Nuclear energy spectra and form factors from Skyrmions

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The Skyrme model

- nonlinear theory of pions in 3D, defined in terms of $U(t, \mathbf{x}) \in SU(2)$.

The Skyrme Lagrangian

$$\mathcal{L} = \frac{F_\pi^2}{16} \operatorname{Tr} \partial_\mu U \partial^\mu U^\dagger + \frac{1}{32e^2} \operatorname{Tr} [\partial_\mu U U^\dagger, \partial_\nu U U^\dagger][\partial^\mu U U^\dagger, \partial^\nu U U^\dagger] + \frac{1}{8} m_\pi^2 F_\pi^2 \operatorname{Tr} (U - 1_2) ,$$

where F_π is the pion decay constant, e is a dimensionless parameter and m_π is the pion mass.

- using energy and length units of $F_\pi/4e$ and $2/eF_\pi$ respectively,

$$L = \int \left\{ -\frac{1}{2} \operatorname{Tr} (R_\mu R^\mu) + \frac{1}{16} \operatorname{Tr} ([R_\mu, R_\nu][R^\mu, R^\nu]) + m^2 \operatorname{Tr} (U - 1_2) \right\} d^3x ,$$

where $R_\mu = (\partial_\mu U) U^\dagger$, and $m = 2m_\pi / eF_\pi$.

- $U : \mathbb{R}^3 \to S^3$, at fixed time. $U \to 1_2$ at spatial infinity $\implies U : S^3 \to S^3$.
- $\pi_3(S^3) = \mathbb{Z}$, so maps between 3-spheres fall into homotopy classes indexed by an integer, B:

$$B = \int B_0(\mathbf{x})\, d^3x\,,$$

  where

$$B_\mu(\mathbf{x}) = \frac{1}{24\pi^2}\,\epsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\,\partial^\nu UU^{-1}\partial^\alpha UU^{-1}\partial^\beta UU^{-1}\,.$$

- Isospin transformation: $U \mapsto AUA^\dagger$, where $A \in SU(2)$.

Semiclassical collective coordinate quantization

- Given a static Skyrmion U_0, there is a nine-parameter set of configurations, all degenerate in energy:

$$U(\mathbf{x}) = A_1 U_0(D(A_2)(\mathbf{x} - \mathbf{X}))A_1^\dagger \,,$$

  where $A_1, A_2 \in SU(2)$ and $D(A_2)_{ij} = \frac{1}{2}\mathrm{Tr}(\tau_i A_2 \tau_j A_2^\dagger) \in SO(3)$.
- Semiclassical quantization performed by promoting collective coordinates $A_1, A_2, \mathbf{X}$ to dynamical degrees of freedom.

- Dynamical ansatz: $\hat{U}(\mathbf{x}, t) = A_1(t) U_0(D(A_2(t))\mathbf{x}) A_1(t)^\dagger$. Inserting this into L, we obtain the kinetic energy

$$T = \frac{1}{2} a_i U_{ij} a_j - a_i W_{ij} b_j + \frac{1}{2} b_i V_{ij} b_j \, ,$$

where b_i and a_i are the angular velocities in space and isospace respectively, and inertia tensors U_{ij}, W_{ij} and V_{ij} are given by:

$$\begin{aligned} U_{ij} &= -\int \mathrm{Tr}\left(T_i T_j + \frac{1}{4}[R_k, T_i][R_k, T_j] \right) d^3x \, , \\ W_{ij} &= \int \epsilon_{jlm} x_l \, \mathrm{Tr}\left(T_i R_m + \frac{1}{4}[R_k, T_i][R_k, R_m] \right) d^3x \, , \\ V_{ij} &= -\int \epsilon_{ilm} \epsilon_{jnp} x_l x_n \, \mathrm{Tr}\left(R_m R_p + \frac{1}{4}[R_k, R_m][R_k, R_p] \right) d^3x \, , \end{aligned}$$

where $R_k = (\partial_k U_0) U_0^\dagger$ and $T_i = \frac{i}{2} [\tau_i, U_0] \, U_0^\dagger$.

- The momenta corresponding to b_i and a_i are the body-fixed spin and isospin angular momenta L_i and K_i:

$$
\begin{aligned}
L_i &= -W^T_{ij} a_j + V_{ij} b_j \, , \\
K_i &= U_{ij} a_j - W_{ij} b_j \, .
\end{aligned}
$$

- The space-fixed spin and isospin angular momenta are denoted J_i and I_i respectively.
- Regard L_i, K_i, J_i and I_i as quantum operators, each satisfying the $\mathfrak{su}(2)$ commutation relations.

Fermionic quantization and Finkelstein-Rubinstein constraints

- FR constraints if the Skyrmion has symmetries - easily determined using the rational map ansatz.
- $R(M_2(z)) = M_1(R(z))$, where M_1 and M_2 are Möbius transformations.
- If a rotation in physical space by an angle θ_2 about an axis $\mathbf{n}_2$ is equivalent to an isorotation by an angle θ_1 about an axis $\mathbf{n}_1$, then up to a crucial base point condition, the corresponding FR constraint is given by Krusch's formula:

$$e^{i\theta_2 \mathbf{n}_2 \cdot \mathbf{L}} e^{i\theta_1 \mathbf{n}_1 \cdot \mathbf{K}} |\Psi\rangle = \chi_{FR} |\Psi\rangle \,,$$

  where $\chi_{FR} = (-1)^{\mathcal{N}}$ and $\mathcal{N} = \frac{B}{2\pi}(B\theta_2 - \theta_1)$.
- Basis for wavefunctions: $|J, J_3, L_3\rangle \otimes |I, I_3, K_3\rangle$.

Reparametrizing the model

- Quantum numbers of the quantized $B = 6$ Skyrmion are the same as those of lithium-6.
- Obtained expressions for the mean charge radius $\langle r^2 \rangle^{1/2}$ and quadrupole moment Q, dependent only on the Skyrme parameters.
- Proposed changing the Skyrme parameters from the traditional set. Obtained specific values for these parameters for $B = 6$, by matching with properties of the lithium-6 nucleus.

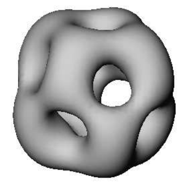


Figure: Baryon density isosurface for the $B = 6$ Skyrmion.

Reparametrizing the model

- Adkins, Nappi and Witten:

$$e = 4.84, \quad F_\pi = 108\,\text{MeV} \;\text{ and }\; m_\pi = 138\,\text{MeV} \;\; (\text{implies } m = 0.528)\,.$$

- Setting $m = 1.125$, we obtain $\langle r^2 \rangle^{1/2}$ in close agreement with that of lithium-6.
- Slightly modifying $R(z)$ we obtain Q in close agreement with experiment.
- By equating E to the mass of lithium-6, we fit $F_\pi/4e$.
- New parameters:

$$e = 3.26, \quad F_\pi = 75.2\,\text{MeV} \;\text{ and }\; m_\pi = 138\,\text{MeV} \;\; (\text{which implies } m = 1.125)\,.$$

- Further support for these parameters is found by reconsidering the deuteron and α-particle.
- N. S. Manton and SWW, Phys. Rev. D74: 125017 (2006).

Energy spectra of light nuclei

- Reconsidered the quantization of Skyrmions with $1 \leq B \leq 8$.
- Obtained novel, general expressions for the elements of U_{ij}, V_{ij} and W_{ij} in terms of the rational map.
- Determined the kinetic energy (T) of the ground and excited states in each case.
- Results correspond well to experiment, and we were able to predict the energies and spins of a few as yet unobserved states.

$B = 4$

- $R(z) = \frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$
- Generating symmetries:

$$R(iz) = 1/R(z)\,, \quad R\left(\frac{iz+1}{-iz+1}\right) = e^{i\frac{2\pi}{3}}R(z)\,.$$

- FR constraints:

$$e^{i\frac{\pi}{2}L_3}e^{i\pi K_1}|\Psi\rangle = |\Psi\rangle\,, \quad e^{i\frac{2\pi}{3\sqrt{3}}(L_1+L_2+L_3)}e^{i\frac{2\pi}{3}K_3}|\Psi\rangle = |\Psi\rangle\,.$$

- Lowest allowed states:

$$|0,0\rangle \otimes |0,0\rangle\,,$$

$$(|2,2\rangle + \sqrt{2}i|2,0\rangle + |2,-2\rangle) \otimes |1,1\rangle - (|2,2\rangle - \sqrt{2}i|2,0\rangle + |2,-2\rangle) \otimes |1,-1\rangle\,,$$

$$\left(|4,4\rangle + \sqrt{\frac{14}{5}}|4,0\rangle + |4,-4\rangle\right) \otimes |0,0\rangle\,.$$

- U_{ij}, V_{ij} and W_{ij} are all diagonal, with $U_{11} = U_{22}$, $V_{ij} = v\delta_{ij}$ and $W_{ij} = 0$.
- Kinetic energy operator:

$$T = \frac{1}{2v}\mathbf{J}^2 + \frac{1}{2U_{11}}\mathbf{I}^2 + \frac{1}{2}\left(\frac{1}{U_{33}} - \frac{1}{U_{11}}\right) K_3^2 \,.$$

-

$$\begin{aligned} E_{J=0,\,I=0} &= \mathcal{M}_4 = 3679\,\mathrm{MeV}\,, \\ E_{J=2,\,I=1} &= \mathcal{M}_4 + 28.7\,\mathrm{MeV} = 3708\,\mathrm{MeV}\,, \\ E_{J=4,\,I=0} &= \mathcal{M}_4 + 39.4\,\mathrm{MeV} = 3718\,\mathrm{MeV}\,. \end{aligned}$$

$B = 4$

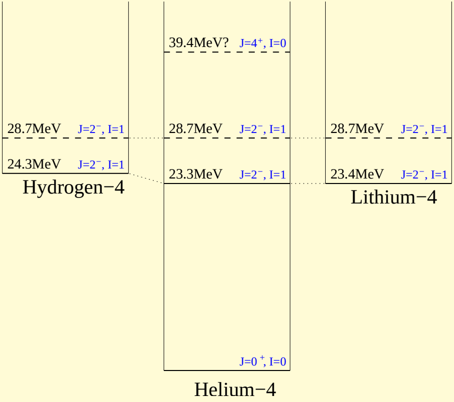


Figure: Solid lines indicate experimentally observed states, while dashed lines indicate our predictions.

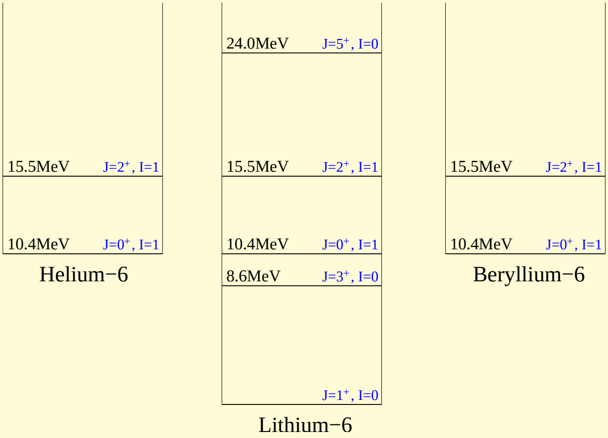
$B = 6$

- $R(z) = \frac{z^4 + ia}{z^2(iaz^4+1)}$, $a = 0.16$
- Lowest allowed states: $|1,0\rangle \otimes |0,0\rangle$, $|3,0\rangle \otimes |0,0\rangle$ and $|0,0\rangle \otimes |1,0\rangle$.
- Kinetic energy operator:

$$T = \frac{1}{2V_{11}}\left[\mathbf{J}^2 - L_3^2\right] + \frac{1}{2U_{11}}\left[\mathbf{I}^2 - K_3^2\right] + \frac{1}{2(U_{33}V_{33} - W_{33}^2)}\left[U_{33}L_3^2 + V_{33}K_3^2 + 2W_{33}L_3K_3\right]$$

- $$E_{J=1,\, I=0} = \mathcal{M}_6 + \frac{1}{V_{11}} = \mathcal{M}_6 + 1.7\,\text{MeV} = 5602\,\text{MeV}$$
  $$E_{J=3,\, I=0} = \mathcal{M}_6 + \frac{6}{V_{11}} = \mathcal{M}_6 + 10.3\,\text{MeV} = 5611\,\text{MeV}$$
  $$E_{J=0,\, I=1} = \mathcal{M}_6 + \frac{1}{U_{11}} = \mathcal{M}_6 + 12.1\,\text{MeV} = 5613\,\text{MeV}$$

$B = 6$



- O. V. Manko, N. S. Manton and SWW, Phys. Rev. C76: 055203 (2007).

Electromagnetic form factors of quantized Skyrmions

- Internal structure of nuclei can be investigated by electron scattering.
- Wish to investigate whether the symmetries of the classical Skyrmions give rise to unusual behaviour in their form factors.

Lithium-6 form factors

- Dynamical ansatz:

$$\hat{U}(\mathbf{x},t) = A_1(t)U_0(D(A_2(t))(\mathbf{x}-\mathbf{X}(t)))A_1(t)^{-1}$$

- Ground state of the quantized $B=6$ Skyrmion:

$$\Psi^{gd}_{J_3}(\mathbf{p}) = \frac{\sqrt{3}}{8\pi^2}D^1_{0J_3}(\phi,\theta,\psi)D^0_{00}(\alpha,\beta,\gamma)e^{i\mathbf{p}\cdot\mathbf{X}}$$

- Form factors calculated by inserting $\hat{U}(\mathbf{x},t)$ into the electromagnetic current density:

$$J_\mu(x) = \frac{1}{2}B_\mu(x) + I^3_\mu(x)\,,$$

  and determining matrix elements of the resulting operator.

Lithium-6 charge and quadrupole form factors $G_C(q^2)$ and $G_Q(q^2)$

- Defined in the Breit frame ($\mathbf{p}+\mathbf{p}'=0$):

$$\langle\Psi^{gd}_{J_3'}(\mathbf{p}')|J_0(\mathbf{x}=0)|\Psi^{gd}_{J_3}(\mathbf{p})\rangle = G_C(q^2)\delta_{J_3'\,J_3}+\frac{1}{6\mathcal{M}^2}G_Q(q^2)U_{J_3'\,a}(3q^aq^b-q^2\delta^{ab})U^\dagger_{b\,J_3}\,,$$

  where $\mathbf{q}=\mathbf{p}'-\mathbf{p}$ is the momentum transfer, $\mathcal{M}$ is the lithium-6 mass and U_{ij} is the unitary matrix relating the Cartesian and spherical bases.

- Braaten and Carson obtained a general expression for the matrix element of $J_0(\mathbf{x}=0)$ - for the lithium-6 nucleus:

$$\langle\Psi^{gd}_{J_3'}(\mathbf{p}')|J_0(\mathbf{x}=0)|\Psi^{gd}_{J_3}(\mathbf{p})\rangle$$

$$=\delta_{J_3'\,J_3}\frac{1}{2}\int j_0(qr)B_0(\mathbf{x})d^3x+U_{J_3'\,a}(3q^aq^b-q^2\delta^{ab})U^\dagger_{b\,J_3}\frac{1}{4}\int(1-3\cos^2\theta)j_2(qr)B_0(\mathbf{x})d^3x\,,$$

  where the $j_n(qr)$ are spherical Bessel functions.

Lithium-6 charge and quadrupole form factors

- We obtain:

$$\begin{aligned} G_C(q^2) &= \frac{1}{2}\int j_0(qr)B_0(\mathbf{x})d^3x\,, \\ \frac{1}{\mathcal{M}^2}G_Q(q^2) &= \frac{3}{2}\frac{1}{q^2}\int (1-3\cos^2\theta)j_2(qr)B_0(\mathbf{x})d^3x\,. \end{aligned}$$

- As $q^2 \to 0$, $G_C(q^2) \to 1 - \frac{1}{6}q^2\langle r^2\rangle$ and $G_Q(q^2) \to \mathcal{M}^2 Q$.
- Quadrupole contribution to electron scattering is negligible compared to monopole contribution - due to extremely small Q.

Lithium-6 magnetic moment

- Classical magnetic moment:

$$\mu_a = \frac{1}{2} \int \epsilon_{abc} x_b J_c \, d^3x = \frac{1}{4} \int \epsilon_{abc} x_b B_c \, d^3x$$

- Obtain, for the rotated classical Skyrmion:

$$\hat{\mu}_a = D(A_2)^T_{a\alpha} \left(M_{\alpha k} a_k + N_{\alpha k} b_k \right) ,$$

where

$$\begin{aligned} M_{\alpha k} &= \frac{1}{32\pi^2} \int x_\beta \, \mathrm{Tr} \left(T_k [R_\alpha, R_\beta] \right) d^3x \,, \\ N_{\alpha k} &= \frac{1}{32\pi^2} \int \epsilon_{krs} x_\beta x_r \, \mathrm{Tr} \left(R_s [R_\alpha, R_\beta] \right) d^3x \,. \end{aligned}$$

Lithium-6 magnetic moment

- For the $B = 6$ Skyrmion, both $M_{\alpha k}$ and $N_{\alpha k}$ are diagonal, $M_{11} = M_{22} = 0$, $N_{11} = N_{22}$ and $N_{33} = -2M_{33}$.
- Therefore
$$\hat{\mu}_a = -\frac{N_{11}}{V_{11}} J_a + \text{terms proportional to } K_3 \,.$$
- Magnetic moment of the quantized $B = 6$ Skyrmion:
$$\mu = \langle \Psi_{J_3=1} | \hat{\mu}_3 | \Psi_{J_3=1} \rangle = -\frac{N_{11}}{V_{11}}$$
- μ is calculated to be 0.54 nm. The experimental value is 0.82 nm.

Lithium-6 magnetic form factor $G_M(q^2)$

- Defined in the Breit frame:

$$\langle \Psi^{gd}_{J_3'}(\mathbf{p}')|J_i(\mathbf{x}=0)|\Psi^{gd}_{J_3}(\mathbf{p})\rangle = \frac{1}{2\mathcal{M}} G_M(q^2) U_{J_3'\,a}(q^a\delta^{ib} - \delta^{ai}q^b)U^\dagger_{b\,J_3}\,.$$

- For the lithium-6 nucleus:

$$\langle \Psi^{gd}_{J_3'}(\mathbf{p}')|J_i(0)|\Psi^{gd}_{J_3}(\mathbf{p})\rangle = U_{J_3'\,a}(\hat{q}^a\delta^{ib} - \delta^{ai}\hat{q}^b)U^\dagger_{b\,J_3}\frac{3}{8V_{11}}\int r\left(1+\cos^2\theta\right) j_1(qr)B_0(\mathbf{x})d^3x\,,$$

and so

$$\frac{1}{2\mathcal{M}} G_M(q^2) = \frac{3}{8V_{11}}\frac{1}{q}\int r\left(1+\cos^2\theta\right) j_1(qr)B_0(\mathbf{x})d^3x\,.$$

- As $q^2 \to 0$, $G_M(q^2) \to 2\mathcal{M}\mu$, where μ is the magnetic moment of lithium-6.

Lithium-6 charge form factor

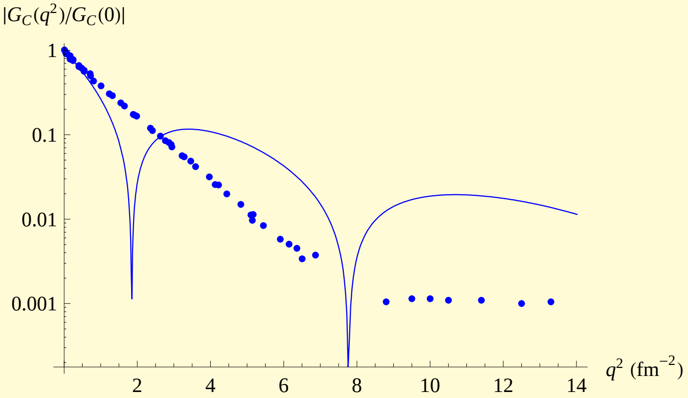


Figure: Absolute values of the charge form factor of the quantized $B = 6$ Skyrmion (solid) compared with experimental data for lithium-6 (dots).

Lithium-6 magnetic form factor

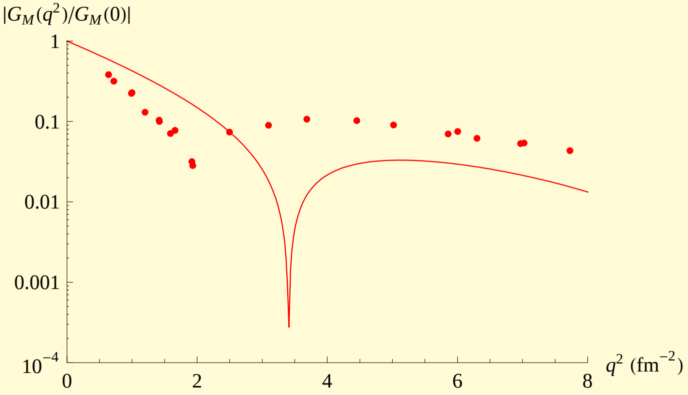


Figure: Absolute values of the magnetic form factor of the quantized $B = 6$ Skyrmion (solid) compared with experimental data for lithium-6 (dots).

Structure of the α-particle

- Charge form factor:

$$G_C(q^2) = \frac{1}{2}\int j_0(qr)B_0(\mathbf{x})d^3x\,.$$

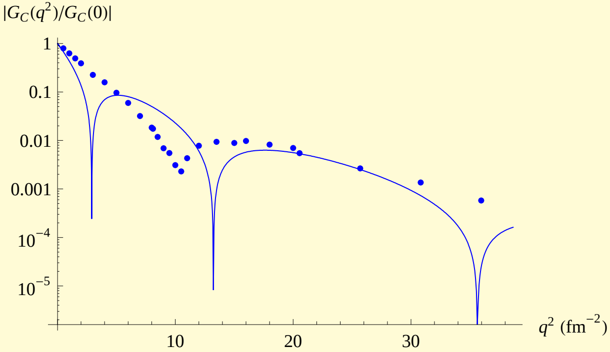


Figure: Absolute values of $G_C(q^2)$ for the quantized $B = 4$ Skyrmion (solid) compared with experimental data for helium-4 (dots).

Outlook and future work

- Work done so far has certainly provided support for the interpretation of nucleons and nuclei as quantized Skyrmions.
- New parameter set has provided better results than the traditional parameter set for larger values of B. However, one set of Skyrme parameters will not accurately describe all sectors of the model, if we restrict ourselves to the semiclassical rigid body quantization described here. Inclusion of further degrees of freedom would be a significant refinement.
- Recent work inspired by string theory and the AdS/CFT correspondence gives further credence to the idea that at large N_c, baryons and nuclei are described by some variant of the Skyrme model.