Quark confinement from dyons

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Abstract

We present a semiclassical approach to the $SU(N)$ Yang–Mills theory whose partition function at nonzero temperatures is approximated by a saddle point – an ensemble of an infinite number of interacting dyons of N kinds. Surprisingly, known criteria of confinement are satisfied in this semiclassical approximation:
(i) the average Polyakov line is zero below some critical temperature, and nonzero above it,
(ii) a quark-antiquark pair has linear rising potential energy,
(iii) the average spatial Wilson loop falls off exponentially with the area,
(iv) N^2 gluons are canceled out from the spectrum.
We find that the critical deconfinement temperature is in good agreement with lattice data.

1 How dyons explain confinement, qualitatively

We shall be considering the quantum $SU(N)$ Yang–Mills theory (without dynamical quarks) at nonzero temperatures T below and up to the critical deconfinement temperature T_c. It is in this range of temperatures that the four remarkable phenomena specified in the Abstract and called together "confinement" take place. We suggest an explanation of all four, based on a simple semiclassical picture.

The main idea is that quantum zero-point oscillations of the Yang–Mills (YM) fields occur not around zero but rather about certain classical configurations of the YM potentials that are called dyons and which are saddle points of the YM partition function at nonzero T.

If a semiclassical approach to the YM theory makes any sense at all, it becomes clear after some considerations that dyons suit perfectly the aim of explaining the four confinement criteria. Dyons or Bogomolny–Prasad–Sommerfield (BPS) monopoles [1] are gluon field configurations whose chromomagnetic and chromoelectric fields are Coulomb at large distances from the cores (hence the name "dyon"). In addition, dyons are characterized by the value of the A_4 component of the YM potential reached far from the cores. A gauge invariant version of it is that dyons are characterized by nontrivial eigenvalues of the Polyakov loop at spatial infinity, where the

$$\text{Polyakov loop} \quad L = P\,\exp\left(i\int_0^{\frac{1}{T}} A_4 dx^4\right) \stackrel{\text{eigenvalues}}{\longrightarrow} \text{diag}\left(e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, \ldots, e^{2\pi i\mu_N}\right). \qquad (1)$$

The gauge invariant phases satisfy $\mu_1+\ldots+\mu_N=0$, and we shall assume that they are ordered: $\mu_1 \le \mu_2 \le \ldots \le \mu_N \le \mu_{N+1} \equiv \mu_1+1$. We shall call the set of N phases $\{\mu_m\}$ the "holonomy" for short.

What holonomy or what set of μ_m's is preferred by the quantum YM system is a dynamical question but let us for the time being assume that μ_m's are equidistant,

$$\mu_m^{\text{conf}} = -\frac{1}{2} - \frac{1}{2N} + \frac{m}{N} \quad \Longrightarrow \quad \text{Tr}L = 0. \qquad (2)$$

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For example, in $SU(3)$ it is

$$\mu_1 = -\frac{1}{3}, \mu_2 = 0, \mu_3 = \frac{1}{3} \longrightarrow L = \mathrm{diag}\left(e^{-\frac{2\pi i}{3}}, 1, e^{\frac{2\pi i}{3}}\right), \quad \mathrm{Tr}L = 0. \tag{3}$$

Then one expects that in an ensemble of dyons one would have on the average $<\mathrm{Tr}L>= 0$ which is the 1st criterion of confinement mentioned in the Abstract. We shall see that the "confining" holonomy (2) is indeed dynamically favoured by the ensemble of dyons, and for a rather simple reason.

Next comes the second basic property of dyons, namely their Coulomb asymptotics both for the electric and magnetic fields. An ensemble of such objects is expected to experience Debye screening, meaning the appearance of a mass gap and an exponential decrease of correlation functions both for the electric and magnetic sources. In particular, the correlation function of two Polyakov lines (being the source of electric field) is expected to fall off exponentially with the Debye length. At the same time this correlation function defines the static potential energy of a probe quark and antiquark:

$$\left\langle \mathrm{Tr}L(\mathbf{z}_1)\ \ \mathrm{Tr}L^\dagger(\mathbf{z}_2)\right\rangle = \mathrm{const.}\ \exp\left(-\frac{V(\mathbf{z}_1-\mathbf{z}_2)}{T}\right) \sim \exp\left(-\frac{\sigma|\mathbf{z}_1-\mathbf{z}_2|}{T}\right). \tag{4}$$

Such behaviour means linear confining potential, with a string tension σ proportional to the Debye mass. This is the 2nd criterion of confinement.

Large spatial Wilson loops exhibit the area behaviour basically by the same mechanism as discovered 30 years ago by A. Polyakov [2] in the $3d$ Georgi–Glashow model: It is due to the magnetic monopoles and the Debye screening in the monopole plasma. Therefore, the 3d criterion of confinement is satisfied by dyons, too. We stress that to obtain both the electric string (created by two Polyakov lines going in the time direction) and the magnetic string (created by a spatial Wilson loop) one needs a plasma of electric *and* magnetic charges, *i.e.* dyons.

Confinement implies that below T_c there are no gluons in the spectrum but only glueballs. Meanwhile, in perturbation theory one gets the Stefan–Boltzmann law for the gluons free energy

$$F_{\mathrm{SB}} = -\frac{\pi^2}{45}\,T^4\,V\,(N^2-1). \tag{5}$$

It is proportional to the number of gluons N^2-1 and has the T^4 behaviour characteristic of massless particles. In the confinement phase, if only glueballs are left in the spectrum the free energy must be $\mathcal{O}(N^0)$. Therefore, confinement implies a massive cancelation in the gluons free energy. The "confining" holonomy (2) provides such a cancelation.

Indeed, one can compute the vacuum energy in a constant field A_4 with the result [3, 4]

$$F_{\mathrm{pert}} = \frac{(2\pi)^2\,T^4\,V}{3}\sum_{m>n}^{N}(\mu_m-\mu_n)^2[1-(\mu_m-\mu_n)]^2\Big|_{\mathrm{mod}\,1}. \tag{6}$$

It has N zero minima when all μ_m's are equal *modulo* unity, see Fig. 1. The confining holonomy (2) corresponds to the non-degenerate *maximum* of Eq. (6) equal to

$$F_{\mathrm{pert,\,max}} = \frac{\pi^2}{45}\,T^4\,V\left(N^2-\frac{1}{N^2}\right). \tag{7}$$

We see that the leading $\mathcal{O}(N^2)$ term in the Stefan–Boltzmann law is canceled by the vacuum energy precisely at the confining holonomy point, such that N^2 gluons do not appear in the spectrum and the 4th criterion of confinement is fulfilled.

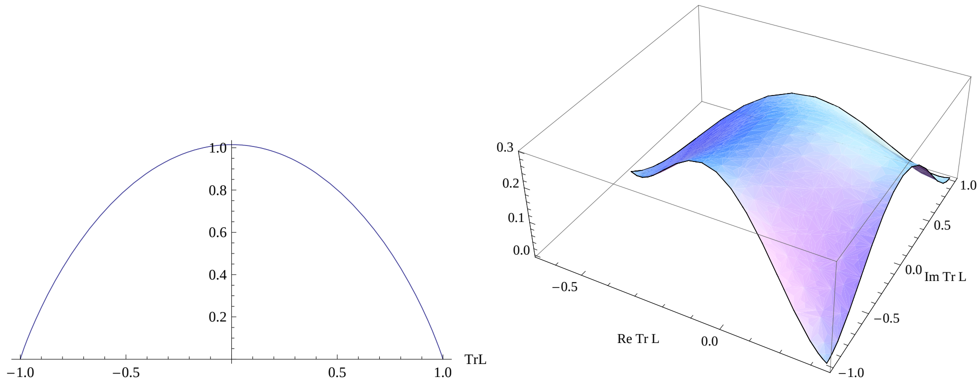

Figure 1: The perturbative potential energy as function of the Polyakov line for the $SU(2)$ (*left*) and $SU(3)$ (*right*) groups. It has minima where the Polyakov loop is one of the N elements of the center Z_N and is maximal at the "confining" holonomy.

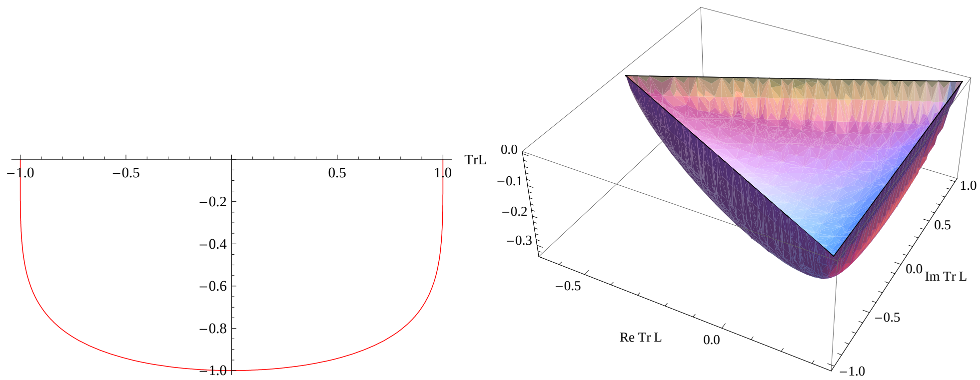

Figure 2: The dyon-induced nonperturbative potential energy as function of the Polyakov line for the $SU(2)$ (*left*) and $SU(3)$ (*right*) groups. Contrary to the perturbative potential energy, it has a single and non-degenerate minimum at the confining holonomy corresponding to $\mathrm{Tr}L = 0$.

The question is, why the YM systems prefers dynamically the confining holonomy (2). It corresponds not to the minima but to the maximum of the perturbative free energy (6) and hence confinement is counterintuitive from the perturbative point of view.

It is interesting that in the supersymmetric $\mathcal{N}=1$ version of the YM theory (where in addition to gluons there are spin-$\frac{1}{2}$ gluinos in the adjoint representation) the perturbative potential energy (6) is absent in all orders owing to fermion-boson cancelation, but the nonperturbative potential energy is nonzero. Moreover, it is known exactly as function of μ's [5]: it has a single minimum at precisely the "confining" holonomy (2). The result can be traced to the semiclassical contribution of dyons, which turns out to be exact owing to supersymmetry.

In the non-supersymmetric pure YM theory, the dyon-induced contribution cannot be computed exactly but only in the semiclassical approximation (this is what the talk is about), and the perturbative contribution (6) is present, too. We shall show below that a semiclassical configuration − an ensemble of dyons with quantum fluctuations about it − generates a nonperturbative free energy shown in Fig. 2. It has the opposite behaviour of the perturbative one, having the minimum at the equidistant (confining) values of the μ's. There is a fight between the perturbative and nonperturbative contributions to the free energy [6]. Since the perturbative contribution to the free energy is $\sim T^4$ with respect to the nonperturbative one, it certainly wins when temperatures are high enough, and the system is then forced into one of the N vacua thus breaking spontaneously the Z_N symmetry. At low temperatures the nonperturbative contribution prevails forcing the system into the confining vacuum. At a critical T_c

there is a confinement-deconfinement phase transition. It turns out to be of the second order for $N=2$ but first order for $N=3$ and higher, in agreement with lattice findings.

This talk is based on a paper with Victor Petrov [7].

2 Dyons and instantons with a nontrivial holonomy

Dyons or BPS monopoles [1] are (anti) self-dual solutions of the nonlinear Maxwell equations, $D^{ab}_\mu F^b_{\mu\nu} = 0$. In $SU(N)$ there are exactly N kinds of fundamental dyons characterized by Coulomb asymptotics for both electric and magnetic fields:

$$\pm\mathbf{E} = \mathbf{B} \stackrel{|\mathbf{x}|\to\infty}{=} \frac{1}{2}\frac{\mathbf{x}}{|\mathbf{x}|^3} \times \begin{cases} \mathrm{diag}(1,-1,0,...,0,0) \\ \mathrm{diag}(0,1,-1,...,0,0) \\ \quad\quad\ldots \\ \mathrm{diag}(0,0,0,...,1,-1) \\ \mathrm{diag}(-1,0,0,...,0,1) \end{cases} . \tag{8}$$

Dyon of the $m^{\rm th}$ kind ($m=1,\ldots,N$) is the one whose asymptotic field has "1" on the $m^{\rm th}$ place on the diagonal and "-1" on the $(m+1)^{\rm st}$ place.

Dyon solutions are also labeled by the holonomy or the set of μ_s's at spatial infinity:

$$A_4(|\mathbf{x}|\to\infty) \to 2\pi T \mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_N). \tag{9}$$

The explicit expressions for the solutions in various gauges can be found *e.g.* in the Appendix of Ref. [8]. Inside the cores which are of the size $\sim 1/(T\nu_m)$, the fields are large, nonlinearity is essential. The action density is time-independent everywhere and is proportional to the temperature. Isolated dyons are thus $3d$ objects but with finite action independent of temperature:

$$S_{\rm dyon} = \frac{2\pi}{\alpha_s}\nu_m, \quad \nu_m \equiv \mu_{m+1}-\mu_m, \quad \sum_m \nu_m = 1, \tag{10}$$

(here $\mu_{N+1}\equiv\mu_1+1$). The full action of all N kinds of well-separated dyons together is that of one standard instanton: $S_{\rm inst}=2\pi/\alpha_s$.

In the semiclassical approach, one has first of all to find the statistical weight with which a given classical configuration enters the partition function. It is given by $\exp(-\text{Action})$, times the determinant$^{-1/2}$ from small quantum oscillations about the saddle point. For an isolated dyon as a saddle-point configuration, this factor diverges linearly in the infrared region owing to the slow Coulomb decrease of the dyon field (8). It means that isolated dyons are not acceptable as saddle points: they have zero weight, despite finite classical action. However, one may look for classical solutions that are superpositions of N fundamental dyons, with zero net magnetic charge. The small-oscillation determinant must be infrared-finite for such classical solutions, if they exist.

The needed classical solution has been found a decade ago by Kraan and van Baal [9] and independently by Lee and Lu [10], see also [11]. We shall call them for short the "KvBLL instantons"; an alternative name is "calorons with nontrivial holonomy". The solution was first found for the $SU(2)$ group but soon generalized to an arbitrary $SU(N)$ [12], see [13] for a review.

The general solution $A^{\rm KvBLL}_\mu$ depends on Euclidean time t and space $\mathbf{x}$ and is parameterized by $3N$ positions of N kinds of 'constituent' dyons in space $\mathbf{x}_1,\ldots,\mathbf{x}_N$ and their $U(1)$ phases $\psi_1,\ldots,\psi_N$. All in all, there are $4N$ collective coordinates characterizing the solution (called the moduli space), of which the action $S_{\rm inst}=2\pi/\alpha_s$ is in fact independent, as it should be for a general solution with a unity topological charge. The solution also depends explicitly on temperature T and on the holonomy $\mu_1,\ldots,\mu_N$:

$$A^{\rm KvBLL}_\mu = \bar{A}^a_\mu(t,\mathbf{x};\,\mathbf{x}_1,\ldots,\mathbf{x}_N,\psi_1,\ldots,\psi_N;\,T,\mu_1,\ldots,\mu_N). \tag{11}$$

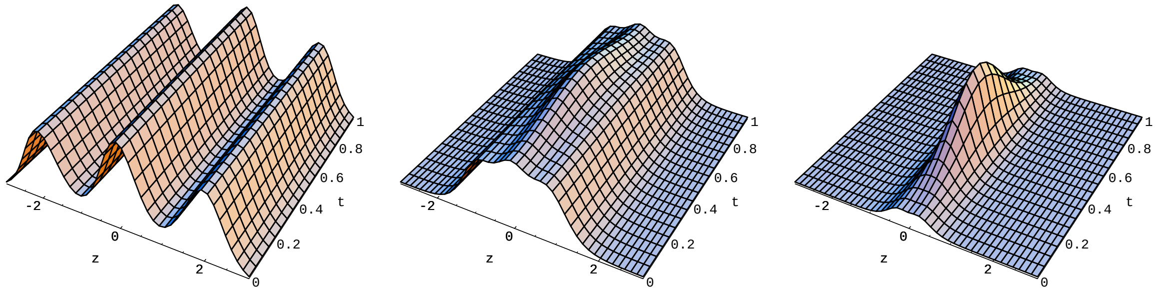

Figure 3: Action density inside the $SU(3)$ KvBLL instanton as function of time and one space coordinate, for large (*left*), intermediate (*middle*) and small (*right*) separations between the three constituent dyons.

The solution is a relatively simple expression given by elementary functions. If the holonomy is trivial (all μ's are equal *modulo* unity) the expression takes the form of the strictly periodic $O(3)$ symmetric caloron [14] reducing further to the standard $O(4)$ symmetric BPST instanton [15] in the $T \to 0$ limit. At small temperatures but arbitrary holonomy, the KvBLL instanton also has only a small $\mathcal{O}(T)$ difference with the standard instanton.

One can plot the action density of the KvBLL instanton in various corners of the parameter (moduli) space, see Fig. 3.

When all dyons are far apart one observes N static (*i.e.* time-independent) objects, the isolated dyons. As they merge, the configuration is not static anymore, it becomes a *process* in time. In the limiting case of a complete merger, the configuration becomes a $4d$ lump resembling the standard instanton. The full (integrated) action is exactly the same $S_{\rm inst} = 2\pi/\alpha_s$ for any choice of the dyon separations. It means that classically dyons do not interact. However, they do experience a peculiar interaction at the quantum level to which we proceed.

3 Quantum weight of many dyons

Remarkably, the small-oscillation determinant about a single KvBLL $SU(N)$ instanton made of N different-kind dyons can be computed exactly [16, 17]. With this experience, the quantum weight of an arbitrary number of dyons of N kinds has been suggested in Ref. [7]. In the YM partition function, there are saddle points corresponding to any set of K_m dyons. In the thermodynamic limit $V \to \infty$ one needs to take a saddle point with $\mathcal{O}(V)$ dyons. Let K_m be the number of dyons of kind m ($m = 1 \ldots N$) and let $\mathbf{x}_{mi}$ be the coordinate of the $i^{\rm th}$ dyon of kind m ($i = 1 \ldots K_m$). In the semiclassical approximation the YM partition function is approximated by the partition function of a grand canonical ensemble of $K_1 + K_2 + \ldots + K_N$ dyons,

$$\mathcal{Z} = \sum_{K_1 \ldots K_N} \frac{1}{K_1! \ldots K_N!} \prod_{m=1}^{N} \prod_{i=1}^{K_m} \int (d\mathbf{x}_{mi}\, f) \sqrt{\det g(\mathbf{x}_{mi})}, \tag{12}$$

where $g(\mathbf{x}_{mi})$ is a $4(K_1 + \ldots + K_N) \times 4(K_1 + \ldots + K_N)$ metric tensor of the dyons' moduli space, composed by the overlaps of zero modes of individual dyons, and f is the fugacity,

$$f = \frac{N^2}{16\pi^3 \lambda^2} \frac{\Lambda^4}{T} = \mathcal{O}(N^2). \tag{13}$$

The bare 't Hooft coupling constant λ is renormalized and starts to "run" only at the 2-loop level not considered here. Eventually, its argument will be the largest scale in the problem, be it the temperature or the equilibrium density of dyons.

It is not difficult to find the metric tensor $g(\mathbf{x}_{mi})$ for well-separated dyons. In this case the four zero modes $\phi^{(\kappa)}_\mu (\kappa = 1,2,3,4)$ of individual dyons are given by the components of the field strength: $\phi^{(\kappa)}_\mu = F_{\mu\kappa}$. The zero modes for the $m^{\rm th}$ kind of dyon are normalized to its action, $\int {\rm Tr}\phi^{(\kappa)}_\mu \phi^{(\lambda)}_\mu \sim \delta^{\kappa\lambda}\nu_m$ (see Eq. (10)) and hence depend on the holonomy. Since the field strengths decay as $1/r^2$ (see Eq. (8)) the overlaps between zero modes are Coulomb-like, and only those that are nearest neighbors in m do interact. In fact, the diagonal components of the metric tensor also acquire Coulomb-like corrections since the action of individual dyons is actually normalized to its asymptotic field A_4 that gets Coulomb corrections from other dyons.

As a result, we obtain the $4(K_1 + \ldots + K_N) \times 4(K_1 + \ldots + K_N)$ metric tensor $g(\mathbf{x}_{mi})$ with Coulomb interactions as entries, and the ν_m's on the diagonal. It turns out that its determinant is a square of the determinant of a related matrix, $\sqrt{\det g} = \det G$ where G is a $(K_1 + \ldots + K_N) \times (K_1 + \ldots + K_N)$ matrix:

$$G_{mi,nj} = \delta_{mn}\delta_{ij}\left(4\pi\nu_m + \sum_k \frac{1}{T|\mathbf{x}_{mi}-\mathbf{x}_{m-1,k}|} + \sum_k \frac{1}{T|\mathbf{x}_{mi}-\mathbf{x}_{m+1,k}|} - 2\sum_{k\neq i}\frac{1}{T|\mathbf{x}_{mi}-\mathbf{x}_{mk}|}\right)$$
$$-\frac{\delta_{m,n-1}}{T|\mathbf{x}_{mi}-\mathbf{x}_{m+1,j}|} - \frac{\delta_{m,n+1}}{T|\mathbf{x}_{mi}-\mathbf{x}_{m-1,j}|} + 2\left.\frac{\delta_{mn}}{T|\mathbf{x}_{mi}-\mathbf{x}_{mj}|}\right|_{i\neq j}, \qquad (14)$$

where $\mathbf{x}_{mi}$ is the coordinate of the $i^{\rm th}$ dyon of kind m. The matrix G has the following nice properties:

- symmetry: $G_{mi,nj} = G_{nj,mi}$
- overall "neutrality": the sum of Coulomb interactions in non-diagonal entries cancel those on the diagonal: $\sum_{nj} G_{mi,nj} = 4\pi\nu_m$
- identity loss: dyons of the same kind are indistinguishable, meaning mathematically that $\det G$ is symmetric under permutation of any pair of dyons $(i \leftrightarrow j)$ of the same kind m. Dyons do not 'know' to which instanton they belong to
- attraction/repulsion: if one decreases the separation between same-kind dyons or increases the separation between different-kind dyons, the $\det G$ decreases. It means that same-kind dyons repulse each other whereas different-kind dyons attract each other. The $\det G$ measure favors formation of neutral clusters with N different kinds of dyons
- factorization: in the geometry when dyons fall into K well separated neutral clusters of N dyons of different kinds, $\det G$ factorizes into a product of *exact* integration measures for K KvBLL instantons [18, 19] valid for *any* separations between different-kind dyons, including their strong overlap
- last but not least, the metric g corresponding to G is hyper-Kähler, as it should be for the moduli space of a self-dual classical field [20]. In fact, it is a severe restriction on the metric.

An overall constant factor depending on the holonomy and temperature, $\exp(-F_{\rm pert}V)$, is understood in Eq. (12), where $F_{\rm pert}$ is the perturbative gluon loop (6) in the background of a constant field A_4 (9). This factor arises from the non-zero modes in the fluctuation determinant about dyons and is necessarily present as most of the $3d$ space outside the dyons' core is just a constant A_4 background. Indeed the calculation [16, 17] exhibits this factor which is the only one proportional to the 3-volume V.

The ensemble defined by a determinant of a matrix whose dimension is the number of particles, is not a usual one. More customary, the interaction is given by the Boltzmann factor

$\exp\left(-U_{\rm int}(\mathbf{x}_1,\ldots)\right)$. Of course, one can always present the determinant in that way using the identity $\det G = \exp(\mathrm{Tr}\log G) \equiv \exp(-U_{\rm int})$ but the interactions will then include three-, four-, five-... body forces. At the same time, it is precisely the determinant form of the interaction that makes the statistical physics of dyons an exactly solvable problem.

4 Statistical physics of dyons as a Quantum Field Theory

It is possible to present the grand canonical ensemble of dyons, governed by the interaction (14) as an equivalent $3d$ quantum field theory. This will enable us to compute various correlation functions of interest.

To proceed to the quantum field theory description we use two mathematical tricks.

1. "Fermionization" (Berezin [21]). It is helpful to exponentiate the Coulomb interactions rather than keeping them in $\det G$. To that end one presents the determinant of a matrix as an integral over a finite number of anticommuting Grassmann variables ($\{\psi_A^\dagger\,\psi_B\} = \delta_{AB}$):

$$\det(G_{AB}) = \int \prod_A d\psi_A^\dagger\, d\psi_A\, \exp\left(\psi_A^\dagger\, G_{AB}\, \psi_B\right)\,.$$

Now we have the two-body Coulomb interactions in the exponent and it is possible to use the second trick.

2. "Bosonization" (Polyakov [2]). One can present the Coulomb interactions in the exponent with the help of a Gaussian integral over an auxiliary field ϕ:

$$\exp\left(\sum_{m,n}\frac{Q_m Q_n}{|\mathbf{x}_m-\mathbf{x}_n|}\right) = \int D\phi\,\exp\left[-\int d\mathbf{x}\left(\frac{1}{16\pi}\partial_i\phi\partial_i\phi + \rho\phi\right)\right] = \exp\left(\int \rho\frac{4\pi}{\triangle}\rho\right),$$
$$\rho = \sum Q_m\,\delta(\mathbf{x}-\mathbf{x}_m).$$

After applying the first trick the "charges" Q_m become Grassmann variables but after applying the second one, it becomes easy to integrate them out since the square of a Grassmann variable is zero. In fact one needs $2N$ boson fields $\mathrm{v}_m, \mathrm{w}_m$ to reproduce diagonal elements of G and $2N$ anticommuting ("ghost") fields $\chi_m^\dagger, \chi_m$ to present the non-diagonal elements. The chain of identities is accomplished in Ref. [7] and the result for the partition function for the dyon ensemble (12) is, identically, a path integral defining a quantum field theory in 3 dimensions:

$$\mathcal{Z} = \int D\chi^\dagger\, D\chi\, D\mathrm{v}\, D\mathrm{w}\,\exp\int d^3x\left\{\frac{T}{4\pi}\left(\partial_i\chi_m^\dagger\partial_i\chi_m + \partial_i\mathrm{v}_m\partial_i\mathrm{w}_m\right)\right.$$
$$\left.+f\left[(-4\pi\mu_m+\mathrm{v}_m)\frac{\partial\mathcal{F}}{\partial\mathrm{w}_m} + \chi_m^\dagger\frac{\partial^2\mathcal{F}}{\partial\mathrm{w}_m\partial\mathrm{w}_n}\chi_n\right]\right\},\quad \mathcal{F} = \sum_{m=1}^N e^{\mathrm{w}_m-\mathrm{w}_{m+1}}\,. \tag{15}$$

The fields v_m have the meaning of the asymptotic Abelian electric potentials of dyons,

$$\begin{aligned}(A_4)_{mn} &= \delta_{mn}\,A_{m\,4},\\ A_{m\,4}(\mathbf{x})/T &= 2\pi\mu_m - \tfrac{1}{2}\mathrm{v}_m(\mathbf{x}),\quad \mathbf{E}_m = \nabla A_{m\,4},\end{aligned} \tag{16}$$

while w_m have the meaning of the dual (or magnetic) Abelian potentials. Note that the kinetic energy for the $\mathrm{v}_m, \mathrm{w}_m$ fields has only the mixing term $\partial_i\mathrm{v}_m\partial_i\mathrm{w}_m$ which is nothing but the Abelian duality transformation $\mathbf{E}\cdot\mathbf{B}$. The function $\mathcal{F}(\mathrm{w})$ in (15) where one assumes a cyclic summation over m, is known as the periodic (or affine) Toda lattice.

Although the Lagrangian in Eq. (15) describes a highly nonlinear interacting quantum field theory, it is in fact exactly solvable! To prove it, one observes that the fields v_m enter the

Lagrangian only linearly, therefore one can integrate them out. It leads to a functional δ-function:

$$\int D\mathrm{v}_m \quad \longrightarrow \quad \delta\left(-\frac{T}{4\pi}\partial^2 \mathrm{w}_m + f\frac{\partial \mathcal{F}}{\partial \mathrm{w}_m}\right). \tag{17}$$

This δ-function restricts possible fields w_m over which one still has to integrate in eq. (15). Let $\bar{\mathrm{w}}_m$ be a solution to the argument of the δ-function. Integrating over small fluctuations about $\bar{\mathrm{w}}$ gives the Jacobian

$$\mathrm{Jac} = \det{}^{-1}\left(-\frac{T}{4\pi}\partial^2\delta_{mn} + f\frac{\partial^2\mathcal{F}}{\partial \mathrm{w}_m \partial \mathrm{w}_n}\bigg|_{\mathrm{w}=\bar{\mathrm{w}}}\right). \tag{18}$$

Remarkably, exactly the same functional determinant but in the numerator arises from integrating over the ghost fields, for any background $\bar{\mathrm{w}}$. Therefore, all quantum corrections cancel *exactly* between the boson and ghost fields (a characteristic feature of supersymmetry), and the ensemble of dyons is basically governed by a classical field theory.

To find the ground state we examine the fields' potential energy being $-4\pi f\mu_m\partial\mathcal{F}/\partial \mathrm{w}_m$ which we prefer to write restoring $\nu_m = \mu_{m+1} - \mu_m$ and $\mathcal{F}$ as

$$F_{\mathrm{dyon}} = -4\pi f V \sum_m \nu_m\, e^{\mathrm{w}_m - \mathrm{w}_{m+1}} \tag{19}$$

(the volume factor arises for constant fields w_m). One has first to find the stationary point in w_m for a given set of ν_m's. It leads to the equations

$$\frac{\partial \mathcal{P}}{\partial \mathrm{w}_m} = 0$$

whose solution is

$$e^{w_1 - w_2} = \frac{(\nu_1\nu_2\nu_3 ... \nu_N)^{\frac{1}{N}}}{\nu_1}, \quad \text{etc.} \tag{20}$$

Putting it back into eq. (19) we obtain

$$F_{\mathrm{dyon}} = -4\pi f V N (\nu_1\nu_2 ... \nu_N)^{\frac{1}{N}}, \quad \nu_1 + ... + \nu_N = 1. \tag{21}$$

The minimum equal $F_{\mathrm{dyon,\,min}} = -4\pi f V$ is achieved at $\nu_1 = \ldots = \nu_N = \frac{1}{N}$, that is at equidistant, confining value of the holonomy! Cf. Eq. (2). We have also proven that the result is exact, as all potential quantum corrections cancel in the partition function (15).

Given this cancelation, the key finding – that the dyon-induced free energy has the minimum at the confining value of holonomy – is trivial. If all Coulomb interactions cancel after integration over dyons' positions, the weight of a many-dyon configuration is the same as if they were infinitely dilute (although they are not). Then the weight, what concerns the holonomy, is proportional to the product of diagonal matrix elements of G in the dilute limit, that is to the normalization integrals for dyon zero modes, that is to the product of the dyon actions $\sim \nu_m$ where $\nu_m = \mu_{m+1} - \mu_m$ and $\nu_N = \mu_1 + 1 - \mu_N$ such that $\nu_1 + \nu_2 + \ldots + \nu_N = 1$. The sum of all N kinds of dyons' actions is fixed and equal to the instanton action, however, it is the *product* of actions that defines the weight. The product is maximal when all actions are equal, hence the equidistant or confining μ's are statistically preferred. Thus, the average Polyakov line is zero, $< \mathrm{Tr}L >= 0$.

5 Heavy quark potential

The field-theoretic representation of the dyon ensemble enables one to compute various YM correlation functions in the semiclassical approximation. The key observables relevant to confinement are the correlation function of two Polyakov lines (defining the heavy quark potential), and the average of large Wilson loops. A detailed calculation of these quantities is performed in Ref. [7]; here we only present the results and discuss the meaning.

5.1 N-ality and k-strings

From the viewpoint of confinement, all irreducible representations of the $SU(N)$ group fall into N classes: those that appear in the direct product of any number of adjoint representations, and those that appear in the direct product of any number of adjoint representations with the irreducible representation being the rank-k antisymmetric tensor, $k = 1, \ldots, N-1$. "N-ality" is said to be zero in the first case and equal to k in the second. N-ality-zero representations transform trivially under the center of the group Z_N; the rest acquire a phase $2\pi k/N$.

One expects that there is no asymptotic linear potential between static color sources in the adjoint representation as such sources are screened by gluons. If a representation is found in a direct product of some number of adjoint representations and a rank-k antisymmetric representation, the adjoint ones "cancel out" as they can be all screened by an appropriate number of gluons. Therefore, from the confinement viewpoint all N-ality $= k$ representations are equivalent and there are only $N-1$ string tensions $\sigma_{k,N}$ being the coefficients in the *asymptotic* linear potential for sources in the antisymmetric rank-k representation. They are called "k-strings".

The value $k=1$ corresponds to the fundamental representation whereas $k = N$–1 corresponds to the representation conjugate to the fundamental [quarks and anti-quarks]. In general, the rank-$(N-k)$ antisymmetric representation is conjugate to the rank-k one; it has the same dimension and the same string tension, $\sigma_{k,N} = \sigma_{N-k,N}$.

The behaviour of $\sigma_{k,N}$ as function of k and N is an important issue as it discriminates between various confinement mechanisms. On general N-counting grounds one can only infer that at large N and $k \ll N$, $\sigma_{k,N}/\sigma_{1,N} = (k/N)(1 + \mathcal{O}(1/N^2))$. Important, there should be no $\mathcal{O}(N^{-1})$ correction [22]. A popular version called "Casimir scaling", according to which the string tension is proportional to the Casimir operator for a given representation (it stems from an idea that confinement is somehow related to the modification of a one-gluon exchange at large distances), does not satisfy this restriction.

5.2 Correlation function of Polyakov lines

To find the potential energy $V_{k,N}$ of static "quark" and "antiquark" transforming according to the antisymmetric rank-k representation, one has to consider the correlation of Polyakov lines in the appropriate representation:

$$\left\langle \mathrm{Tr} L_{k,N}(\mathbf{z}_1)\ \mathrm{Tr} L^\dagger_{k,N}(\mathbf{z}_2)\right\rangle = \mathrm{const.}\ \exp\left(-\frac{V_{k,N}(\mathbf{z}_1 - \mathbf{z}_2)}{T}\right). \tag{22}$$

Far away from dyons' cores the field is Abelian and in the field-theoretic language of Eq. (15) is given by Eq. (16). Therefore, the Polyakov line in the fundamental representation is

$$\mathrm{Tr} L(\mathbf{z}) = \sum_{m=1}^{N} Z_m, \quad Z_m = \exp\left(2\pi i \mu_m - \frac{i}{2}\mathrm{v}_m(\mathbf{z})\right). \tag{23}$$

In the general antisymmetric rank-k representation

$$\mathrm{Tr} L_{k,N}(\mathbf{z}) = \sum_{m_1<m_2<\ldots<m_k}^{N} Z_{m_1} Z_{m_2} \ldots Z_{m_k} \tag{24}$$

where cyclic summation from 1 to N is assumed.

The average (22) can be computed from the quantum field theory (15). Inserting the two Polyakov lines (24) into Eq. (15) we observe that the Abelian electric potential v_m enters linearly in the exponent as before. Therefore, it can be integrated out, leading to a δ-function for the dual field w_m, which is now shifted by the source (cf. Eq. (17)):

$$\int D\mathrm{v}_m \quad \longrightarrow \quad \prod_m \delta\left(-\frac{T}{4\pi}\partial^2 \mathrm{w}_m + f\frac{\partial\mathcal{F}}{\partial \mathrm{w}_m}\right.$$
$$\left. -\frac{i}{2}\,\delta(\mathbf{x}-\mathbf{z_1})(\delta_{mm_1}+\ldots+\delta_{mm_k}) + \frac{i}{2}\,\delta(\mathbf{x}-\mathbf{z_2})(\delta_{mn_1}+\ldots+\delta_{mn_k})\right).$$

One has to find the dual field $\mathrm{w}_m(\mathbf{x})$ nullifying the argument of this δ-function, plug it into the action

$$\exp\left(\int d\mathbf{x}\,\frac{4\pi f}{N}\mathcal{F}(w)\right), \tag{25}$$

and sum over all sets $\{m_1<m_2<...<m_k\}$, $\{n_1<n_2<...<n_k\}$ with the weight $\exp\left(2\pi i\,(m_1+\ldots+m_k-n_1-\ldots-n_k)\,/N\right)$. The Jacobian from resolving the δ-function again cancels exactly with the determinant arising from ghosts. Therefore, the calculation of the correlator (22), sketched above, is exact.

At large separations between the sources $|\mathbf{z}_1-\mathbf{z}_2|$, the fields w_m resolving the δ-function are small and one can expand the Toda chain:

$$\mathcal{F}(w)=\sum_m e^{\mathrm{w}_m-\mathrm{w}_{m+1}} \approx N+\frac{1}{2}\,\mathrm{w}_m\,\mathcal{M}_{mn}\,\mathrm{w}_n, \tag{26}$$

where

$$\mathcal{M}=\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & -1\\ -1 & 2 & -1 & \ldots & 0 & 0\\ 0 & -1 & 2 & -1 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ -1 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}. \tag{27}$$

As apparent from Eq. (26), the eigenvalues of $\mathcal{M}$ determine the spectrum of the dual fields w_m. There is one zero eigenvalue which decouples from everywhere, and $N-1$ nonzero eigenvalues

$$\mathcal{M}^{(k)}=\left(2\sin\frac{\pi k}{N}\right)^2, \qquad k=1,...,N-1. \tag{28}$$

Certain orthogonality relation imposes the selection rule: the asymptotics of the correlation function of two Polyakov lines in the antisymmetric rank-k representation is determined by precisely the k^{th} eigenvalue. We obtain [7]

$$\left\langle \mathrm{Tr}L_{k,N}(\mathbf{z}_1)\;\;\mathrm{Tr}L^\dagger_{k,N}(\mathbf{z}_2)\right\rangle \stackrel{z_{12}\to\infty}{=} \text{const.}\,\exp\left(-|\mathbf{z}_1-\mathbf{z}_2|\,M\sqrt{\mathcal{M}^{(k)}}\right) \tag{29}$$

where M is the 'dual photon' mass,

$$M=\sqrt{\frac{4\pi f}{T}}=\frac{N\Lambda^2}{2\pi\lambda T}=\mathcal{O}(N). \tag{30}$$

Comparing it with the definition of the heavy quark potential (22) we find that there is an asymptotically linear potential between static "quarks" in any N-ality nonzero representation, with the k-string tension

$$\sigma_{k,N}=MT\sqrt{\mathcal{M}^{(k)}}=2MT\,\sin\frac{\pi k}{N}=\frac{\Lambda^2}{\lambda}\frac{N}{\pi}\,\sin\frac{\pi k}{N}. \tag{31}$$

This is the so-called 'sine regime': it has been found before in certain supersymmetric theories [23]. Lattice simulations [24] support this regime, whereas another lattice study [25] gives somewhat smaller values but within two standard deviations from the values following from eq. (31).

We see that at large N and $k\ll N$, $\sigma_{k,N}/\sigma_{1,N}=(k/N)(1+\mathcal{O}(1/N^2))$, as it should be on general grounds, and that all k-string tensions have a finite limit at zero temperature.

6 Area law for large Wilson loops

The magnetic field of dyons beyond their cores is Abelian and is a superposition of the Abelian fields of individual dyons. For large Wilson loops we are interested in, it is this superposition field of a large number of dyons that contributes most as they have a slowly decreasing $1/|\mathbf{x}-\mathbf{x}_i|$ asymptotics, hence the use of the field outside the cores is justified. Owing to self-duality,

$$[B_i(\mathbf{x})]_{mn} = [\partial_i A_4(\mathbf{x})]_{mn} = -\frac{T}{2}\,\delta_{mn}\,\partial_i \mathrm{v}_m(\mathbf{x}), \tag{32}$$

cf. eq. (16). Since A_i is Abelian beyond the cores, one can use the Stokes theorem for the spatial Wilson loop:

$$W \equiv \mathrm{Tr}\,\mathcal{P}\exp i\oint A_i dx^i = \mathrm{Tr}\exp i\int B_i\, d^2\sigma^i = \sum_m \exp\left(-i\frac{T}{2}\int d^2\sigma^i\,\partial_i \mathrm{v}_m\right).$$

Eq. (**??**) may look contradictory as we first use $B_i = \mathrm{curl}\,A_i$ and then $B_i = \partial_i A_4$. Actually there is no contradiction as the last equation is true up to Dirac string singularities which carry away the magnetic flux. If the Dirac string pierces the surface spanning the loop it gives a quantized contribution $\exp(2\pi i\cdot\mathrm{integer}) = 1$; one can also use the gauge freedom to direct Dirac strings parallel to the loop surface in which case there is no contribution from the Dirac strings at all.

Let us take a flat Wilson loop lying in the (xy) plane at $z=0$. Then eq. (**??**) is continued as

$$W = \sum_m \exp\left(-i\frac{T}{2}\int_{x,y\in\mathrm{Area}} d^3x\,\partial_z \mathrm{v}_m \delta(z)\right) = \sum_m \exp\left(i\frac{T}{2}\int_{x,y\in\mathrm{Area}} d^3x\,\mathrm{v}_m\,\partial_z\delta(z)\right).$$

It means that the average of the Wilson loop in the dyon ensemble is given by the partition function (15) with the source

$$\sum_m \exp\left(i\frac{T}{2}\int d^3x\;\mathrm{v}_m\,\frac{d\delta(z)}{dz}\,\theta(x,y\in\mathrm{Area})\right)$$

where $\theta(x,y\in\mathrm{Area})$ is a step function equal to unity if x,y belong to the area inside the loop and zero otherwise.

As in the case of the Polyakov lines the presence of the Wilson loop shifts the argument of the δ-function arising from the integration over the v_m variables, and the ghost determinant cancels exactly the Jacobian from the fluctuations of w_m's, therefore the classical-field calculation is exact.

One has to solve the non-linear Toda equations on w_m's with a source along the surface of the loop,

$$-\partial^2 \mathrm{w}_m + M^2\left(e^{\mathrm{w}_m-\mathrm{w}_{m+1}} - e^{\mathrm{w}_{m-1}-\mathrm{w}_m}\right) = -2\pi i\,\delta_{mm_1}\,\frac{d\delta(z)}{dz}\,\theta(x,y\in\mathrm{Area}), \tag{33}$$

for all m_1, plug it into the action $(4\pi f/N)\mathcal{F}(w)$, and sum over m_1. In order to evaluate the average of the Wilson loop in a general antisymmetric rank-k representation, one has to take the source in eq. (33) as $-2\pi i\,\delta'(z)\,(\delta_{mm_1}+\ldots+\delta_{mm_k})$ and sum over $m_1<\ldots<m_k$ from 1 to N, see eq. (24).

Contrary to the case of the Polyakov lines, one cannot, generally speaking, linearize eq. (33) in w_m but has to solve the non-linear equations as they are. The Toda equations (33) with a $\delta'(z)$ source in the r.h.s. define "pinned soliton" solutions $\mathrm{w}_m(z)$ that are $1d$ functions in the direction transverse to the surface spanning the Wilson loop but do not depend on the coordinates x,y provided they are taken inside the loop. Beyond that surface $\mathrm{w}_m = 0$. Along the perimeter of the loop, w_m interpolate between the soliton and zero. For large areas, the action (25) is therefore proportional to the area of the surface spanning the loop, which gives the famous area law for the average Wilson loop. The coefficient in the area law, the 'magnetic' string tension, is found from integrating the action density of the soliton $\mathrm{w}_m(z)$ in the z direction.

The exact solutions of Eq. (33) for any N and any representation k have been found in Ref. [7], and the resulting 'magnetic' string tension turns out to be

$$\sigma_{k,N} = \frac{\Lambda^2}{\lambda}\frac{N}{\pi}\sin\frac{\pi k}{N}\,, \tag{34}$$

which coincides with the 'electric' string tension (31) found from the correlators of the Polyakov lines, for all k-strings!

Several comments are in order here.

- The 'electric' and 'magnetic' string tensions should coincide only in the limit $T \to 0$ where the Euclidean $O(4)$ symmetry is restored. Both calculations have been in fact performed in that limit as we have ignored the temperature-dependent perturbative potential (6). If it is included, the 'electric' and 'magnetic' string tensions split.
- despite that the theory (15) is 3-dimensional, with the temperature entering just as a parameter in the Lagrangian, it "knows" about the restoration of Euclidean $O(4)$ symmetry at $T \to 0$.
- the 'electric' and 'magnetic' string tensions are technically obtained in very different ways: the first is related to the mass of the elementary excitation of the dual fields w_m, whereas the latter is related to the mass of the dual field soliton.

Dyons force the system to have the "most nontrivial" holonomy (2). For that holonomy, the perturbative potential energy (6) is at its maximum equal to

$$\frac{F_{\text{pert, max}}}{V} = \frac{\pi^2}{45}T^4\left(N^2 - \frac{1}{N^2}\right). \tag{35}$$

The full free energy is the sum of the three terms above.

We see that the leading $\mathcal{O}(N^2)$ term in the Stefan–Boltzmann law is *canceled* by the potential energy precisely at the confining holonomy point and nowhere else! In fact it seems to be the only way how $\mathcal{O}(N^2)$ massless gluons can be canceled out of the free energy, and the main question shifts to why does the system prefer the "most nontrivial" holonomy. Dyons seem to answer that question.

7 Deconfinement phase transition

The nonperturbative free energy corresponding to the minimum of the dyon-induced potential energy as function of the holonomy (21) is

$$\frac{F_{\text{dyon}}}{V} = -\frac{N^2}{2\pi^2}\frac{\Lambda^4}{\lambda^2}. \tag{36}$$

We have doubled the minimum from eq. (21) keeping in mind that there are also anti-dyons and assuming that their interactions with dyons is not as strong as the interactions between dyons and anti-dyons separately, as induced by the determinant measure (14), therefore treating dyons and anti-dyons as two independent "liquids". (By the same logic, the string tension (31) has to be multiplied by $\sqrt{2}$ as due to anti-dyons.)

	$SU(3)$	$SU(4)$	$SU(6)$	$SU(8)$
$T_c/\sqrt{\sigma}$, theory	0.6430	0.6150	0.5967	0.5906
$T_c/\sqrt{\sigma}$, lattice	0.6462(30)	0.6344(81)	0.6101(51)	0.5928(107)

As the temperature rises, the perturbative free energy (35) grows as T^4 and eventually it overcomes the negative nonperturbative free energy (36), see Figs. 1,2. At this point, the trivial holonomy for which both the perturbative and nonperturbative free energy are zero, becomes favourable. Therefore an estimate of the critical deconfinement temperature comes from equating the sum of Eq. (36) and Eq. (35) to zero, which gives

$$T_c^4 = \frac{45}{2\pi^4}\,\frac{N^4}{N^4-1}\,\frac{\Lambda^4}{\lambda^2}\,. \tag{37}$$

As expected, it is stable in N. A more robust quantity, both from the theoretical and lattice viewpoints, is the ratio $T_c/\sqrt{\sigma}$ where σ is the string tension in the fundamental representation, since in this ratio the poorly known parameters Λ and λ cancel out:

$$\frac{T_c}{\sqrt{\sigma}} = \left(\frac{45}{4\pi^4}\,\frac{\pi^2 N^2}{(N^4-1)\sin^2\frac{\pi}{N}}\right)^{\frac{1}{4}} \quad \stackrel{N\to\infty}{\longrightarrow} \quad \frac{1}{\pi}\left(\frac{45}{4}\right)^{\frac{1}{4}}. \tag{38}$$

In the Table, we compare the values from Eq. (38) to those measured in lattice simulations of the pure $SU(N)$ gauge theories [26]; there is a surprisingly good agreement. A detailed study of the thermodynamics of the phase transition will be published elsewhere.

8 Summary

What happens in the semiclassical approximation based on dyons, can be summarized as follows:

- The ensemble of dyons favours dynamically the confining value of the holonomy. This is almost clear, given that the weight is proportional to the product of individual actions of fundamental dyons, and it is maximal when the actions are equal. Such holonomy corresponds to the zero of the Polyakov line
- Dyons form a sort of Coulomb plasma (but an exactly solvable variant of it) with an appearance of the Debye mass both for "electric" and "magnetic" (dual) photons. The first gives rise to the exponential fall-off of the correlation of two Polyakov lines, *i.e.* to the linear heavy-quark potential, the second yields the area law for spatial Wilson loops
- $\mathcal{O}(N^2)$ massless gluons cancel out from the free energy, and only massive (string?) excitations are left.

The reason why a semiclassical approximation works well for strong interactions (where all dimensionless quantities are, generally speaking, of the order of unity) is not altogether clear. The following considerations provide a possible explanation. After UV renormalization is performed about the classical saddle points and the scale parameter Λ appears as the result of the dimensional transmutation, further quantum corrections to the saddle point are a series in the running 't Hooft coupling λ whose argument is typically the largest scale in the theory, in this case $\max(T, n^{1/4})$ where n is the $4d$ density of dyons. An estimate shows that the running λ is between 1/4 at zero temperature and 1/7 or less at critical temperature. Therefore, although these numbers are "of the order of unity", in practical terms they indicate that high order loop corrections are not too large.

Unfortunately, approximations made in Ref. [7] and reproduced above are not limited to neglecting higher loop corrections. We have (i) ignored dyon interactions induced by the small oscillation determinant over nonzero modes (although we did take into account that it renormalizes the gauge coupling giving rise to the scale parameter Λ, and that it leads to the perturbative potential energy as function of the holonomy), (ii) neglected the interactions of dyons of opposite duality, treating them as two noninteracting "liquids", (iii) conjectured a simple form of the dyon measure which may be incorrect when two *same-kind* dyons come close. Although certain justification for these approximations can be put forward [7] it is desirable not to use them at all, and that may be possible.

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References

[1] E.B. Bogomolny, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976)]; M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975)

[2] A. Polyakov, Nucl. Phys. **B120**, 429 (1977)

[3] D.J. Gross, R.D. Pisarski and L.G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981)

[4] N. Weiss, Phys. Rev. **D24**, 475 (1981); Phys. Rev. **D25**, 2667 (1982)

[5] N.M. Davies, T.J. Hollowood, V.V. Khoze and M.P. Mattis, Nucl. Phys. **B559**, 123 (1999), arXiv:hep-th/9905015

[6] D. Diakonov, Prog. Part. Nucl. Phys. **51**, 173 (2003), arXiv:hep-ph/0212026

[7] D. Diakonov and V. Petrov, Phys. Rev. **D76**, 056001 (2007), arXiv:0704.3181

[8] D. Diakonov and V. Petrov, Phys. Rev. **D67**, 105007 (2003), arXiv:hep-th/0212018

[9] T.C. Kraan and P. van Baal, Phys. Lett. **B428**, 268 (1998), arXiv:hep-th/9802049; Nucl. Phys. **B533**, 627 (1998), arXiv:hep-th/9805168

[10] K. Lee and C. Lu, Phys. Rev. **D58**, 025011 (1998), arXiv:hep-th/9802108

[11] K. Lee and P. Yi, Phys. Rev. **D56**, 3711 (1997), arXiv:hep-th/9702107

[12] T.C. Kraan and P. van Baal, Phys. Lett. **B435**, 389 (1998), arXiv:hep-th/9806034

[13] F. Bruckmann, D. Nogradi and P. van Baal, Acta Phys. Polon. **B34**, 5717 (2003), arXiv:hep-th/0309008

[14] B.J. Harrington and H.K. Shepard, Phys. Rev. **D17**, 2122 (1978); Phys. Rev. **D18**, 2990 (1978)

[15] A. Belavin, A. Polyakov, A. Shvarts and Yu. Tyupkin, Phys. Lett. **59**, 85 (1975)

[16] D. Diakonov, N. Gromov, V. Petrov and S. Slizovskiy, Phys. Rev. **D70**, 036003 (2004), arXiv:hep-th/0404042; N. Gromov, in: Proc. NATO Advanced Study Institute and EU Hadron Physics 13 Summer Institute, St. Andrews, Scotland, 22-29 Aug 2004, p. 411, arXiv:hep-th/0701192

[17] S. Slizovkiy, Phys. Rev. **D76**, 085019 (2007), arXiv:0707.0851 [hep-th]

[18] T.C. Kraan, Commun. Math. Phys. **212**, 503 (2000), arXiv:hep-th/9811179

[19] D. Diakonov and N. Gromov, Phys. Rev. **D72**, 025003 (2005), arXiv:hep-th/0502132

[20] M.F. Atiyah and N.J. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton University Press (1988)

[21] F.A. Berezin, *Second Quantization Method*, Nauka, Moscow (1965)

[22] M. Shifman, Acta Phys. Polon. **B6**, 3805 (2005), arXiv:hep-th/0510098

[23] M.R. Douglas and S.H. Shenker, Nucl. Phys. **B447**, 271 (1995); A. Hanany, M. Strassler and A. Zaffaroni, Nucl. Phys. **B513**, 87 (1998), arXiv:hep-th/9707244

[24] L. Del Debbio, H. Panagopoulos, P. Rossi and E. Vicari, JHEP **0201**, 009 (2002), arXiv:hep-th/0111090

[25] B. Lucini, M. Teper and U. Wenger, JHEP **0406**, 012 (2004), arXiv:hep-lat/0404008

[26] B. Lucini, M. Teper and U. Wenger, JHEP **0401**, 061 (2003), arXiv:hep-lat/0307017; B. Lucini, M. Teper and U. Wenger, arXiv:hep-lat/0502003