\int f' \sin^2 f \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2 \frac{2i\,dz\,d\bar{z}}{(1+|z|^2)^2}\,dr\,,$$

and equals the (polynomial) degree of the rational map. Here, the profile function $f(r)$ satisfies the boundary conditions $f(0)=\pi$ and $f(\infty)=0$. The energy can be written as

$$E = 4\pi\int_0^\infty \left(r^2 f'^2 + 2B\sin^2 f\,(f'^2+1) + \mathcal{I}\,\frac{\sin^4 f}{r^2} + 2m^2 r^2\,(1-\cos f)\right)dr\,,$$

where

$$\mathcal{I} = \frac{1}{4\pi}\int \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^4 \frac{2i\,dz\,d\bar{z}}{(1+|z|^2)^2}\,.$$

In order to minimise E one first minimises $\mathcal{I}$ over all maps of degree B. The profile function f is then found by solving the Euler-Lagrange equation for E with B and $\mathcal{I}$ fixed. For small B the symmetry is sufficient to determine the relevant rational map. For larger B, the parameters have to be found by minimizing $\mathcal{I}$ numerically. The rational maps which minimize $\mathcal{I}$ for $m=0$ have been determined numerically in [5, 20] for all $B \leq 40$. The rational map ansatz has proven to be very successful for predicting energy and symmetries of Skyrmions (in particular for $m=0$). Table 1 gives some examples of rational maps, and figure 1 shows the corresponding level sets of constant energy density.

B	symmetry	$R(z)$
1	$O(3)$	z
2	$D_{\infty h}$	z^2
3	T_d	$\frac{\sqrt{3}iz^2-1}{z^3-\sqrt{3}iz}$
4	O_h	$\frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$

Table 1: Rational maps for the minimal energy solutions for $B=1,\ldots,4$.

The restriction that that $R(z)=p(z)/q(z)$ is a quotient of two holomorphic polynomials can be relaxed. By allowing p and q to be functions of z and $\bar{z}$ the rational map ansatz can be "improved". This leads to best approximations to the numerical solutions for $B=3$ and 4 known to date [22]. This approach also provides insight into the singularity structure of Skyrmions. However, the price to pay is that the energy now contains two functions which have to be minimized ($\mathcal{I}$ and $\mathcal{J}$). Another way to generalize the rational map ansatz is to "deform the sphere", so that the energy density in localized around a squashed sphere. The baryon density remains the same, but the energy density picks up even more terms (work in progress). For yet another generalization of the rational map ansatz see [23].

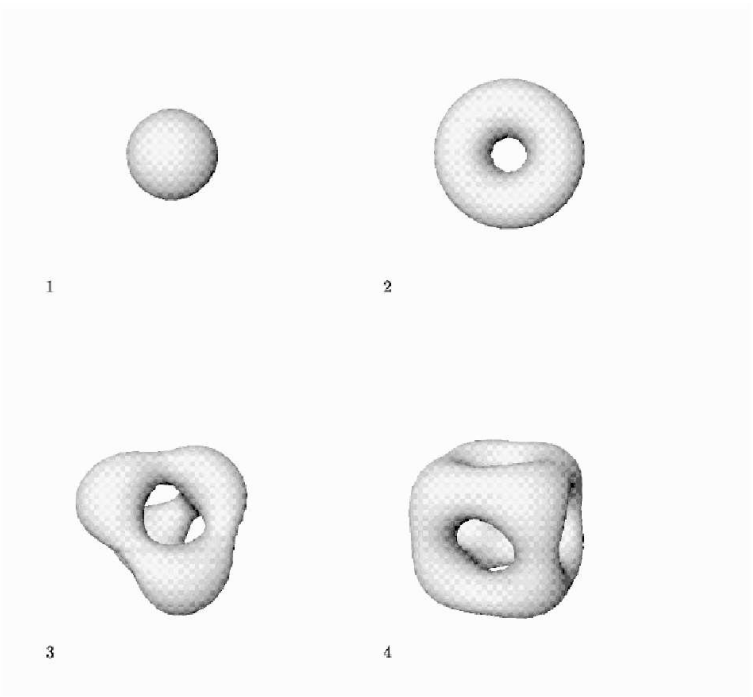

Figure 1: Energy density level sets for $B = 1, \ldots, 4$. (Figure taken from [21]).

2.2 Quantization of Skyrmions

In the following, we describe how to quantize the Skyrme model, which is a a scalar field theory, and obtain fermions following the approach of Finkelstein and Rubinstein [6]. Then we show how to calculate the Finkelstein-Rubinstein constraints using the rational map ansatz [8, 24].

So, how do we quantize a scalar field theory and obtain fermions? The key observation is that the configuration space of the Skyrme model is topologically non-trivial, namely,

$$\pi_1(Q_B) = \mathbb{Z}_2, \tag{3}$$

where Q_B denotes the space of Skyrme configurations with topological charge B. So, rather than defining the wavefunction on the configuration space Q_B we can define the wavefunctions ψ on the covering space of configuration space:

$$\psi : CQ_B \to \mathbb{C}.$$

Note that (3) implies that CQ_B is a double cover. In order to have fermionic quantization we have to impose the following condition,

$$\psi(\tilde{q}_1) = -\psi(\tilde{q}_2), \tag{4}$$

where $\tilde{q}_1$ and $\tilde{q}_2 \in CQ_B$ project to the same point $q \in Q_B$. In other words, $\tilde{q}_1$ and $\tilde{q}_2$ are related by a non-contractible loop in configuration space. Such loops naturally arise as symmetries of Skyrmions. For example, consider a rotation in space followed by a rotation in target space which leaves a given Skyrme configuration invariant. Assuming that the wavefunction ψ is localised at this Skyrme configuration leads to the following induced action of the $SO(3) \times SO(3)$ symmetry on ψ :

$$\exp\left(-i\alpha\, \mathbf{n} \cdot \mathbf{L}\right) \exp\left(-i\beta\, \mathbf{N} \cdot \mathbf{K}\right) \psi(\tilde{q}) = \chi_{FR} \psi(\tilde{q}), \tag{5}$$

$$\text{where } \chi_{FR} = \begin{cases} 1 & \text{if the induced loop is contractible,} \\ -1 & \text{otherwise.} \end{cases}$$

Here α, $\mathbf{n}$, β and $\mathbf{N}$ are the angles and axis of rotation in space and target space, $\mathbf{L}$ and $\mathbf{K}$ are the body-fixed angular momentum operators in space and target space, respectively. The key question is can we calculate $\chi_{FR} \in \pi_1(Q_B)$? The answer is yes, and relies on the following theorem.

Theorem 2.1 (S.K) *The rational map ansatz induces a surjective homomorphism* $\pi_1(Rat_B) \to \pi_1(Q_B)$.

This theorem implies that if we can calculate the Finkelstein-Rubinstein constraints for rational maps $R(z) \in Rat_B$, where Rat_B is the space of rational maps of degree B, then we also know the constraints for the full configuration space Q_B of the Skyrme model. Fortunately, we can calculate the constraints for rational maps, and hence obtain the Finkelstein-Rubinstein constraints

$$\chi_{FR} = (-1)^{\mathcal{N}} \quad \text{where} \quad \mathcal{N} = \frac{B}{2\pi}\left(B\alpha - \beta\right),$$

for the symmetry given by (5).

This approach is very general, and it is worth emphazising that the formula for the Finkelstein-Rubinstein constraints are exact, provided the (numerically calculated) exact Skyrme configuration can be deformed into a rational map Skyrmion while preserving the respective symmetries. However, in order to calculate the wavefunctions, we have to make some approximations. The simplest is the semiclassical collective coordinate quantization which we will describe in the following.

Starting from a static Skyrme configuration $U_0(\mathbf{x})$ we can generate a set of Skyrme configurations $U(\mathbf{x})$ which have the same energy as $U_0(\mathbf{x})$ via

$$U(\mathbf{x}) = A\, U_0\left(D\left(A'\right)\mathbf{x}\right) A^{\dagger}$$

where A and A' are constant $SU(2)$ matrices and $D(A')$ is the associated $SO(3)$ rotation. Inserting this ansatz into the Skyrme Lagrangian (1) we obtain the following kinetic energy

$$T = \frac{1}{2}a_i U_{ij} a_j - a_i W_{ij} b_j + \frac{1}{2} b_i V_{ij} b_j\,, \tag{6}$$

where $a_k = -i\mathrm{Tr}\left(\tau_k A^{\dagger}\dot{A}\right)$ and $b_k = -i\mathrm{Tr}\left(\tau_k \dot{A}' A'^{\dagger}\right)$. Here U_{ij}, V_{ij} and W_{ij} are integrals involving U_0. The conjugate momenta corresponding to b_i and a_i are the body-fixed spin and isospin angular momenta L_i and K_i:

$$\begin{aligned} L_i &= -W_{ij}^T a_j + V_{ij} b_j\,, \\ K_i &= U_{ij} a_j - W_{ij} b_j\,. \end{aligned}$$

We denote the space-fixed spin and isospin angular momenta by J_i and I_i respectively. Note $\mathbf{J}^2 = \mathbf{L}^2$ and $\mathbf{I}^2 = \mathbf{K}^2$. We now regard L_i, K_i, J_i and I_i as quantum operators, each individually satisfying the $\mathfrak{su}(2)$ commutation relations[1].

The idea is now to find the lowest values of spin and isospin which are compatible with all the Finkelstein-Rubinstein constraints, then calculate the energy of these states. The Finkelstein-Rubinstein constraints arise from the symmetries of the classical minimal energy configuration. A basis for the wavefunctions is given by $|J, J_3, L_3\rangle \otimes |I, I_3, K_3\rangle$. Via Legendre transform, the kinetic energy T can be expressed in terms of angular momentum operators. The matrices U_{ij}, V_{ij} and W_{ij} can be evaluated using

$$\Sigma_{ij} = 2\int \sin^2 f \frac{C_{\Sigma_{ij}}}{(1+|R|^2)^2}\left(1 + f'^2 + \frac{\sin^2 f}{r^2}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2\right) d^3x\,,$$

where $\Sigma = (U, V, W)$ and $C_{\Sigma_{ij}}$ only depends on angular variables [27]. A similar expression was derived in [28].

[1] For a discussion of the more subtle points about body-fixed and space-fixed angular momenta in this context see for example [25, 26]

2.3 Results and Challenges

The approach in the previous section has been carried out quite successfully by various authors. The groundstates and possible excited states have been calculated using Finkelstein-Rubinstein constraints for $B \leq 22$, [7, 8], leading to satisfactory results for even B, but the predictions for odd B are not as good. There are also some general results for even-even and odd-odd nuclei, which agree with experiment [24]. Manko, Manton and Wood [27] calculated the energy levels for $1 \leq B \leq 8$. The results correspond well to experiment, and energies and spins of a few as yet unobserved states have been predicted.

However, the quantitative results are not very accurate. For odd $B = 5$ and 7, the correct ground states can only be calculated by deforming the minimal energy solutions significantly, [29]. For $B = 10$, 18 and 22, the approach gives the wrong groundstates.

Despite these seemingly disappointing results there is cause for optimism. There are various ways in the above approximations can be improved. The zero-mode quantization is the simplest approximation to the quantum states (also known as rigid rotator approximation). A more sophisticated quantization can lead to better results [14, 30, 31]

There is also an ongoing discussion of how best to fix the three physical parameters (f_π, e and m) in the Skyrme model. The original, and most widely used, set of Skyrme parameters has been proposed in [2, 3], by matching to the proton and the Delta mass. However, it has been shown that this matching condition can be considered to be an artifact of the rigid rotator approximation, [9]. The studies in [32] suggest that the effective pion mass m should have the rather large value of $m \approx 1$. In [33], a 30% lower value of the Skyrme parameter was suggested in order to match a large range of nuclei masses to experimental data. A similar conclusion was reached in [10] by considering the electromagnetic properties of the quantized $B = 6$ Skyrmion, describing ${}^{6}_{3}\mathrm{Li}$.

Another recent development is the realization that the pion mass which occurs in the last term in (1) has a larger influence than expected. At the physical value of the pion mass, shell-like configurations become unstable to squashing [11]. While the minimals for small $B < 8$ hardly change, many new (local) minimal energy configurations have been found for larger B, see [12, 32, 13]. These new configurations are no longer shell-like, but often seem to contain several $B = 4$ cubes. The relation to the phenomenological alpha particle model is discussed in [32]. There is hope that the Skyrme model will one day be able to describe the low energy excited states for example for $B = 12$ or $B = 16$ as rotational bands related to different local minima in the same way as in the alpha particle model. In order to achieve this aim, we need a better understanding of the classical solutions, possibly including saddle point solutions in the Skyrme model [34].

Finally, we have to take into account that Skyrmions deform when they are spinning. This can lead to changes in the symmetries of minimal energy solutions, and different symmetries can lead to different allowed quantum states!

3 Relative equilibria

The zero mode quantization has been quite successful. However, it is clear that Skyrmions are deforming when they are spinning, and this effect needs to be taken into account. The theory of relative equilibria was developed for rotating molecules. It gives strong theorems as to what kind of symmetry bifurcations can occur. Numerical simulation of spinning Skyrmions is very time-consuming. Therefore, a good understanding of what kind of behaviour is to be expected is very valuable.

3.1 Theoretical Background

Let $\mathcal{C}$ be a smooth manifold with a smooth action of a finite dimensional Lie group G. Let $< .,. >_q$ denote the G-invariant inner product on $T_q\mathcal{C}$. Consider the G-invariant Lagrangian

$$L = \tfrac{1}{2} < \dot{q}, \dot{q} >_q - V(q),$$

where $V(q)$ is also G-invariant. Let $q(t) = \exp(t\xi)r(t)$. This induces the Lagrangian L_ξ

$$L_\xi = \tfrac{1}{2} < \dot{r}, \dot{r} > - < \dot{r}, \xi \cdot r > + \tfrac{1}{2} < \xi \cdot r, \xi \cdot r > - V(r),$$

where

$$\xi \cdot r = \frac{d}{dt} \left(\exp(t\xi)r\right)_{t=0}.$$

An equilibrium point of L_ξ is called a relative equilibrium of L. A good choice of coordinates are the *slice coordinates,* splitting the dynamics into components along the group orbit $G \cdot q_0$ and transverse to it where q_0 is any point in $\mathcal{C}$. In a neighbourhood of q_0, we can write $q = g \cdot s$ where $g \in G$ and s is in the "slice" S transverse to the group action. For the tangent vectors, we obtain

$$g^{-1}\dot{q} = \xi \cdot s + \dot{s},$$

where $\xi = g^{-1}\dot{g}$. These coordinates have the advantage that rotational and vibrational modes decouple at the point q_0.

Theorem 3.1 *1. In a neighbourhood of an equilibrium point q_0 the corresponding Hamiltonian H can be expressed as*

$$H = h(\mu) + Q(s, \sigma),$$

where μ and σ are the momenta conjugate to ξ and s, respectively.

2. The function $h(\mu)$ is even and its Taylor series at $0 \in \mathfrak{g}^*$ is

$$h(\mu) = V(q_0) + \tfrac{1}{2}\mu^T \mathbf{I}(q_0)^{-1}\mu - \tfrac{1}{16}V_2^{-1}\left(\mu^T \mathbf{I}_s^{-1}(q_0)\mu, \mu^T \mathbf{I}_s^{-1}(q_0)\mu\right) + O(\mu^6),$$

where $\mathbf{I}$ is the locked inertia tensor, and V_2 and $\mathbf{I}_s^{-1}$ are certain derivatives with respect to s.

This theorem gives us a way to systematically explore relative equilibria in the Skyrme model. Here, μ corresponds to the angular momenta $\mathbf{K}$ and $\mathbf{L}$, and s parametrizes the vibrations/deformations, which we have ignored so far. A better understanding of the classical behaviour will help us to derive a more sophisticated quantization of Skyrmions. Note that the zero-mode quantization corresponds to only considering the quadratic approximation to $h(\mu)$. Symmetries place further restrictions on $h(\mu)$ and enable us to make more general statements about possible bifurcation patterns.

4 Results and Outlook

In this talk, we described how to use the Skyrme model to calculate ground states and excited states of atomic nuclei. We also discussed various effects that have to be taken into account. One important effect is that Skyrmions deform when they are spinning. The theory of relative equilibria, which has been very useful for understanding molecular spectra [15, 16, 17], will help us to find the answer to various questions.

The first set of questions is about classical behaviour. We want to understand the families of relative equilibria which bifurcate from a given equilibrium point. Furthermore, we want to understand the symmetries of various bifurcations given the symmetry of the equilibrium points

and the locked inertia tensor (which is related to the matrices U_{ij}, V_{ij} and W_{ij} in equation (6)). We are planning to calculate these symmetry bifurcations explicitly for various baryon numbers. Using a generalization of the rational map ansatz it might be possible to evaluate higher order terms in the function $h(\mu)$, which contains a lot of information about the bifurcations. This approach should at least enable us to identify interesting configurations together with the most appropriate rotation axes in space and target space. The corresponding relative equilibria could then be calculated numerically. For higher baryon numbers there is often a multitude of different local minima. The knowledge of how for a given B the bifurcation pattern of these various families evolves as the angular velocity increases will give us a better insight into which of these minima play the most important role.

Finally, there is an extension of the theory of relative equilibria to quantum mechanical problems which can also be applied to the Skyrme model. Hopefully, this theory, together with the insight we gained from the classical problem, will allow us to calculate ground state and excited states of light atomic nuclei with more confidence, including those with half-integer spin.

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