Multiloop Gluon Amplitudes

and

AdS/CFT

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Outline

- Motivations.
- Cusp anomalous dimension and integrability.
- Refresher on the calculation of gluon amplitudes.
- Basis of integrals and dual conformal invariance.
- BDS iterative ansatz for multiloop amplitudes.
- Strong and weak coupling tests and troubles.
- MHV amplitudes and Wilson loops.
- Two loop six point amplitude.
- Conclusions and open questions.

Motivations I

● Gluon scattering amplitudes in QCD and supersymmetric gauge theories are very difficult to compute, so this is a fertile ground for new insights and methods.

● Experimental program at the LHC requires many new calculations of QCD-associated processes. Development of new tools for computing scattering amplitudes is an important topic.

Motivations II

- Over the past several years we have learned a lot about remarkably rich mathematical structures in Yang-Mills theories.

- On one hand, in the context of AdS/CFT a lot of work has been done exploring Yang-Mills integrable structures and computing anomalous dimensions. Beisert, Eden and Staudacher proposed an exact 'S-matrix' for planar $\mathcal{N} = 4$ Yang-Mills theory.

- On the other hand, following the discovery of twistor string theory, we have seen a lot of progress in Yang-Mills scattering amplitudes. Of particular interest are Bern, Dixon and Smirnov iterative relations for planar $\mathcal{N} = 4$ Yang-Mills amplitudes.

- Cusp anomalous dimension is a player in both games.

- Alday and Maldacena proposed a prescription for computing scattering amplitudes using AdS string theory at strong coupling.

Integrability and Cusp Anomalous Dimension

- **Integrability is a very powerful tool for computing anomalous dimensions of operators.**

- **The cusp anomalous dimension $f(\lambda)$ governs the behavior of twist-two operators in the limit of very large spin:**

$$\Delta\left(\mathrm{Tr}[ZD^S Z]\right) = S + f(\lambda)\log S + \mathcal{O}(S^0), \qquad S \gg 1.$$

This quantity has long played an important role in quantitative checks of AdS/CFT.

- **Guess-S-matrix for planar $\mathcal{N} = 4$ Yang-Mills, has been proposed by Beisert, Eden, Staudacher. Via Bethe ansatz equations it gives the entire Yang-Mills spectrum in large J limit and in particular, it implies a (complicated!) integral equation $f(\lambda)$ valid for all λ.**

The Cusp Anomalous Dimension

At strong coupling, **the solution exhibits remarkable agreement with AdS string**

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi}(1 - \frac{3\log 2}{\sqrt{\lambda}} - \frac{K}{\lambda} + \cdots),$$

[Gubser, Klebanov, Polyakov; Frolov, Tseytlin] [Alday, Arutyunov, Benna, Eden, Kelbanov; Benna, Benvenuti, Eden, Klebanov, Scardicchio; Casteill, Kristjansen; Basso, Korchemsky, Kotanski; Roiban, Tseytlin].

At weak coupling, **the solution agrees with perturbative Yang-Mills**

$$f(\lambda) = 8\left(\frac{\lambda}{16\pi^2}\right) - \frac{8\pi^2}{3}\left(\frac{\lambda}{16\pi^2}\right)^2 + \frac{88\pi^4}{45}\left(\frac{\lambda}{16\pi^2}\right)^3 -$$

$$\left(\frac{584\pi^6}{315} + 128\zeta_3^2\right)\left(\frac{\lambda}{16\pi^2}\right)^4 + \mathcal{O}(\lambda^5)$$

[Bern, Czakon, Dixon, Kosower, Smirnov; Cachazo, Spradlin, AV]

The rest of the talk will be devoted to the calculation of this and other quantities from perturbative scattering amplitudes in $\mathcal{N} = 4$ YM.

Gluon Scattering Amplitudes

We have learned that Feynman diagrams are not the most efficient way to calculate scattering amplitudes: too messy+too many terms+hide the structure of amplitudes.

There has been a lot of progress on tree amplitude calculations stimulated by twistor string theory. [Witten]

MHV rules, recursion relations, etc. [Cachazo, Svrcek, Witten] [Britto, Cachazo, Feng, Witten] [Roiban, Spradlin, AV] [Brandhuber, Spence, Travaglini] [Dixon, Glover, Khoze] [Bern, Dixon, Kosower] [Berkovitz, Motl] [Gukov, Motl, Neitzke] [Arkani-Hamed, Kaplan] [many others]

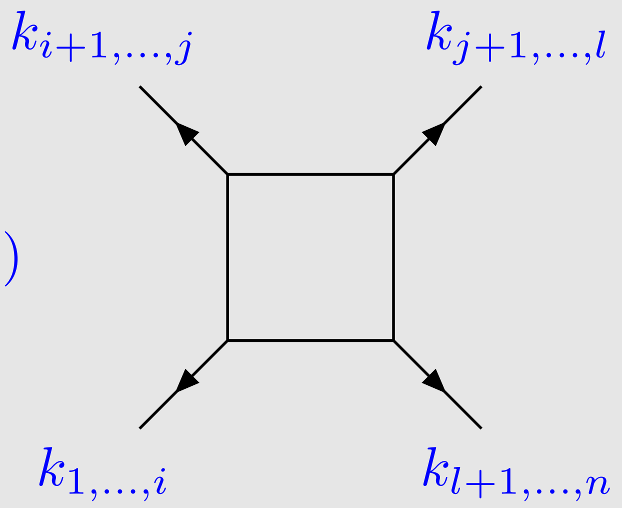
All tree level perturbative amplitides are under control.

One-Loop Amplitudes

In the $\mathcal{N} = 4$ theory, all one-loop integrals which appear in any Feynman diagram calculation can be reduced to a set of scalar box integrals.

In other words, scalar box integrals provide a complete basis for all one-loop gluon amplitudes in $\mathcal{N} = 4$ [Bern, Dixon, Kosower].

$$\mathcal{A}^{1-\mathrm{loop}} = \sum_{\mathrm{boxes}} (\mathrm{coefficient})$$



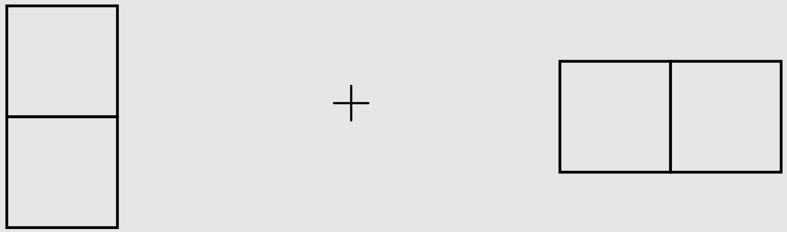
Unitarity methods can be used to determine the coefficients for a desired amplitude [Britto, Cachazo, Feng].

Higher Loops

Unitarity based methods for computing the coefficients can be generalized to higher loop amplitudes [Cachazo, Buchbinder] [Bern, Dixon, Smirnov] [Bern, Carrasco, Johansson, Kosower]

The problem is that the complete basis of integrals is not known even for all two-loop amplitudes!

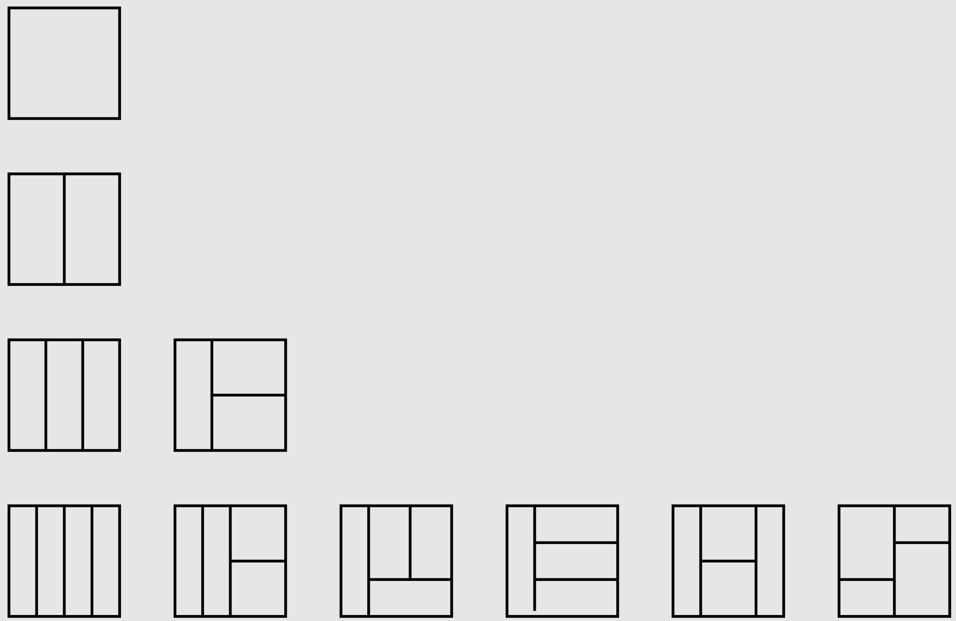
For example, the two-loop four-particle amplitude is given by the sum of only two scalar integrals



But in general it is very difficult to determine which integrals contribute to any particular amplitude.

The Rung Rule

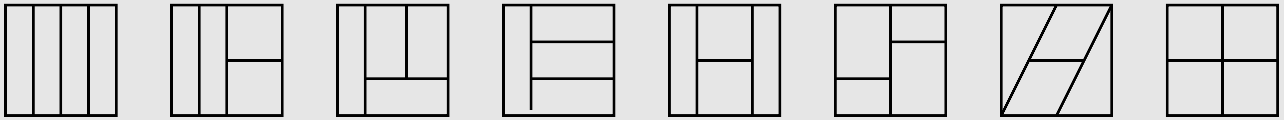
Cuts involving only two particles in an intermediate channel are particularly easy to analyze; and in fact the relevant algebra extrapolates nicely to all loops—this analysis led to what is called the 'rung rule' [Bern, Rozowsky, Yan] which is easiest to explain in pictures.



At two and three loops, this is all there is.

Rung Rule=Wrong Rule

However at four loops the amplitude contains two additional integrals.



[Bern, Czakon, Dixon, Kosower, Smirnov]

The integrand of the five-loop amplitude involves the 21 rung rule integrals and an additional 13 non-rung-rule contributions. [Bern, Carrasco, Johansson, Kosower]

It seems that non-rung rule contributions can be determined by 'dual conformality'–conformal invariance in momentum space. [Drummond, Henn, Smirnov, Sokachev]

Dual Conformal Invariance at One Loop

The one-loop four-particle amplitude contains the integral

$$\mathcal{I}^{(1)}(k_1,k_2,k_3,k_4) = \int d^4p_1 \frac{(k_1+k_2)^2(k_2+k_3)^2}{p_1^2(p_1-k_1)^2(p_1-k_1-k_2)^2(p_1+k_4)^2}.$$

Now, pass to dual coordinates by defining

$$k_1 = x_{12}, \qquad k_2 = x_{23}, \qquad k_3 = x_{34}, \qquad k_4 = x_{41}, \qquad p_1 = x_{15},$$

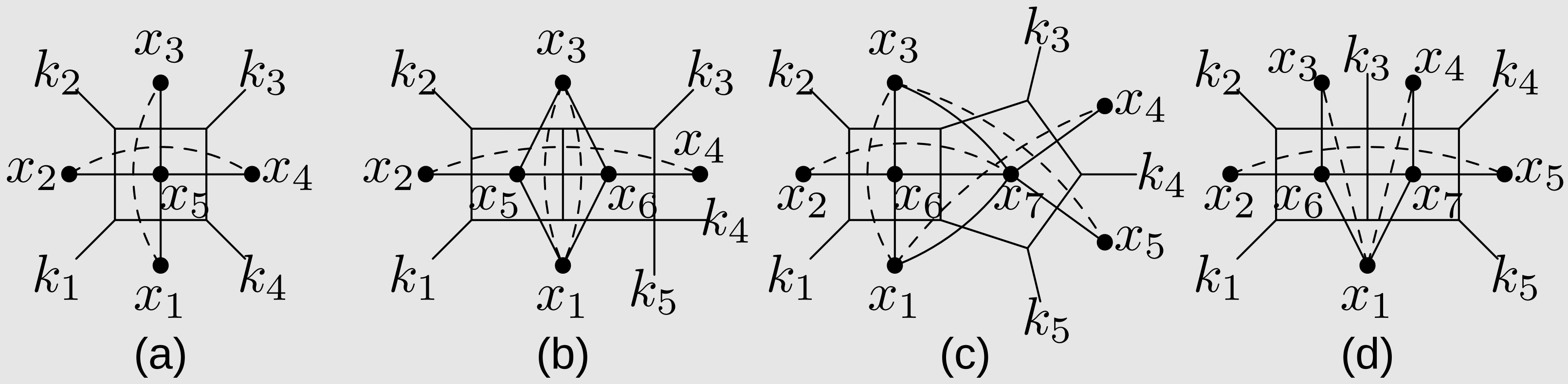
where $x_{ij} = x_i - x_j$.

In these new variables,

$$\mathcal{I}^{(1)}(x_1,x_2,x_3,x_4) = \int d^4x_5 \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}.$$

which is easily seen to be invariant under arbitrary conformal transformations on the x_i**!**

Examples of Dual Conformal Integrals

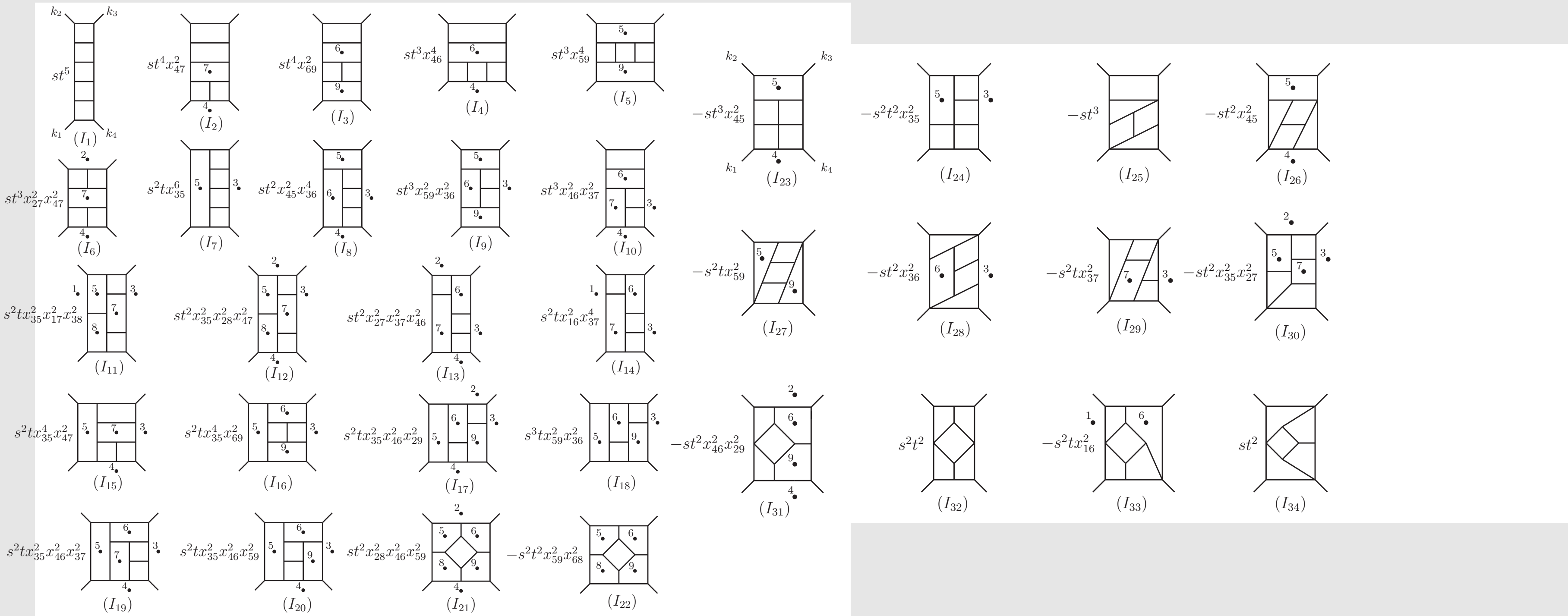


- Points x_i label the vertices of the dual graph, a solid line connecting two points x_i and x_j corresponds to a factor of $1/x_{ij}^2$, while a dashed line corresponds to a factor of x_{ij}^2.

- An integral is dual conformal invariant if the difference between the number of solid lines and dashed lines at a vertex equals 4 at the internal vertices and 0 at the external vertices.

Dual Conformal Invariant Diagrams at Five Loops

[Bern, Carrasco, Johansson, Kosower]



Iteration Relation for Two-loop Amplitudes

- In planar $\mathcal{N}=4$ Yang-Mills, MHV amplitudes $M_n^{(L)}(\epsilon) = \frac{A_n^{(L)}}{A_n^{(0)}}$ computed to date satisfy an iteration relation.

- At two-loops, the iteration conjecture expresses n-point amplitudes entirely in terms of one-loop amplitudes and a set of constants. For two-loop MHV amplitude, Anastasiou, Bern, Dixon and Kosower conjecture reads

$$M_n^{(2)}(\epsilon) = \frac{1}{2}(M_n^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)M_n^{(1)}(2\epsilon) + C^{(L)} + \mathcal{O}(\epsilon),$$ where

$$f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots), \qquad C^{(2)} = -\zeta_2^2/2.$$

This form is based on explicit computations of two-loop amplitudes for four particles [Anastasiou, Bern, Dixon, Kosower]. For five-point amplitude, it has been confirmed by direct calculation [Cachazo, Spradlin, AV] [Bern, Dixon, Kosower, Roiban, Smirnov].

BDS Iteration Relations for Multiloop Amplitudes

- This iterative structure together with the exponential nature of IR divergences suggests suggests an all-orders resummation should be possible.

- Bern, Dixon, Smirnov found three-loop generalization for four-particle amplitude by direct calculation, guiding the all-loop order BDS proposal

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\right)$$

where

$$M_n = \sum_{L=0}^{\infty} a^L M_n^{(L)}(\epsilon),$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)},$$

$$a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}.$$

Scattering Amplitudes and Anomalous Dimension

- The iterative relations imply that vast majority of the rational coefficients which specify the L-loop amplitude are completely determined in terms of lower loop amplitudes.

- The first unfixed number at order $\frac{1}{\epsilon^2}$ is the L-loop planar cusp anomalous dimension $f(\lambda)$, computed perturbatively up to four-loops [Bern, Czakon, Dixon, Kosower, Smirnov] [Cachazo, Spradlin, AV]

$$f(\lambda) = 4\sum_{l=0}^{\infty} a^l f_0^{(l)} = \frac{\lambda}{2\pi^2}\left(1 - \frac{\lambda}{48} + \frac{11\lambda^2}{11520} - \left(\frac{73}{1290240} + \frac{\zeta_3^2}{512\pi^6}\right)\lambda^3 + \cdots\right)$$

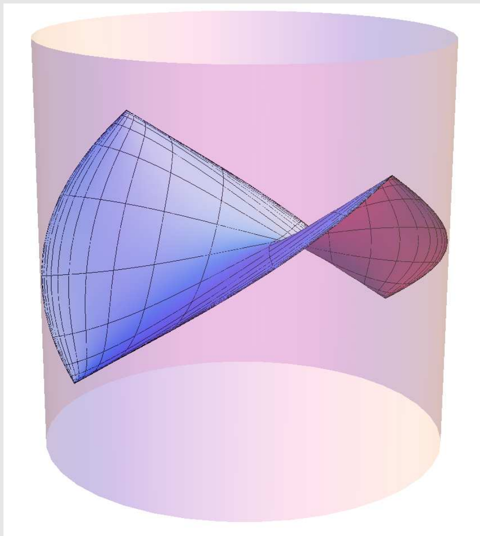
- The second unfixed number at order $\frac{1}{\epsilon}$ is the L-loop planar collinear anomalous dimension $g(\lambda)$, computed perturbatively up to four-loops [Cachazo, Spradlin, AV]

$$g(\lambda) = 2\sum_{l=0}^{\infty} a^l f_1^{(l)l}/l = -\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2 + \frac{2}{3}(6\zeta_5 + 5\zeta_2\zeta_3)\left(\frac{\lambda}{8\pi^2}\right)^3 - 77.56\left(\frac{\lambda}{8\pi^2}\right)^4 \cdots$$

Gluon Scattering at Strong Coupling

Alday and Maldacena have given a prescription for using AdS/CFT to calculate gluon scattering amplitudes at strong coupling.

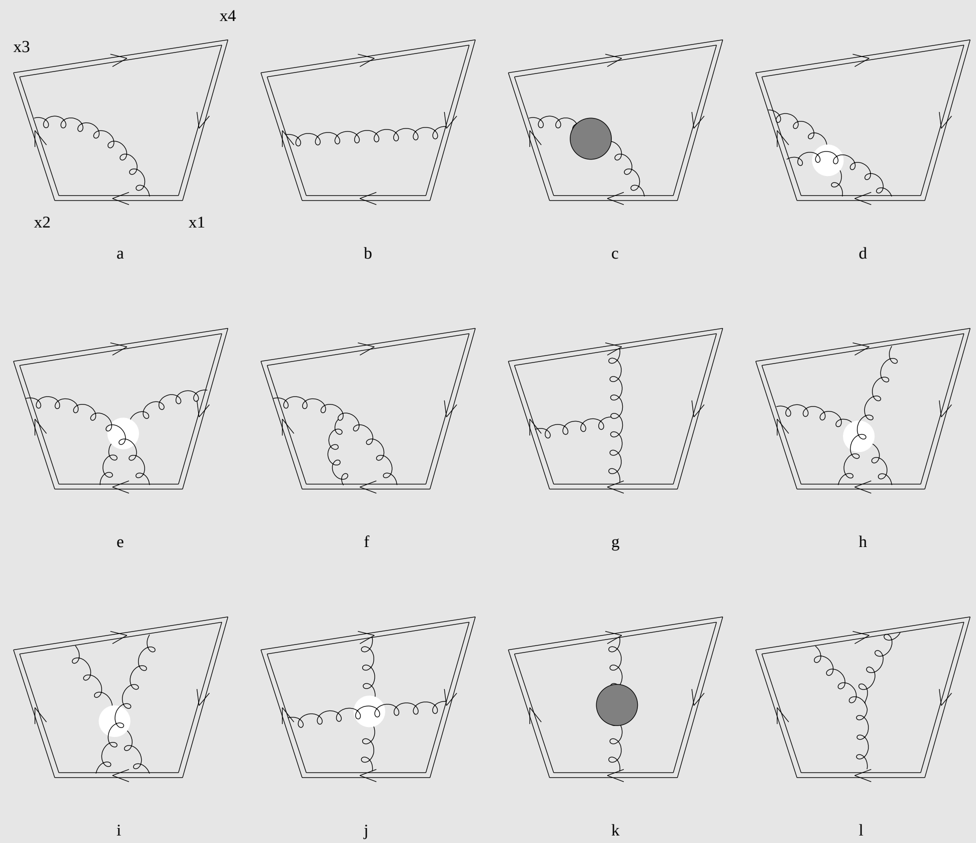
The prescription is computationally equivalent to evaluating a certain Wilson loop composed of null line segments. For four gluons, the relevant classical worldsheet is:



This calculation confirmed the strong coupling prediction from the BDS iteration ansatz for the four-point amplitude.

Scattering Amplitudes and Wilson loops

Drummond, Korchemsky, Sokatchev and Brandhuber, Heslop, Travaglini showed that lowest-order contributions to a light-like rectangular Wilson loop agrees with BDS ansatz for gauge theory amplitudes.



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Relation between MHV amplitudes and Wilson loops is very surprising!

Works for four and five-point amplitudes at one and two loops.

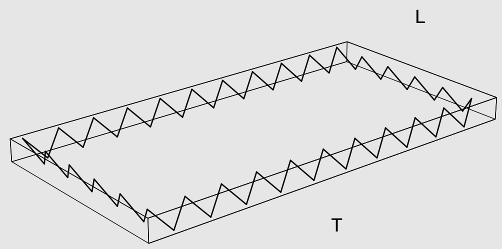
But four and five-point results are fixed by dual conformal symmetry....

What happens with a larger number of legs?

It seems very difficult to find explicit string solution beyond four points corresponding to strong coupling...

Trouble at Large Number of Legs

Alday and Maldacena (at strong coupling) have shown that in the limit of a large number of legs, the Wilson loop calculation does not agree with the BDS ansatz.



In the limit of very large T and L and for $T >> L$, the expectation value of the rectangular Wilson loop

$$\log\langle W^{AM}_{\rm rect}\rangle = \frac{4\sqrt{\lambda}\pi^2}{\Gamma(1/4)^4}\frac{T}{L} \neq \log\langle W^{BDS}_{\rm rect}\rangle = \frac{\sqrt{\lambda}}{4}\frac{T}{L}.$$

Either the connection between Wilson loops and the amplitudes breaks down? Or BDS ansatz breaks down beyond five-point amplitudes? To answer, one needs six-point Wilson loop and amplitude calculations!

Two Loops Six Point Amplitude

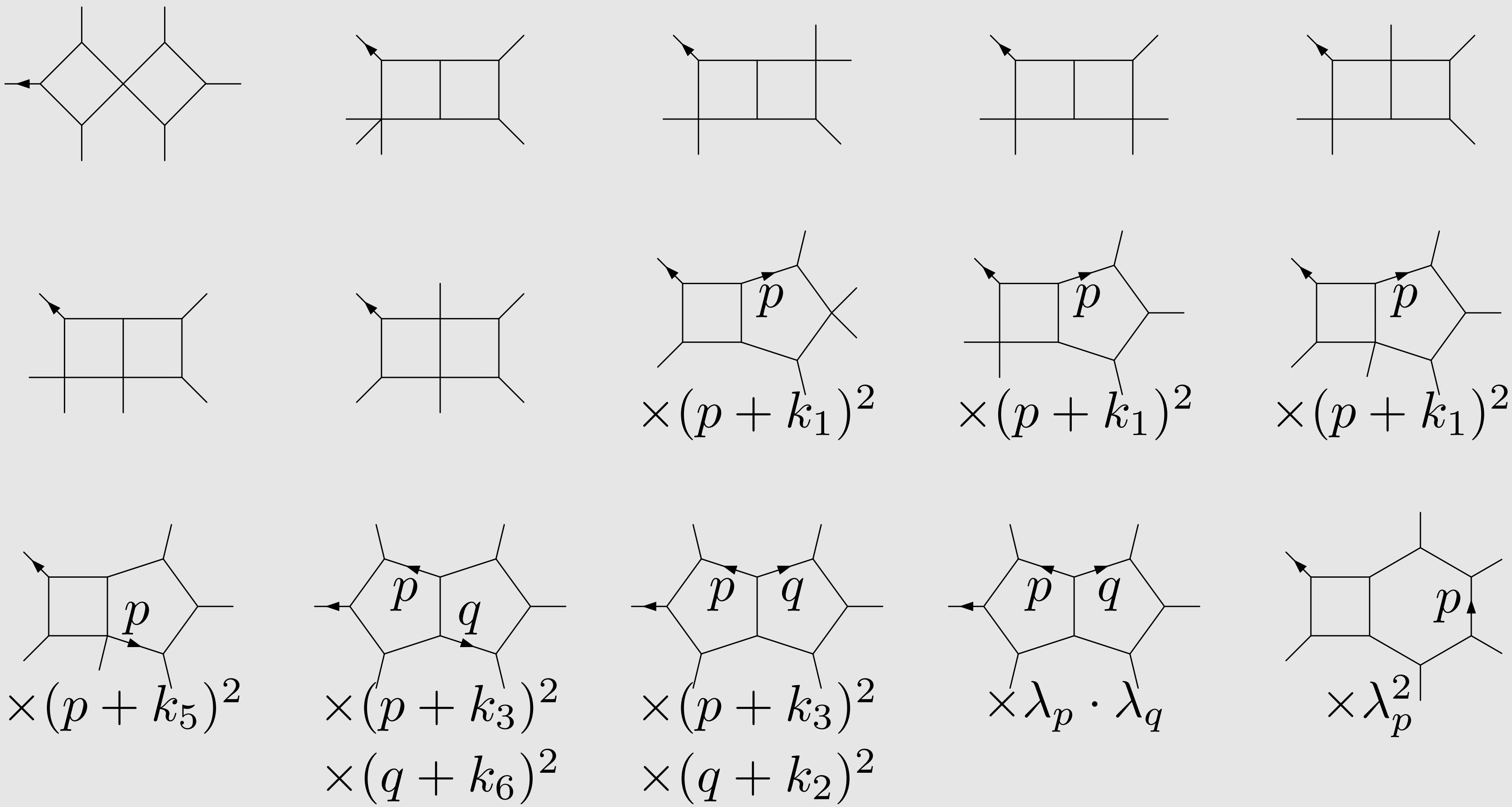
We find the complete expression for the parity-even part of the two-loop six-particle amplitude [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV]

We performed the calculation using the unitarity-based method, employing a variety of cuts to express the amplitude in terms of selected set of six-point two-loop Feynman integrals.

$$M_6^{(2),D=4-2\epsilon}(\epsilon) = \frac{1}{16}\sum_{i=1}^{15} c_i I^{(i)}(\epsilon)$$

We evaluated the integrals using AMBRE and MB packages and computed the amplitude numerically against BDS ansatz, and against values for the corresponding Wilson loop.

Two Loops Six Point Amplitude: integrals



Two Loops Six Point Amplitude: coefficients

$$
\begin{aligned}
c_1 &= s_{61}s_{34}s_{123}s_{345} + s_{12}s_{45}s_{234}s_{345} + s_{345}^2\big(s_{23}s_{56} - s_{123}s_{234}\big)\,,\\
c_2 &= 2s_{12}s_{23}^2\,,\\
c_3 &= s_{234}\big(s_{123}s_{234} - s_{23}s_{56}\big)\,,\\
c_4 &= s_{12}s_{234}^2\,,\\
c_5 &= s_{34}\big(s_{123}s_{234} - 2s_{23}s_{56}\big)\,,\\
c_6 &= -s_{12}s_{23}s_{234}\,,\\
c_7 &= 2s_{123}s_{234}s_{345} - 4s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}\,,\\
c_8 &= 2s_{61}\big(s_{234}s_{345} - s_{61}s_{34}\big)\,,\\
c_9 &= s_{23}s_{34}s_{234}\,,\\
c_{10} &= s_{23}\big(2s_{61}s_{34} - s_{234}s_{345}\big)\,,\\
c_{11} &= s_{12}s_{23}s_{234}\,,\\
c_{12} &= s_{345}\big(s_{234}s_{345} - s_{61}s_{34}\big)\,,\\
c_{13} &= -s_{345}^2s_{56}\,,\\
c_{14} &= -2s_{126}\big(s_{123}s_{234}s_{345} - s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}\big)\,,\\
c_{15} &= 2s_{61}\big(s_{123}s_{234}s_{345} - s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}\big)\,.
\end{aligned}
\tag{1}
$$

An example of an integral

$$I^{(12)} = \frac{(-1)^{1+2\eta}e^{2\epsilon\gamma}}{\Gamma(-1-2\epsilon-\eta)\Gamma(\eta)} \int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} \prod_{j=1}^{18} \frac{dz_j}{2\pi i} \Gamma(-z_j)$$
$$\times \frac{\Gamma(3+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})}{\Gamma(4+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})} \Gamma(1+z_{3,5,9})$$
$$\times(-s_{12})^{z_{8,13}}(-s_{23})^{z_{14}}(-s_{34})^{z_{1,18}}(-s_{45})^{z_{3,15}}(-s_{61})^{z_{11}}$$
$$\times(-s_{123})^{z_{9,16}}(-s_{234})^{z_{17}}(-s_{345})^{z_{2,12}}$$
$$\times(-s_{56})^{-5-2\epsilon-2\eta-z_{1,2,3,8,9,11,12,13,14,15,16,17,18}}$$
$$\times \frac{\Gamma(-3-\epsilon-z_{1,2,3,4,5,6,7})}{\Gamma(-3-3\epsilon-2\eta-z_{1,2,3,8,9,10})}$$
$$\times \frac{\Gamma(5+2\epsilon+2\eta+z_{1,2,3,8,9,10,11,12,13,14,15,16,17,18})}{\Gamma(1-z_4)\Gamma(\eta-z_5)\Gamma(-z_6)\Gamma(1-z_7)}$$
$$\times\Gamma(-5-2\epsilon-2\eta-z_{1,2,3,6,8,9,10,11,12,13,14,15,16})$$
$$\times\Gamma(-1-\epsilon-\eta+z_{4,5,6,7}-z_{11,12,14,15,17,18})$$
$$\times\Gamma(-2-\epsilon-\eta-z_{1,2,3,8,9,10})\Gamma(\eta-z_5+z_{14,15,16})$$
$$\times\Gamma(1-z_4+z_{12,13,18})\Gamma(1+z_{1,2,4,8})$$
$$\times\Gamma(1-z_7+z_{11,12,15})\Gamma(1+z_{1,6,10})\Gamma(1+z_{2,3,7})\Gamma(1+z_{11,14,17}) \quad (2)$$

An example of an integral

$$
\begin{aligned}
I^{(12)} = -\frac{1}{\epsilon^4}\Bigg[& \frac{3 s_{123}}{(-s_{12})^{1+2\epsilon}\, s_{61} s_{34} s_{45}} + \frac{s_{23} s_{56}}{s_{12} s_{61} s_{34} s_{45} (-s_{234})^{1+2\epsilon}} \\
& + \frac{1}{s_{61} s_{34} (-s_{345})^{1+2\epsilon}} \Bigg] + \frac{1}{\epsilon^3}\Bigg[\frac{s_{123}}{s_{12} s_{61} s_{34} s_{45}} \ln\left(\frac{s_{234}^2 s_{345}^6}{s_{23} s_{34}^3 s_{45}^3 s_{56}} \right) \\
& + \frac{s_{23} s_{56}}{s_{12} s_{61} s_{34} s_{45} s_{234}} \ln\left(\frac{s_{23} s_{56} s_{345}^2}{s_{12}^2 s_{34} s_{45}} \right) + \frac{1}{s_{61} s_{34} s_{345}} \ln\left(\frac{s_{45} s_{234} s_{345}}{s_{23} s_{34} s_{56}} \right) \\
& + \frac{1}{s_{61} s_{34} - s_{234} s_{345}}\, \frac{s_{45} s_{234} s_{12} + 2 s_{345} s_{23} s_{56}}{s_{45} s_{234} s_{12} s_{345}} \ln\left(\frac{s_{61} s_{34}}{s_{234} s_{345}} \right) \\
& + \frac{s_{12} s_{45} s_{234} + (s_{23} s_{56} + 3 s_{123} s_{234}) s_{345}}{s_{12} s_{61} s_{34} s_{45} s_{234} s_{345}} \ln\left(\frac{s_{12}}{s_{61}} \right) \Bigg] + O(\epsilon^{-2}) \,,
\end{aligned}
$$

Two Loops Six Point Amplitude: Results

[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV]

- **Discrepancy with BDS ansatz.**

- **But agreement with Wilson loop calculations by Drummond, Henn, Korchemsky and Sokachev!!!**

kinematics	(u_1, u_2, u_3)	Δ_A	Δ_W
$K^{(1)}$	$(1/4, 1/4, 1/4)$	-0.0181 ± 0.017	$< 10^{-5}$
$K^{(2)}$	$(0.547253,\ 0.203822,\ 0.88127)$	-2.753 ± 0.012	-2.7553
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.74445 ± 0.00653	-4.7446
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.1161 ± 0.10	4.0914
$K^{(5)}$	$(4/8, 4/81, 4/81)$	9.9963 ± 0.50	9.7255

Open Questions

How far need one calculate before unlocking all the structure?

How much is gained by adding one more loop, or one more leg?

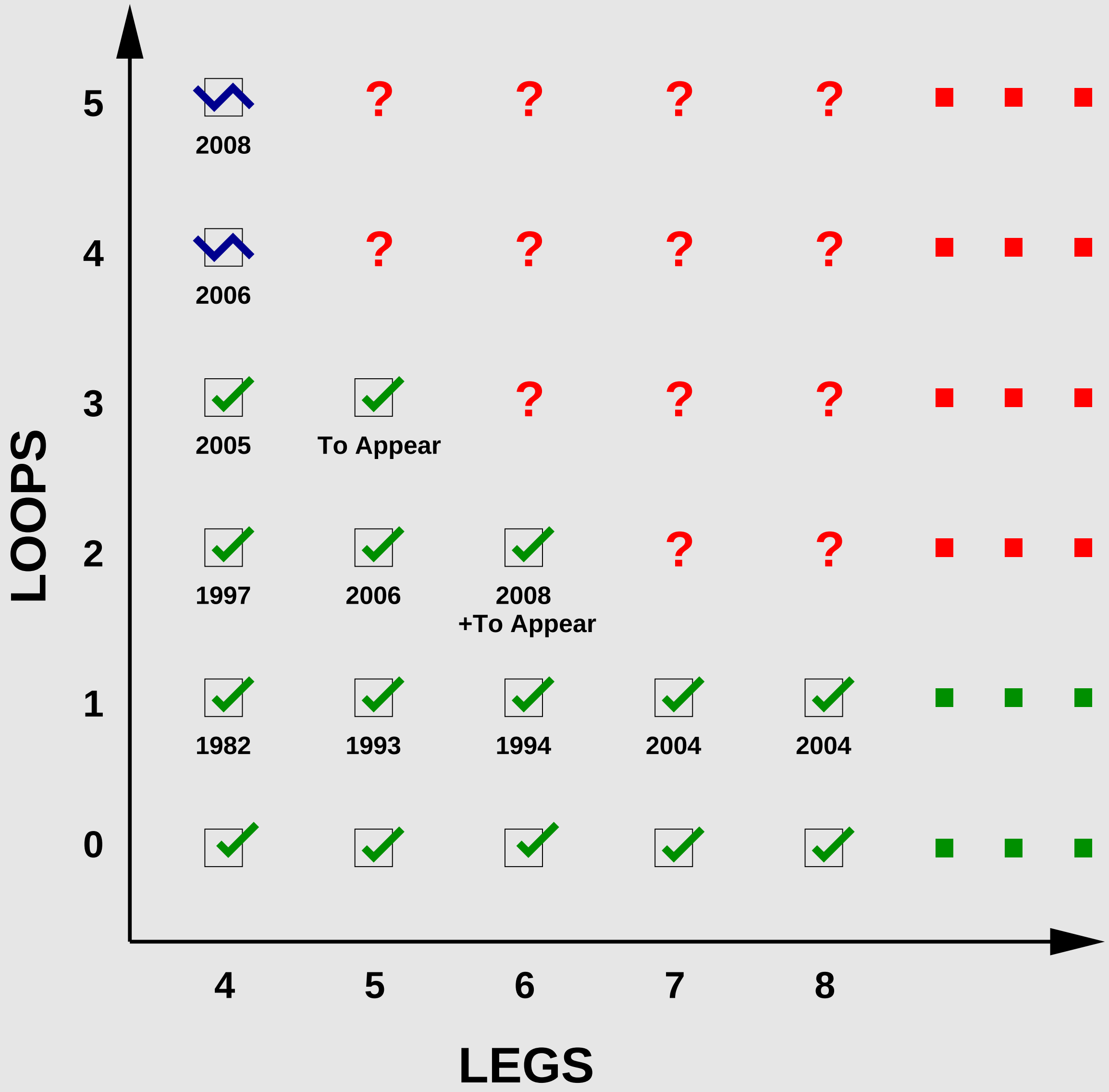
Every new calculation has led to a new surprise!

In the case of loops, there were strong reasons to suspect that special things would start happening at four loops (and they did!) so there was great interest in the calculation of the four-loop cusp anomalous dimension. Five loops: cancellation of $\zeta(6,2)$?

In the case of legs, starting at six-points BDS ansatz breaks down while Wilson loop/amplitude duality holds, suggesting that there should be an additional mechanism besides dual conformal symmetry.

Resummations, non-MHV, non-planar, connection to integrability, other quantities, string theory side, etc.

$\mathcal{N} = 4$ Yang-Mills Status Report



Conclusion

We developed some techniques to aid in direct tests of the conjectured planar $\mathcal{N} = 4$ Yang-Mills S-matrix and multiloop iterative relations.

The motivation behind this research is the desire to explore and uncover the rich mathematical structure underlying $\mathcal{N} = 4$ Yang-Mills theory.

Discovering such structures also has the pleasant side benefit of making previously difficult calculations much simpler.

Prospects are great for continued progress, both in supersymmetric gauge theories as well as QCD. There is definitely a lot more to learn and discover.