Rigorous Definition of Quantum Field Operators and Test Functions Space in Noncommutative Quantum Field Theory

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1 Introduction

Quantum field theory (QFT) as a mathematically consistent theory was formulated in the framework of the axiomatic approach in the works of Wightman, Jost, Bogoliubov, Haag and others.

Noncommutative quantum field theory (NC QFT) being one of the generalizations of standard QFT is intensively developed during the last years. The idea of such a generalization of QFT ascends still to Heisenberg. It was actively developed after Snyder's work.

The present development in this direction is connected with the construction of noncommutative geometry and new physical arguments in favour of such a generalization of QFT. Essential interest in NC QFT is also connected with the fact that in some cases it is obtained as a low-energy limit from the strings theory.

The simplest and at the same time most studied version of noncommutative theory is based on the following Heisenberg-like commutation relations between coordinates:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\theta_{\mu\nu}, \tag{1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix.

NC QFT can be formulated also in commutative space by replacing the usual product of operators by the star (Moyal-type) product:

$$\varphi(x) \star \varphi(x) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}\right)\varphi(x)\varphi(y)|_{x=y}. \tag{2}$$

This product of operators can be extended to the corresponding product of operators in different points:

$$\varphi(x_1)\star\cdots\star\varphi(x_n) = \prod_{a<b\leq n} \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x_a^\mu}\,\frac{\partial}{\partial x_b^\nu}\right)\varphi(x_1)\ldots\varphi(x_n).$$

Let us stress that actually the field operator given at a point cannot be a well-defined operator. Well-defined operators are the smoothed operators:

$$\varphi_f \equiv \int \varphi(x) f(x) d\,x, \tag{3}$$

where $f(x)$ is a test function. In QFT the standard assumption is that $f(x)$ are test functions of tempered distributions.

Wightman approach in NC QFT was formulated in

L. Álvarez-Gaumé and M. A. Vázquez-Mozo, *Nucl. Phys. B*, **668, 293, (2003)**;

M. Chaichian, M. N. Mnatsakanova, K. Nishijima, A. Tureanu and Yu. S. Vernov, *Towards an axiomatic formulation of noncommutative quantum field theory*, **hep-th/0402212**;

Yu.S. Vernov, M.N. Mnatsakanova, *Theor. Math.Phys.*, **142, 337 (2005)**.

Formally the Wightman functions can be written down as follows:

$$W_\star(x_1, x_2, \ldots, x_n) = \langle \Psi_0, \varphi(x_1) \star \cdots \star \varphi(x_n)\Psi_0 \rangle, \qquad (4)$$

where Ψ_0 is a vacuum vector. The formal expression (4) actually means that the scalar product of the vectors $\Phi_k = \varphi_{f_k} \cdots \varphi_{f_1} \Psi_0$ and $\Psi_n = \varphi_{f_{k+1}} \cdots \varphi_{f_n} \Psi_0$ is the following:

$$\langle \Phi_k, \Psi_n \rangle =$$

$$\int W(x_1, \ldots, x_n) \overline{f_1(x_1)} \star \cdots \star \overline{f_k(x_k)} \star f_{k+1}(x_{k+1}) \star \cdots \star f_n(x_n)$$

$$d\,x_1 \ldots d\,x_n, \quad W(x_1, \ldots, x_n) = \langle \Psi_0, \varphi(x_1) \cdots \varphi(x_n)\Psi_0 \rangle. \quad (5)$$

This choice of the product of operators φ_{f_1} and φ_{f_2} reflects the natural physical assumption, that noncommutativity should change the product of operators not only in coinciding points, but also in different ones.

In this report we give a rigorous definition of quantum field operator in NC QFT. For this purpose we have to define the class of test functions, for which the $\star$-product is well defined.

We extend the axiomatic construction of field operators on NC QFT and construct the space on the dense domain of which quantum field operator is well defined. We shall prove that the $\star$-multiplication is well defined for the functions $f_i\,(x_i)$, if

$$f_i\,(x_i) \in S^{\beta}, \qquad \beta < 1/2. \tag{6}$$

S^{β} is a Gel'fand-Shilov space. The case $\beta = 1/2$ is not excluded, but requires the additional assumption.

We show that after the $\star$-multiplication we obtain functions which belong to the space S^{β} with the same β as $f_i\,(x_i)$.

2 Definition of Quantum Field Operators in NC QFT

Let us define rigorously quantum field operator φ_f. To this end we construct a closed and nondegenerate space J such that operators φ_f be well defined on dense domain of J.

The difference of noncommutative case from commutative one is that action of the operator φ_f is defined by the $\star$-product.

Construction of space J we shall begin with introduction of set M of breaking sequences of the following kind

$$g = \{g_0, g_1, \dots g_k\}, \tag{7}$$

where $\qquad g_0 \in \mathbb{C}, \qquad g_1 = g_1^1(x_1), x_1 \in \mathbb{R}^4,$

$$g_i = g_i^1(x_1) \star \dots \star g_i^i(x_i), \quad x_j \in \mathbb{R}^4, 1 \le j \le i;$$

$$k \text{ depends on } g.$$

Let us recall that:

$$\varphi(x) \star \varphi(y) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^{\mu}}\,\frac{\partial}{\partial y^{\nu}}\right)\varphi(x)\varphi(y).$$

Addition and multiplication by complex numbers of the above mentioned sequences are defined component by component, i.e.

$$\{g_0, g_1, \ldots g_k\} + \{h_0, h_1, \ldots h_m\} = \{h_0 + g_0, h_1 + g_1, + \ldots\},$$

$$C\,g = \{C\,g_0, C\,g_1, \ldots C\,g_k\}.$$

The every possible finite sums of the sequences belonging M form space J_0' on which action of the operator φ_f, $f = f\,(x)$, $x \in \mathbb{R}^4$ will be determined.

Certainly, to determine $\star$-product, functions g_k should have sufficient smoothness. We prove, that Moyal product is well defined, if g_k belongs to one of Gel'fand-Shilov spaces S^{β}, $\beta < 1/2$.

Moreover, $f \star g_k \in S^\beta$ with the same β, i.e. after star multiplication the function belongs to the initial space.

The operator φ_f is defined as follows

$$\varphi_f\, g = \{f g_0, f \star g_1, \dots f \star g_k\}, \tag{8}$$

where $f \star g_i = f\,(x) \star g_i^1\,(x_1) \star \dots \star g_i^i\,(x_i)$.
As $f \star (g_i + \tilde{g}_i) = f \star g_i + f \star \tilde{g}_i$, and any vector of space J_0' is the sum of the vectors belonging to set M, the operator φ_f is determined on any vector of space J_0' and $\varphi_f \Phi \in J_0', \forall\, \Phi \in J_0'$.

Scalar product of vectors in J_0' we shall define with the help of Wightman functions $W\,(x_1, \dots, x_n) \equiv \langle \Psi_0, \varphi\,(x_1)\, \dots\, \varphi\,(x_n)\, \Psi_0 \rangle$. We shall consider firstly a chain of vectors: vacuum vector $\Psi_0 = \{1, 0, \dots 0\}, \Phi_1 = \varphi_{f_1} \Psi_0, \dots \Phi_k = \varphi_{f_k} \dots \varphi_{f_1} \Psi_0,\; f_i = f_i\,(x_i),\, x_i \in \mathbb{R}^4$. Evidently, $\Phi_k = \{0, \dots f_k \star \dots \star f_1, 0 \dots 0\}$ and

$$\Psi_n = \varphi_{f_{k+1}} \dots \varphi_{f_n} \Psi_0 = \{0, \dots f_{k+1} \star \dots \star f_n, 0 \dots 0\}.$$

It is obvious, that J_0' is a span of the vectors of such type. Scalar product of vectors Φ_k and Ψ_n is

$$\langle\, \Phi_k, \Psi_n \,\rangle = \langle\, \Psi_0, \varphi_{\bar{f}_1} \,\cdots\, \varphi_{\bar{f}_k} \varphi_{f_{k+1}} \cdots\, \varphi_{f_n} \Psi_0 \,\rangle =$$

$$\int d\,x_1 \ldots d\,x_n W\,(x_1, \ldots, x_n) \cdot$$

$$\overline{f_1\,(x_1)} \star \cdots \star \overline{f_k\,(x_k)} \star f_{k+1}\,(x_{k+1}) \star \cdots \star f_n\,(x_n). \qquad (9)$$

The adjoined operator φ_f^* is defined by the standard formula. If operator φ_f is Hermitian then $\varphi_f^* = \varphi_{\bar{f}}$. Here we consider only Hermitian (real) operators, but the construction can be easily extended to complex fields.

Let us point out that a condition

$$\langle\, \Phi_k, \Psi_n \,\rangle = \overline{\langle\, \Psi_n, \Phi_k \,\rangle} \tag{10}$$

is fulfilled, if (as well as in commutative case),

$$W\,(x_1, \ldots, x_n) = \overline{W\,(x_n, \ldots, x_1)}. \tag{11}$$

The required condition is satisfied, owing to antisymmetry of $\theta^{\,\mu\nu}$.

As any vector of space J_0' is a finite sum of the vectors belonging to the set M, we can directly define scalar product of any vectors of space J_0' .

Let us stress that if the $\star$-product acts only in coinciding points and is substituted by usual one in different points then given construction can also be fulfilled, only in the different points we have to put $\theta^{ij} = 0$. But in this case the function $f(x, y) = f(x)f(y), x \neq y, f(x, x) = f(x) \star f(x)$ is not continuous when $x = y$.

The definition of generalized functions on the space of test functions which are not continuous meets serious difficulties. This point can be considered as an additional argument in favor of use in NC QFT the $\star$-product both in different and coinciding points.

As well as in commutative case, we need to pass from J_0' to nondegenerate and closed space J.

The space J_0' can contain isotropic, i.e. orthogonal to J_0' vectors which, as is known, form subspace. Designating isotropic space as $\tilde{J}_0$ and passing to factor-space $J_0 = J_0'/\tilde{J}_0$, we obtain nondegenerate space, i.e. a space which does not contain isotropic vectors. For closure of space J_0 we assume, as well as in commutative case, that J_0 is normalized space. If the metrics of J_0 is positive, norm $\Phi \equiv \|\Phi\|$ can be defined by the formula $\|\Phi\| = (\langle\Phi, \Phi\rangle)^{1/2}$.
$\bar{J}_0$ (a closure of J_0) is carried out with the help of standard procedure - closure to the introduced norm. This space, in turn, can contain isotropic subspace $\tilde{J}$.

Factor-space $J = \bar{J}_0/\tilde{J}$, obviously, will be nondegenerate space.

Thus, we constructed closed and nondegenerate space J such that operators φ_f are obviously determined on dense domain J_0. Hence, every vector of J can be approximated with arbitrary accuracy by the vectors of the type

$$\varphi_{f_1} \cdots \varphi_{f_n} \Psi_0, \tag{12}$$

where Ψ_0 is a vacuum vector. In other words the vacuum vector Ψ_0 is cyclic, i.e. the axiom of cyclicity of vacuum is fulfilled.

Let us point out that in commutative case construction of space J begins with introduction of sequences g determined by the formula

$$g = \{g_0, g_1, \dots g_k\}, \tag{13}$$

in which, however, $g_i \equiv g_i\,(x_1, \dots x_i)$ are smooth functions of variables $x_j \in \mathrm{I\!R}^4$. We shall note that in the commutative case, starting with J_0', we shall come to the same space J. Really, as space of functions of a type $g_i^1\,(x_1)\,g_i^2\,(x_2)\dots g_i^i\,(x_i)$ is dense in space of functions $g_i\,(x_1, \dots x_i)$, we can complete J_0' up to space of the above mentioned sequences and then carry out the standard construction of space J.

3 Test Functions Space

Our aim is to determine the spaces in which the $\star$-multiplication is well-defined. Evidently the space of tempered distributions cannot be the space compatible with the $\star$-multiplication, as each function of this space contains only a finite number of derivatives. Gel'fand and Shilov proved that if $f(x) \in S^{\beta}$ (see ineq. (14)) then the series of derivatives of infinite order can be well-defined in such a space. Thus we assume that $f(x) \in S^{\beta}$ and prove that the $\star$-product is well-defined only if each f_i belongs to the Gel'fand-Shilov space S^{β}, $\beta < 1/2$. The $\star$-product can be also well-defined if $\beta = 1/2$, but only for functions which satisfy inequality (14) with sufficiently small B.

Let us recall the definition and basic properties of Gel'fand-Shilov spaces S^{β}. In the case of one variable $f(x)$, $x \in \mathbb{R}^1$ belongs to the space S^{β}, if the following condition is satisfied:

$$\left| x^k \frac{\partial^q f(x)}{\partial x^q} \right| \leq C_k B^q q^{q\beta}, \quad -\infty < x < \infty, \quad k, q \in \mathbb{N}, \tag{14}$$

where the constants C_k and B depend on the function $f(x)$. Below we use the inequality (14) only at $k = 0$:

$$\left| \frac{\partial^q f(x)}{\partial x^q} \right| \leq C B^q q^{q\beta}, \quad -\infty < x < \infty, \quad q \in \mathbb{N}. \tag{15}$$

In the case of a function of several variables, the latter inequality (15) holds for any partial derivative:

$$\left| \frac{\partial^{\,q} f\,(x^1, \dots x^k)}{(\partial x^i)^q} \right| \le C\, B^{\,q}\, q^{\,q\beta}, \quad -\infty < x_i < \infty, \quad q \in \mathbb{N}. \tag{16}$$

As our results do not depend on constant C, in what follows we put $C = 1$.

We point out that if the $\star$-product is well-defined for

$$f_i\,(x_i) \,\star\, f_{i+1}\,(x_{i+1}),$$

it is also well-defined for product of arbitrary number of functions.

Let us study

$$f\,(x) \,\star\, f\,(y) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}\right) f\,(x)\, f\,(y). \tag{17}$$

We have to find the conditions under which the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial y^{\nu}} \right)^{n} f\,(x)\, f\,(y) \equiv \sum_{n=0}^{\infty} \frac{D_n}{n!} \tag{18}$$

converges. After simple calculations we come to the inequality:

$$|D_n| < \left(4\,\theta\, B^2\right)^n n^{2\,n\,\beta}. \tag{19}$$

Using this inequality and the fact that, according to the Stirling formula, $\frac{1}{n!} < \left(\frac{e}{n}\right)^n$, we come to the estimate

$$\left| \frac{D_n}{n!} \right| < \tilde{B}^n\, n^{-2\,n\,\gamma}, \tag{20}$$

where $\tilde{B} = 4\,e\,\theta\,B^2,\ \gamma = 1 - 2\,\beta.$

For any $\tilde{B}$ the series

$$\sum_{n=0}^{\infty} \tilde{B}^n \, n^{-2\,n\,\gamma} \tag{21}$$

converges if $\gamma > 0$, i.e. $\beta < 1/2$, and diverges if $\beta > 1/2$. If $\beta = 1/2$ the series converges if $\tilde{B} < 1$.

Thus we come to the conclusion that the series (18) for arbitrary B and C is a convergent one if $\beta < 1/2$ and divergent if $\beta > 1/2$. If $\beta = 1/2$ the series converges at sufficiently small B.

Similarly we can prove that the function $f_\star\,(x, y) \equiv f\,(x) \star f\,(y)$ belongs to the same Gel'fand-Shilov space S^β, $\beta < 1/2$ as $f\,(x)$.

4 Conclusions

We have rigorously constructed field operators in NC QFT and have proven that the space of test functions for the Wightman distribution functions corresponding to the NC QFT, in other words, the space of test functions for which the star-product is well-defined, is one of the Gel'fand-Shilov spaces S^{β} with $\beta < 1/2$.

The existence and determination of the class of test functions spaces is important for any rigorous treatment of the axiomatic approach to NC QFT via NC Wightman functions and the derivation of rigorous results such as CPT and spin-statistics theorems.

The carried out construction of the closed and nondegenerate space, such that operators φ_f are determined on its dense domain, opens a way to derivation of the reconstruction theorem in noncommutative field theory, that we are going to make.