On the problem of inflation in non-linear multidimensional cosmological models

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D-dimensional factorizable geometry:

$$g = g^{(0)}(x) + \sum_{i=1}^{n} L_{Pl}^{2} e^{2\beta^{i}(x)} g^{(i)}(y)$$

Internal space scale factor: $a_i(x) \equiv L_{Pl} e^{\beta^i(x)}$

Manifold: $M = M_0 \times \underbrace{M_1 \times \cdots \times M_n}$

External (our) space-time

Internal spaces (Ricci-flat orbifolds with fixed points)

branes in fixed points

Universal Extra Dimension models: *the Standard Model fields are not localized on branes but can move in the bulk*

D – dimensional action:

$$S = \frac{1}{2\kappa_D^2}\int_M d^D x \sqrt{|g|}\left\{R[g] - 2\Lambda_D\right\} + S_m + S_b$$

bulk matter

branes

Monopole form fields:

$$S_m = -\tfrac{1}{2}\int_M d^D x \sqrt{|g|}\underbrace{\sum_{i=1}^{n}\tfrac{1}{d_i!}\left(F^{(i)}\right)^2}$$

Freund-Rubin ansatz

$$\sum_{i=1}^{n}\frac{f_i^{\,2}}{a_i^{2d_i}}$$

const

$$S_b = -\sum_{\text{fixed points}}\int d^4x\sqrt{|g^{(0)}(x)|}\,\tau_{(k)}\Big|_{\text{fixed points}}$$

Induced metric

tensions

Dimensional reduction (Einstein frame):

$$g^{(0)}_{\mu\nu} = \left(\prod_{i=1}^{n} e^{d_i \bar{\beta}^i} \right)^{-2/(D_0-2)} \tilde{g}^{(0)}_{\mu\nu}$$

$$D_0 = 4; \qquad \bar{\beta}^{\,i}(x) = \beta^{\,i}(x) - \beta^{\,i}_{(0)}$$

Internal space stabilization position
(present day value)

$$n = 1 \quad \Downarrow$$

$$S = \frac{1}{2\kappa_0^2} \int\limits_{M_0} d^{D_0} \sqrt{|\tilde{g}^{(0)}|} \left\{ R[\tilde{g}^{(0)}] - \tilde{g}^{(0)\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - 2U_{eff}(\varphi) \right\}$$

$$\varphi \equiv -\sqrt{\frac{d_1(D-2)}{D_0-2}}\,\bar{\beta}^{\,i}$$

Effective potential:

$$U_{eff}(\varphi) = e^{-\sqrt{2d_1/(d_1+2)}\,\varphi}\left[\Lambda_D + \tilde{f}_1^{\,2} e^{-2\sqrt{2d_1/(d_1+2)}\,\varphi} - \lambda e^{-\sqrt{2d_1/(d_1+2)}\,\varphi}\right]$$

$$\tilde{f}_1^{\,2} \equiv \kappa_D^2 f_1^{\,2} / a_{(0)1}^{2d_1}$$

$$\lambda \equiv -\kappa_0^2 \sum_{k=1}^{m} \tau_{(k)}$$

Zero global minimum condition: $\Lambda_D = f_1^{\,2} = \lambda / 2$

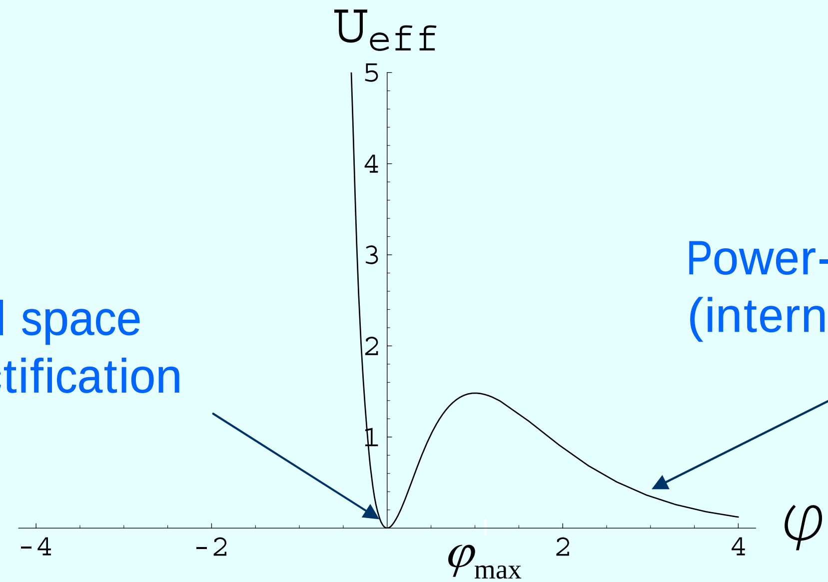
Internal space stabilization

Large φ limit: $U_{eff} \approx \Lambda_D e^{-\sqrt{q}\varphi}, \quad q = \dfrac{2d_1}{d_1+2} < 2$

Power–law inflation: $a(t) = a_0\left[1 + \dfrac{q}{2}\sqrt{\dfrac{8\pi G}{3}\rho_0}\,(t - t_0)\right]^{2/q}$

The form of $U_{eff}(\varphi)$ in the case $d_1 = 3, \Lambda_D = f_1^2 = \lambda/2 = 10$

Internal space compactification



Power-law inflation (internal space decompactification)

Slow-roll parameter in local maximum:

$$\eta_{max} = \frac{1}{U_{eff}} \frac{d^2 U_{eff}}{d\varphi^2}\bigg|_{max} = -\frac{3d_1}{d_1+2} \quad \Rightarrow \quad 1 \le |\eta_{max}| < 3$$

Topological inflation: $\varphi_{max} - \varphi_{min} \ge \varphi_{cr} \approx 1.65$

in our case: $\varphi_{max} = \sqrt{(d_1+2)/2d_1}\,\ln 3 \le 1.35 < 1.65$

Inflation in local maximum is absent

Generalization: additional scalar field ϕ

$U(\phi)$

$$U_{eff}(\varphi,\phi) = e^{-\sqrt{2d_1/(d_1+2)}\,\varphi}\left[\cancel{\Lambda_D} + \tilde{f}_1^{\,2} e^{-2\sqrt{2d_1/(d_1+2)}\,\varphi} - \lambda e^{-\sqrt{2d_1/(d_1+2)}\,\varphi}\right]$$

Origin: nonlinear gravitational models

$$S = \frac{1}{2\kappa_D^2}\int_M d^D x\sqrt{|\bar{g}|}\, f(\bar{R}) \quad - \frac{1}{2}\int_M d^D x\sqrt{|g|}\sum_{i=1}^{n}\frac{1}{d_i!}\left(F^{(i)}\right)^2 - \sum\int d^4x\sqrt{|g^{(0)}(x)|}\,\tau_{(k)}\Big|$$

$$g_{ab} = \Big[\underbrace{df/d\bar{R}}_{\equiv\, e^{\sqrt{(D-2)/(D-1)}\,\phi}}\Big]^{2/(D-2)}\bar{g}_{ab}$$

$$\Downarrow$$

$$\frac{1}{2\kappa_0^2}\int_{M_0} d^{D_0}\sqrt{|\tilde{g}^{(0)}|}\left\{R[\tilde{g}^{(0)}] - \tilde{g}^{(0)\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - \tilde{g}^{(0)\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - 2U_{eff}(\varphi,\phi)\right\}$$

Internal space scale factor

Nonlinearity scalar field

Quadratic model: $f(\bar{R}) = \bar{R} + \xi \bar{R}^2 - 2\Lambda_D$

$$\Rightarrow \quad U(\phi) = \frac{1}{2} e^{-B\phi} \left[\frac{1}{4\xi} \left(e^{A\phi} - 1 \right)^2 + 2\Lambda_D \right], \qquad A = \sqrt{\frac{d_1+2}{d_1+3}}, \qquad B = A \frac{d_1+4}{d_1+2}$$

Non-negative minimum condition of $U_{eff}(\varphi, \phi)$: $\quad \xi, \Lambda_D > 0$

Zero global minimum condition of $U_{eff}(\varphi, \phi)$: $\quad U(\phi_0) = f_1^2 = \lambda/2$

$$\left.\frac{dU(\phi)}{d\phi}\right|_{\phi_0} = 0 \quad \Longrightarrow \quad \exp(A\phi_0) = \left[A - B + \sqrt{A^2 + (2A-B)B8\xi\Lambda_D} \right] / (2A-B)$$

$$\partial_\phi U_{eff}(\varphi, \phi_0) = 0$$

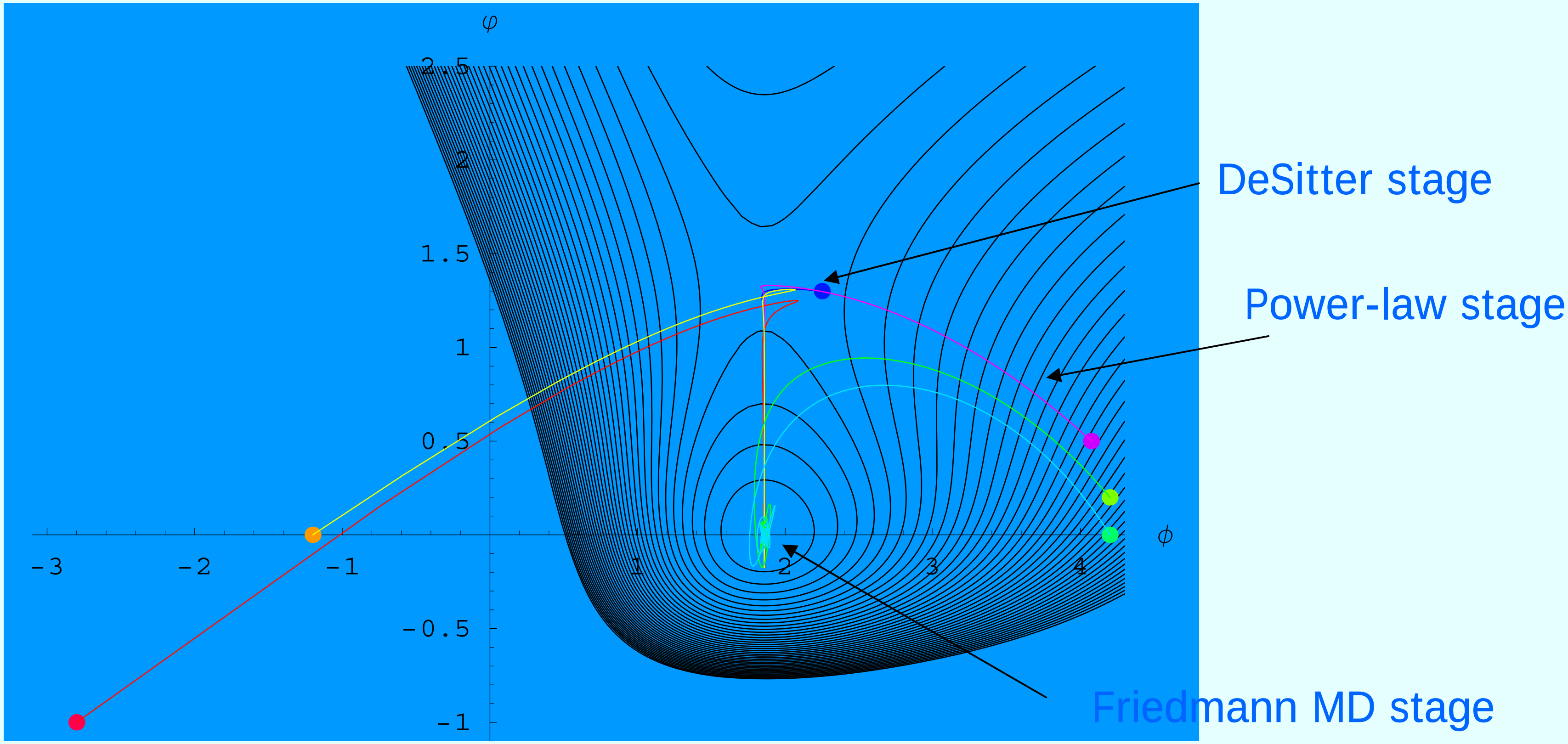
Global minimum of $U_{eff}(\varphi, \phi)$: $\quad (\varphi = 0, \phi = \phi_0)$

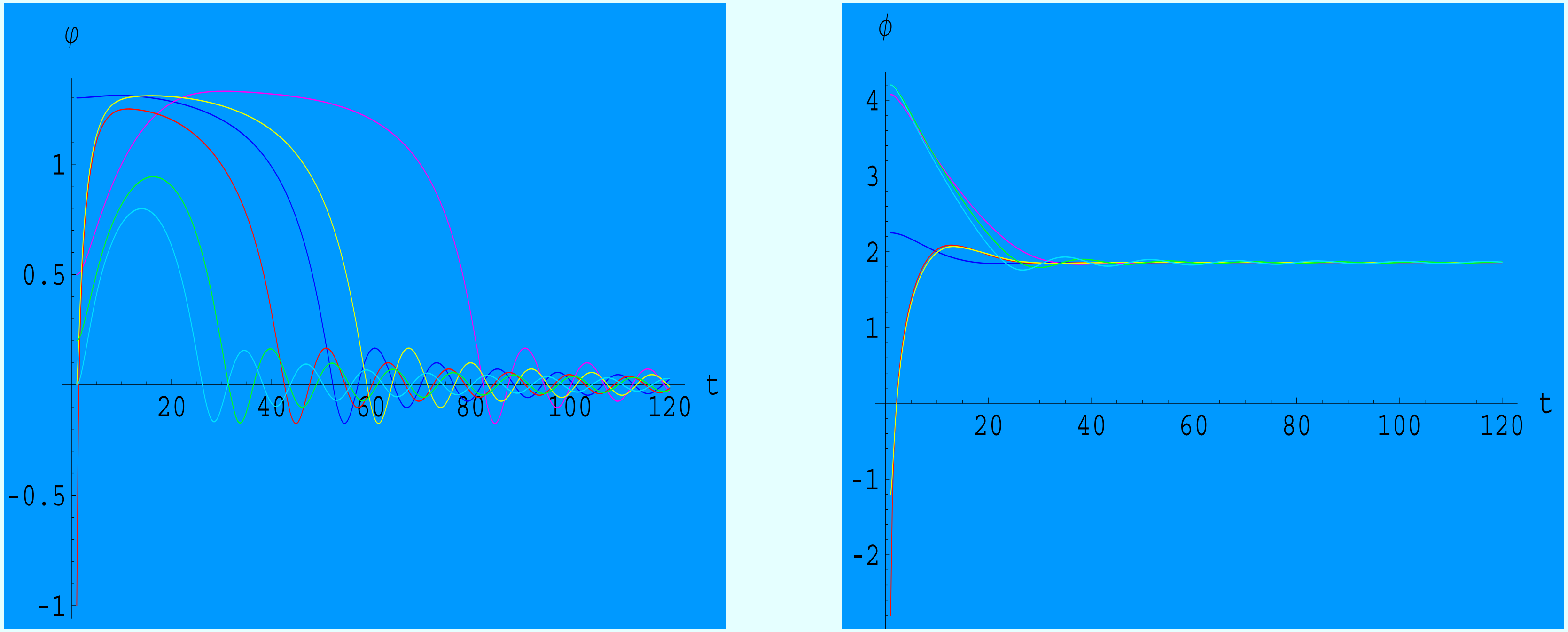
Saddle point of $U_{eff}(\varphi, \phi)$: $\quad (\varphi = \varphi_{max}, \phi = \phi_0)$

$$\varphi_{max} = \sqrt{(d_1+2)/2d_1} \ln 3 \le 1.35 \; < 1.65$$

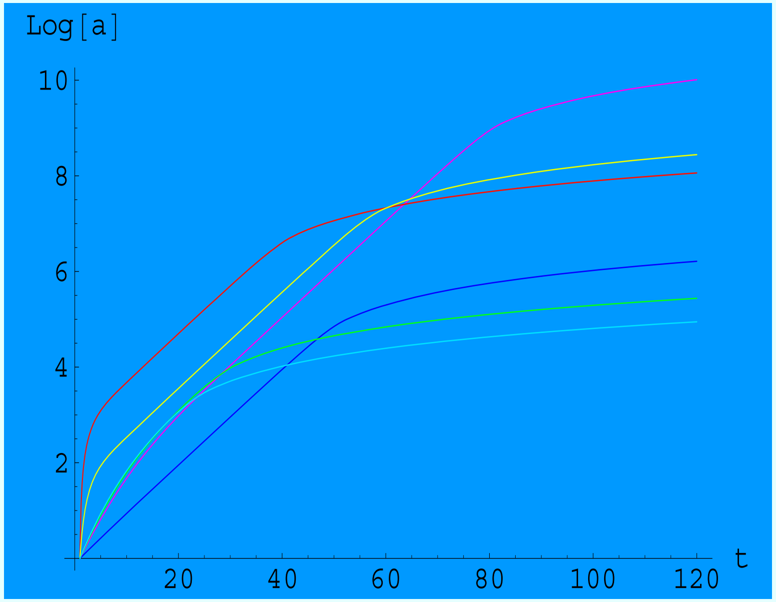
Contour plot of the effective potential $U_{eff}(\varphi,\phi)$ for parameters $d_1 = 1, \xi = \Lambda_D = 1.$

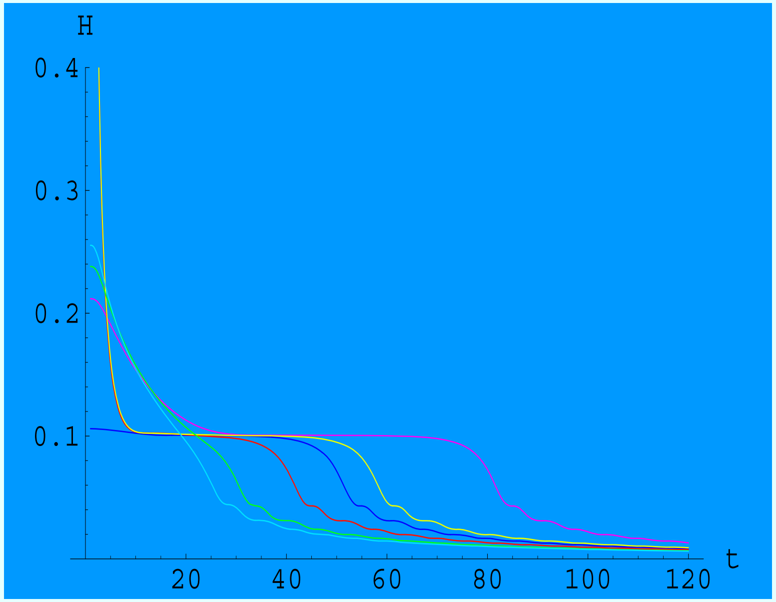
Colour lines describe trajectories for scalar fields starting from different regions of the effective potential.





Internal space scale factor and nonlinear scalar field as functions of time (in Planck units)





Number of e-foldings. We choose the following boundary condition (in Planck units): $a(t=1)=1$

Hubble parameter

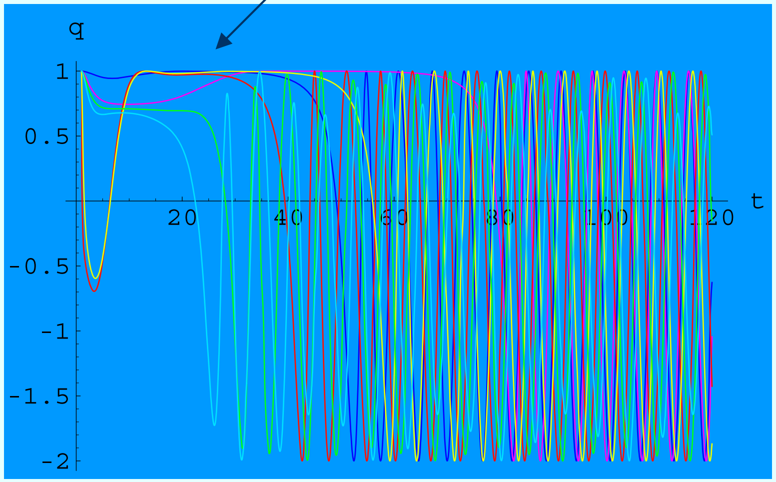
$$\boxed{N_{\max} \approx 9-10}$$

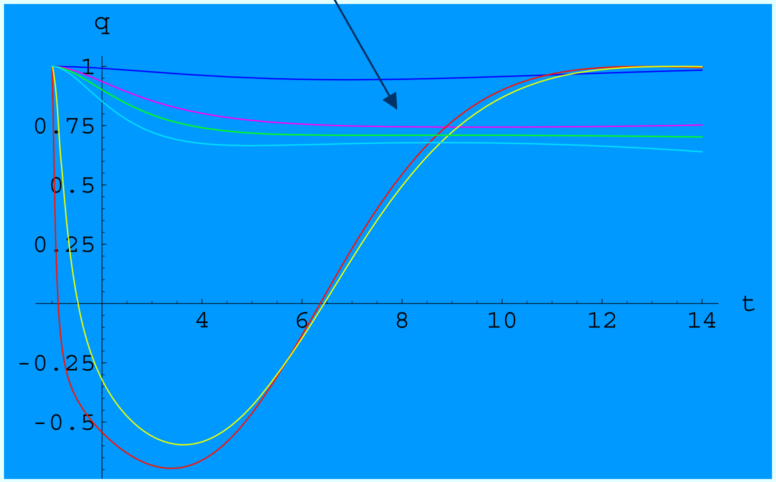
Parameter of acceleration $q = \frac{\ddot{a}}{H^2 a} = \begin{cases} (s-1)/s, & a \propto t^s \\ 1, & a \propto e^{Ht} \end{cases}$ $\quad q = -\frac{1}{2}(1+3\omega)$

EoS parameter

DeSitter stage

Power-law inflation $(s \approx 4)$





Friedmann MD stage $s = 2/3 \quad (\bar{q} = -0.5)$

Power-law inflation:

$$U_{eff} \approx \frac{1}{8\xi} \exp\left(-\sqrt{2d_1/(d_1+2)}\varphi\right) \exp\left((2A-B)\phi\right) \Rightarrow s > \frac{d_1+2}{d_1} = 3\Big|_{d_1=1}$$

Quartic model: $f(\bar{R}) = \bar{R} + \gamma \bar{R}^4 - 2\Lambda_D$

$$\Rightarrow \quad U(\phi) = \frac{1}{2} e^{-B\phi}\left[\frac{3}{4}(4\gamma)^{-1/3}(e^{A\phi}-1)^{4/3} + 2\Lambda_D\right]$$

Asymptotes:

$$\phi \to -\infty \quad \Rightarrow \quad U(\phi) \approx \frac{1}{2} e^{B\phi}\left[\frac{3}{4}(4\gamma)^{-1/3} + 2\Lambda_D\right] \to +\infty \quad if \quad \gamma, \Lambda_D > 0$$

$$\phi \to +\infty \quad \Rightarrow \quad U(\phi) \approx \frac{3}{8}(4\gamma)^{-1/3} e^{\underbrace{(-B+4A/3)}_{\substack{||\\ (D-8)/3\sqrt{(D-2)(D-1)}}}\phi} \to +0 \quad if \quad \gamma > 0, D < 8$$

$D = 8$ - critical value

We shall consider the case $D < 8 \Rightarrow d_1 = 1,2,3$

Non-negative local minimum: $U(\phi_0), \lambda, f_1^2, \gamma, \Lambda_D > 0$

Zero local minimum: $U(\phi_0) = f_1^2 = \lambda / 2$

$U(\phi)$ - extremum condition:

$$\frac{dU}{d\phi}=0 \quad \Rightarrow \quad \overline{R}_{0(1,2)}=\frac{\Lambda_D}{2}\left(\mp\sqrt{\frac{2(2+d_1)}{(4-d_1)k\sqrt{M}}-M}+\sqrt{M}\right)$$

$$\begin{cases} M=-2^{10/3}\dfrac{4+d_1}{\omega^{1/3}}-\dfrac{1}{3\cdot 2^{1/3}k}\dfrac{\omega^{1/3}}{(4-d_1)}, \\ \omega=k\left[-27(4-d_1)(2+d_1)^2+\sqrt{27^2(4-d_1)^2(2+d_1)^4-4\cdot 24^3 k(16-d_1^2)^3}\right], \\ k\equiv\gamma\Lambda_D^3>0 \end{cases}$$

$$k\le\frac{27^2(4-d_1)^2(2+d_1)^4}{4\cdot 24^3(16-d_1^2)^3}\equiv k_0$$

Minimum condition: $\left.\dfrac{d^2U(\phi)}{d\phi^2}\right|_{\phi_0}>0 \quad \Rightarrow \quad \begin{cases}\overline{R}_{0(1)} - \text{ minimum} \\ \overline{R}_{0(2)} - \text{ maximum}\end{cases} \quad \Rightarrow \begin{cases}\phi_{\min=}\dfrac{1}{A}\ln\left[1+4\gamma\overline{R}_{0(1)}^3\right] \\ \phi_{\max=}\dfrac{1}{A}\ln\left[1+4\gamma\overline{R}_{0(2)}^3\right]\end{cases}$

$$U(\phi_{\min})\equiv U_{\min},\, U(\phi_{\max})\equiv U_{\max}$$

$U_{eff}(\varphi,\phi)$ - extremum condition (with respect to φ):

$$\frac{\partial U_{eff}}{\partial\varphi}=0 \quad\Rightarrow\quad \begin{cases}-U_{\min}-3f_1^{\,2}\chi_1^2+2\lambda\chi_1=0; & e^{-b\varphi_1}\equiv\chi_1\\ -U_{\max}-3f_1^{\,2}\chi_2^2+2\lambda\chi_2=0; & e^{-b\varphi_2}\equiv\chi_2\end{cases}$$

Solutions:

$$\chi_{1(\pm)}=\alpha\pm\sqrt{\alpha^2-\beta},\quad \alpha\geq\sqrt{\beta}\equiv\alpha_1$$

$$\chi_{2(\pm)}=\alpha\pm\sqrt{\alpha^2-\beta\frac{U_{\max}}{U_{\min}}},\quad \alpha\geq\sqrt{\beta\frac{U_{\max}}{U_{\min}}}\equiv\alpha_2$$

$$\alpha\equiv\lambda/3f_1^{\,2}$$

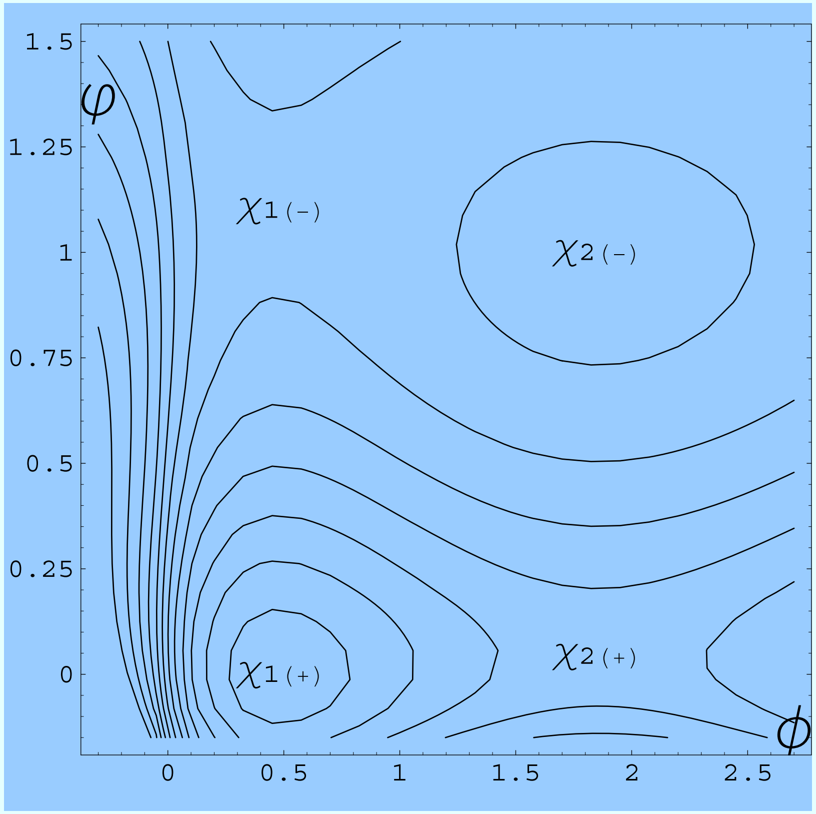
$$\beta\equiv U_{\min}/3f_1^{\,2}$$

$0<\alpha<\alpha_1$	$\alpha=\alpha_1$	$\alpha_1<\alpha<\alpha_2$	$\alpha=\alpha_2$	$\alpha>\alpha_2$
no extrema	one extremum (point of inflection on the line $\phi=\phi_{\min}$)	two extrema (one minimum and one saddle on the line $\phi=\phi_{\min}$)	three extrema (minimum and saddle on the line $\phi=\phi_{\min}$, inflection on the line $\phi=\phi_{\max}$)	four extrema (minimum and saddle on the line $\phi=\phi_{\min}$, maximum and saddle on the line $\phi=\phi_{\max}$)

Four extrema case $(\alpha > \alpha_2)$ with zero local minimum: $U_{eff}(\varphi = 0, \phi = \phi_{\min}) = 0$

$$\underline{\tilde{k}(d_1) < k < k_0} \quad \Leftarrow \quad U_{\max}/U_{\min} < 4/3 \quad \overset{\alpha > \alpha_2}{\Leftarrow} \quad \beta = 1/3, \alpha = 2/3$$

$$k \equiv \gamma \Lambda_D^3 > 0$$



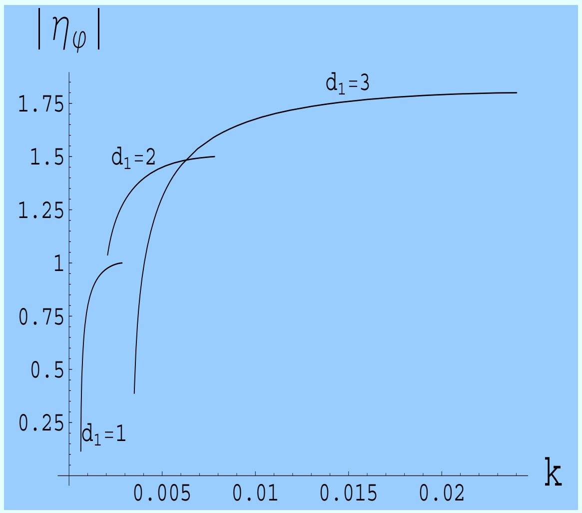
Contour plot of U_{eff} in the case $d_1 = 2, \beta = 1/3, k = 0.004, \Lambda_D = 0.01.$

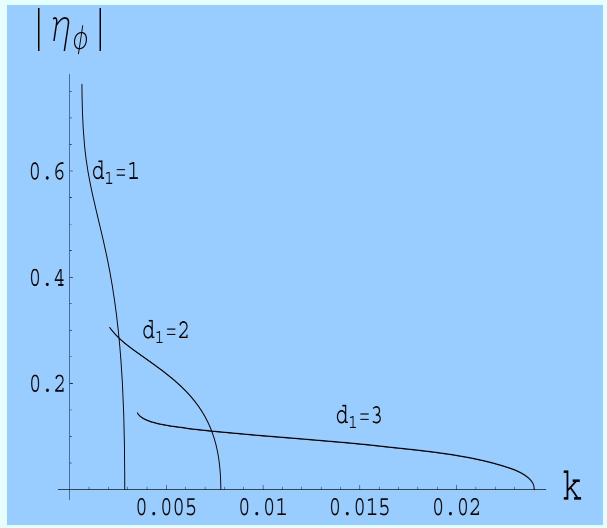
Here, $\alpha_1 = \sqrt{1/3} \approx 0.57, \alpha_2 \approx 0.59$ and $\alpha = 2/3 \approx 0.67 > \alpha_2.$

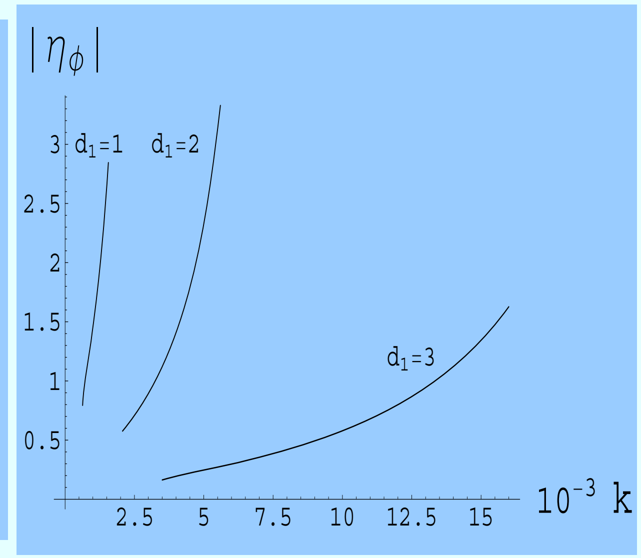
Inflation?

Slow-roll parameters in extrema:

$$\varepsilon = 0$$







Graphs of $|\eta_\varphi|$ and $|\eta_\phi|$ as functions of $k \in (\tilde{k}, k_0)$ for local maximum $\chi_{2(-)}$ and parameters $\beta = 1/3, \Lambda_D = 0.01, d_1 = 1,2,3.$

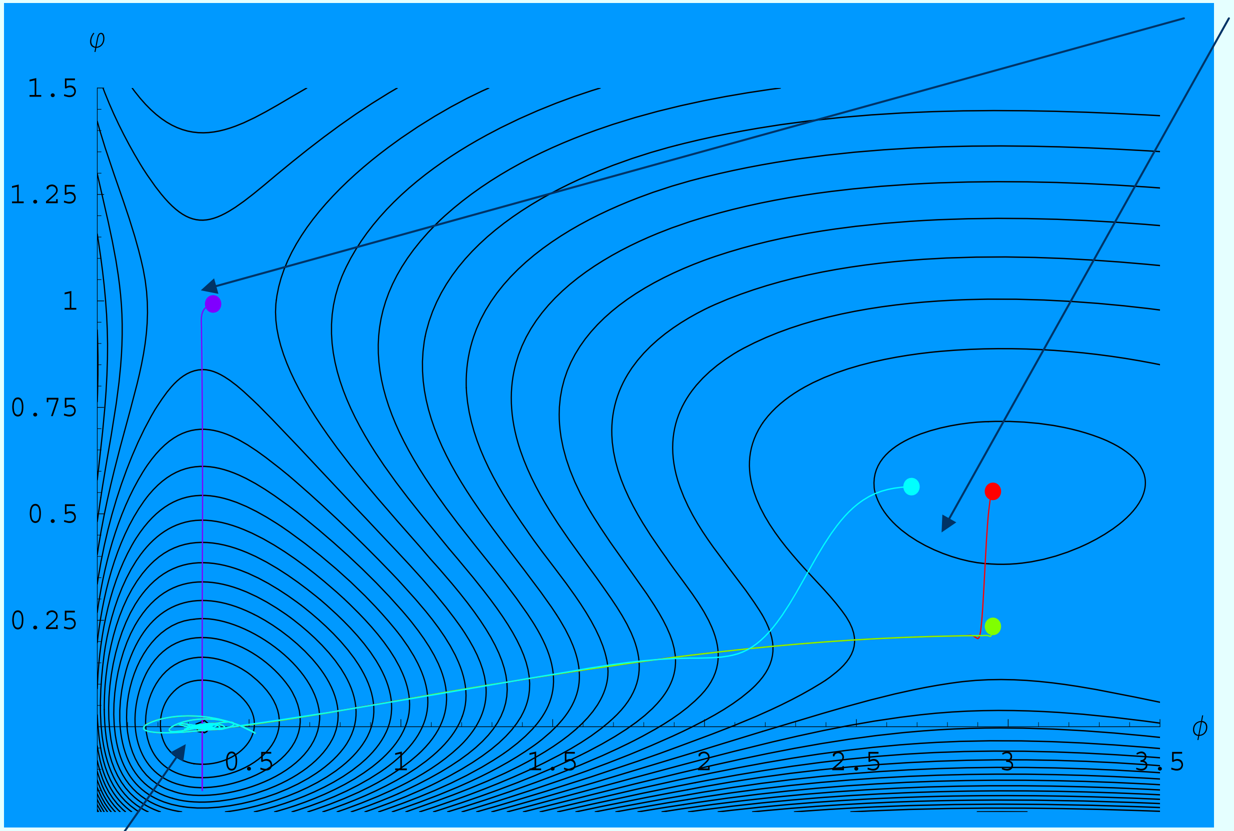
Graph of $|\eta_\phi|$ as fuction of k for saddle point $\chi_{2(+)}$ and parameters $\beta = 1/3, \Lambda_D = 0.01, d_1 = 1,2,3.$

Contour plot of the effective potential $U_{eff}(\varphi, \phi)$ for parameters

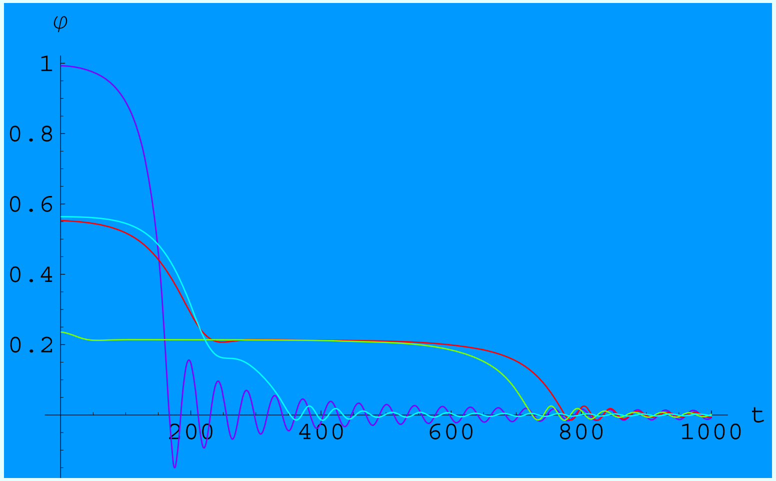
$$d_1 = 3, \beta = 1/3, \alpha = 2/3, k = 0.004, \Lambda_D = 0.01.$$

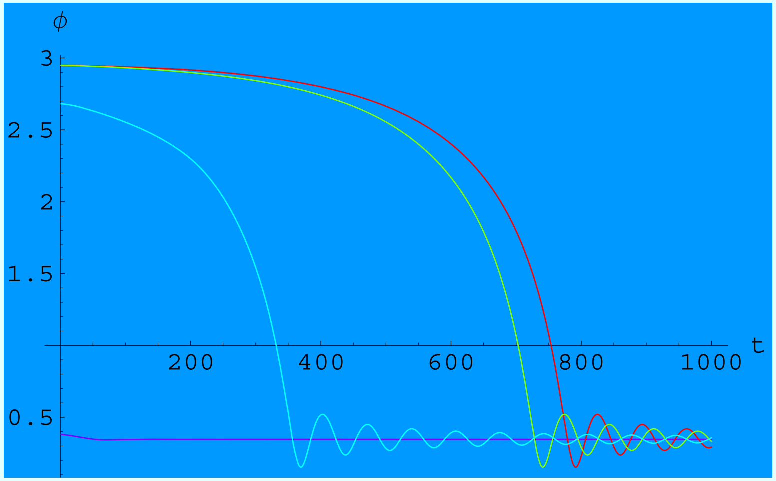
Colour lines describe trajectories for scalar fields starting from different regions of the effective potential.

DeSitter stage

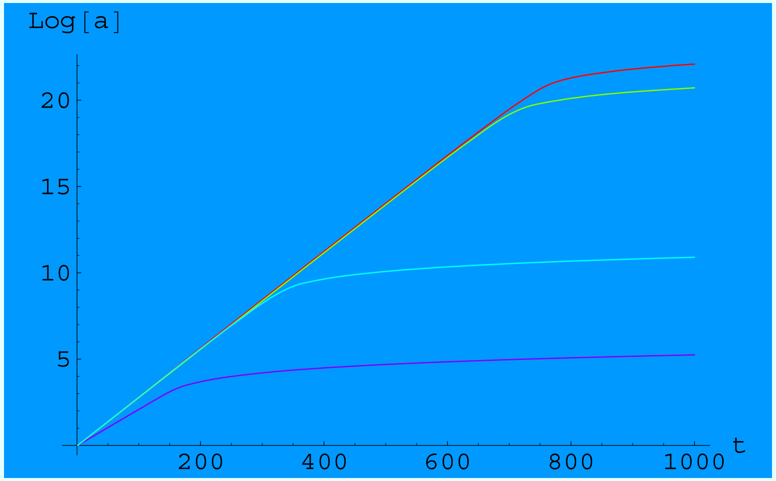


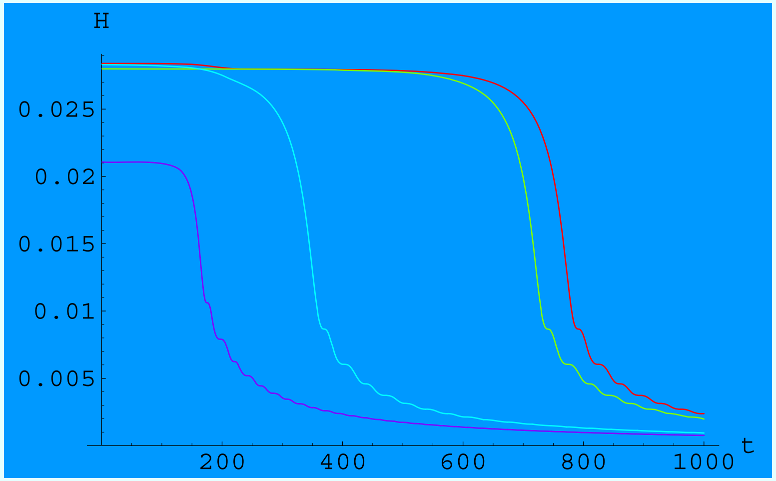
Friedmann MD (dust) stage





Internal space scale factor and nonlinear scalar field as functions of time (in Planck units)





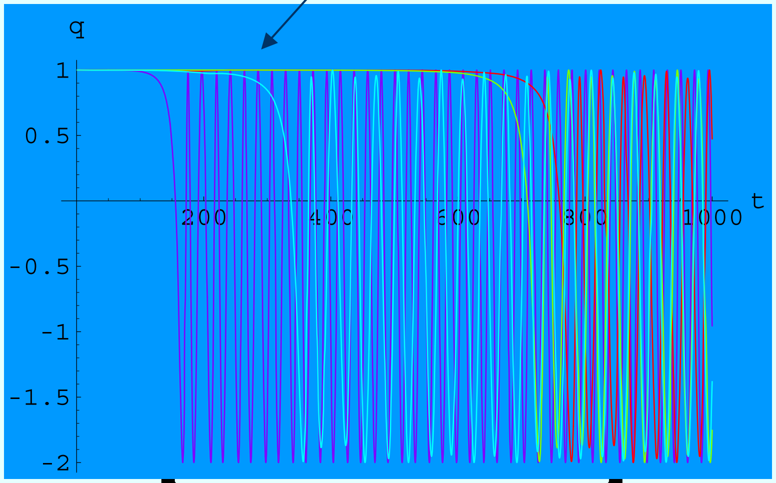
Number of e-foldings

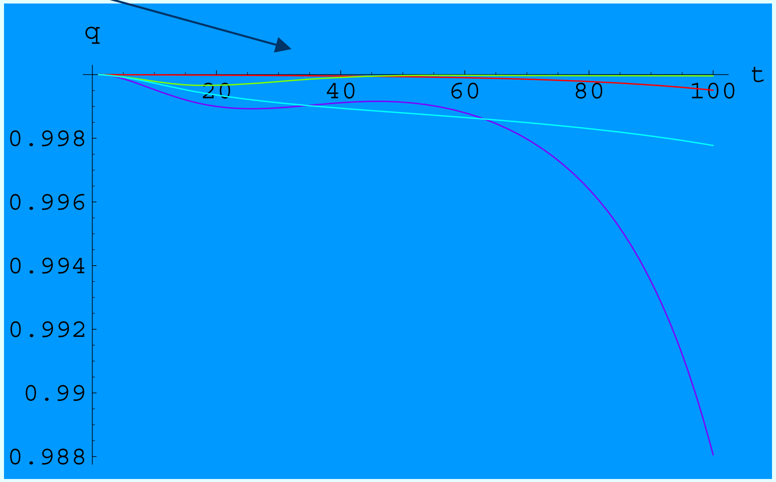
Hubble parameter

$$\boxed{N_{\max} \approx 22}$$

Parameter of acceleration $$q = \frac{\ddot{a}}{H^2 a}$$

DeSitter stage



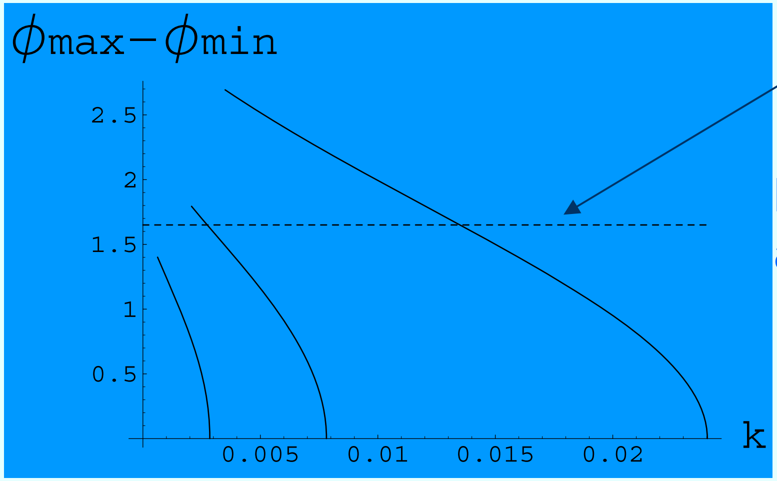


Friedmann MD (dust) stage

$$s = 2/3 \quad (\bar{q} = -0.5)$$

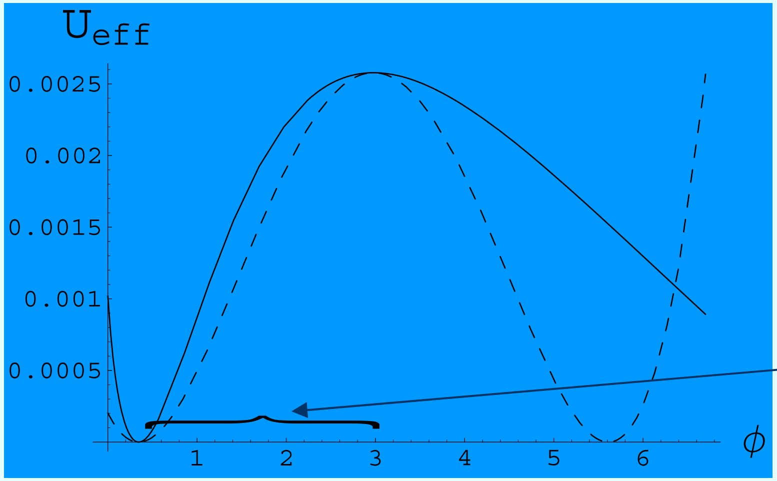
$$a \propto t^{2/3}$$

Topological (eternal) inflation



Critical value $\Delta\phi_{cr} = 1.65$

Plots of $\phi_{\max} - \phi_{\min}$ (for the profile $\varphi = \varphi\big|_{\chi 2(+)}$) as a function of $k \in (\tilde{k}, k_0)$ for parameters $\beta = 1/3, \Lambda_D = 0.01, d_1 = 1,2,3$ (from left to right).



Comparison of the potential $U_{eff}(\varphi\big|_{\chi 2(+)}, \phi)$ with a double well potential for parameters

$\beta = 1/3, \Lambda_D = 0.01, d_1 = 3.$

In this particular case:

$\phi_{\max} - \phi_{\min} = 2.63 > \Delta\phi_{cr}$

The ratio of the characteristic thickness of domain wall to the horizon scale:

$$r_w H \approx \left| U_{eff} / 3\partial_{\phi\phi} U_{eff} \right|^{1/2}_{saddle\ \chi_{2(+)}} \approx 1.30 \quad > \quad r_w H\big|_{cr} \approx 0.48 \quad \Rightarrow$$

thick enough!

Conclusions:

1. For all considered models, there are rangers of parameters where the internal space is **stably compactified**.

2. At the same time nonlinear multidimensional model can provide **inflation** of the external (our) space. For considered models maximal value of e-foldings is $N \approx 22$. This value is not sufficient to explain the horizon and flatness problem but enough for CMB. However, the spectral index is less than 1. For example, in the case of R^4 model $n_s \approx 1 + 2\eta\big|_{\chi 2(+)} \approx 0.61 < 1.$

3. The number of e-foldings of the order of 22 is big enough to encourage the following investigation of the nonlinear multidimensional models to find theories where this number will approach to 50-60.

4. Nonlinear R^4 model can provide conditions for **topological (eternal) inflation**.