Nonlocal linear models in the Friedmann–Robertson–Walker metric

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based on

I.Ya. Aref'eva, L.V. Joukovskaya, S.V.
JHEP 07 (2007) 087, hep-th/0701184;
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To specify different types of cosmic fluids one uses a phenomenological relation between the pressure p and the energy density ϱ

$$p = w\varrho,$$

where w is the state parameter.

$w > 0$ — **Atoms. (4%)**
$w = 0$ — **the Cold Dark Matter. (23%)**
$w < 0$ — **the Dark Energy. (73%)**

$$w(t) = \; -1 - \frac{2}{3}\frac{\dot{H}}{H^2}, \tag{1}$$

Contemporary experiments, including WMAP, give strong support that

$$w_{DE} \approx -1. \tag{2}$$

We consider the case $w_{DE} < -1$. All natural energy conditions are violated and there are problems of instability at classical and quantum levels. A possible way to evade the instability problem for models with $w_{DE} < -1$ is to yield a phantom model as an effective one, arising from a more fundamental theory.

In particular, if we consider a model with higher derivatives such as

$$\phi e^{-\Box}\phi, \tag{3}$$

then in the simplest approximation:

$$\phi e^{-\Box}\phi \simeq \phi^2 - \phi\Box\phi, \tag{4}$$

such a model gives a kinetic term with a ghost sign. Such a possibility does appear in the string field theory framework:

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1 Nonlocal model

We consider a model of gravity coupling with a nonlocal scalar field, which induced by strings field theory

$$S = \int d^4x \sqrt{-g} \left(\frac{m_p^2}{2} R + \frac{\xi^2}{2} \phi \,\Box_g \phi + \frac{1}{2} \left(\phi^2 - c\,\Phi^2 \right) - \Lambda' \right), \qquad (5)$$

where g is the metric,

$$\Box_g = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu,$$

$\Phi = e^{\Box_g} \phi$.

$m_p^2 = g_4 M_p^2 / M_s^2$, where

M_p is a mass Planck,

M_s is a characteristic string scale,

g_4 is a dimensionless effective coupling constant.

$\Lambda = \frac{M_s^4}{g_4} \Lambda'$ is an effective cosmological constant.

ξ and c are positive constants.

We take the metric in the form

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right) \tag{6}$$

and get that the space homogeneous scalar field ϕ satisfies

$$\xi^2 \mathcal{D}\phi + \phi - c\, e^{2\mathcal{D}}\phi = 0, \tag{7}$$

where

$$\Box_g = \mathcal{D} \equiv -\partial_t^2 - 3H(t)\partial_t, \qquad H = \frac{\dot{a}}{a} \qquad \text{and} \qquad \dot{a} \equiv \partial_t a. \tag{8}$$

The Friedmann equations have the following form

$$\begin{aligned} 3H^2 &= \frac{1}{m_p^2}\,\mathcal{E}, \\ 3H^2 + 2\dot{H} &= -\frac{1}{m_p^2}\,\mathcal{P}, \end{aligned} \tag{9}$$

where the energy and the pressure are obtained from the action (5):

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \qquad T_{\mu\nu} = g_{\mu\nu}\,\mathrm{diag}\{\mathcal{E}, -\mathcal{P}, -\mathcal{P}, -\mathcal{P}\}. \tag{10}$$

The energy density and the pressure have additional nonlocal terms:

$$\mathcal{E} = \mathcal{E}_k + \mathcal{E}_p + \mathcal{E}_{nl2} + \mathcal{E}_{nl1} + \Lambda', \qquad \mathcal{P} = \mathcal{E}_k - \mathcal{E}_p + \mathcal{E}_{nl2} - \mathcal{E}_{nl1} - \Lambda', \tag{11}$$

$$\mathcal{E}_k = \frac{\xi^2}{2}(\partial\phi)^2, \qquad \mathcal{E}_p = -\frac{1}{2}\left(\phi^2 - c\left(e^{\mathcal{D}}\phi\right)^2\right),$$

$$\mathcal{E}_{nl1} = c\int_0^1 \left(e^{(1+\rho)\mathcal{D}}\phi\right)\left(-\mathcal{D}e^{(1-\rho)\mathcal{D}}\phi\right)d\rho, \tag{12}$$

$$\mathcal{E}_{nl2} = -c\int_0^1 \left(\partial e^{(1+\rho)\mathcal{D}}\phi\right)\left(\partial e^{(1-\rho)\mathcal{D}}\phi\right)d\rho.$$

From the system of Friedmann equations

$$3H^2 = \frac{1}{m_p^2}\,\mathcal{E}, \qquad 3H^2 + 2\dot{H} = -\frac{1}{m_p^2}\,\mathcal{P}, \tag{13}$$

we obtain the nonlinear integral equation in $H(t)$ and $\phi(t)$:

$$\dot{H} = -\frac{1}{m_p^2}\left(\frac{\xi^2}{2}(\partial_0\phi)^2 - c\int\limits_0^1 \left(\partial_0 e^{(1+\rho)\mathcal{D}}\phi\right)\left(\partial_0 e^{(1-\rho)\mathcal{D}}\phi\right)d\rho\right). \tag{14}$$

Our goal is to localize system (12) and to find its special solutions.

2 Roots of the Characteristic Equation

Let us consider equation of motion:

$$(\xi^2\Box_g + 1)e^{-2\Box_g}\phi = c\,\phi. \tag{15}$$

Really both the metric $g_{\mu\nu}$ and the scalar field ϕ are unknown. We assume that the metric $g_{\mu\nu}$ is given and consider eq. (15) as an equation in ϕ.

The eigenfunctions of the Beltrami-Laplace operator

$$\Box_g\phi = -\,\alpha^2\phi, \tag{16}$$

also represent the solutions of equation of motion (15) with α, which is defined as a solution of the corresponding characteristic equation

$$-\xi^2\alpha^2 + 1 - c\, e^{-2\alpha^2} = 0. \tag{17}$$

The characteristic equation does not depend on metric!

The solutions of (17) are

$$\alpha_n = \pm \frac{1}{2\xi}\sqrt{4 + 2\xi^2 W_n\left(-\frac{2ce^{-2/\xi^2}}{\xi^2}\right)}, \quad n = 0, \pm 1, \pm 2, \ldots$$

where W_n is the n-s branch of the Lambert function satisfying a relation $W(z)e^{W(z)} = z$.

- If $c < 1$, then (17) has two simple real roots for any ξ.
- If $c = 1$, then (17) has a zero root. Nonzero real roots exist at $\xi^2 < 2$.
- If $c > 1$, then (17) has no real roots for $\xi^2 > \xi^2_{max}$, 2 double roots for $\xi^2 = \xi^2_{max}$, and 4 real simple roots for $\xi^2 < \xi^2_{max}$.

3 Energy Density and Pressure

Let us calculate the energy density and the pressure for

$$\phi = \sum_{n=1}^{N} \phi_n, \tag{18}$$

where ϕ_n is a solution of the following equation:

$$\partial_t^2 \phi_n + 3H(t)\partial_t \phi_n = \alpha_n^2 \phi_n, \tag{19}$$

and α_n are solutions to (17).

Without loss of generality we assume that for n and $k \neq n$ the conditions $\alpha_n^2 \neq \alpha_k^2$ are satisfied.

For $N = 1$ we obtain

$$\mathcal{E} \equiv E(\phi_1) + \Lambda' = \frac{\eta_{\alpha_1}}{2}\Big((\partial_0\phi_1)^2 - \alpha_1^2\phi_1^2\Big) + \Lambda', \qquad (20)$$

$$\mathcal{P} \equiv P(\phi_1) - \Lambda' = \frac{\eta_{\alpha_1}}{2}\Big((\partial_0\phi_1)^2 + \alpha_1^2\phi_1^2\Big) - \Lambda', \qquad (21)$$

where for arbitrary α

$$\eta_\alpha \equiv \xi^2 + 2\xi^2\alpha^2 - 2. \qquad (22)$$

We denote the energy density and pressure of function $\phi(t)$ as the functionals $E(\phi)$ and $P(\phi)$, respectively.

For the solution $\phi(t) = \phi_1(t) + \phi_2(t)$ it is convenient to write the energy density in the following form

$$\mathcal{E} = E(\phi_1 + \phi_2) + \Lambda' = E(\phi_1) + E(\phi_2) + E_{cross}(\phi_1, \phi_2) + \Lambda'.$$

We obtain that

$$E_{cross}(\phi_1, \phi_2) = 0 \qquad \text{and} \qquad P_{cross}(\phi_1, \phi_2) = 0, \tag{23}$$

So,

$$E(\phi_1 + \phi_2) = E(\phi_1) + E(\phi_2), \qquad P(\phi_1 + \phi_2) = P(\phi_1) + P(\phi_2).$$

For the case of N summands we obtain:

$$\mathcal{E} = E\left(\sum_{n=1}^{N} \phi_n\right) + \Lambda' = \sum_{n=1}^{N} E(\phi_n) + \Lambda', \tag{24}$$

$$\mathcal{P} = P\left(\sum_{n=1}^{N} \phi_n\right) - \Lambda' = \sum_{n=1}^{N} P(\phi_n) - \Lambda'. \tag{25}$$

4 Construction of solutions in the Friedmann–Robertson–Walker metric

4.1 Equations of motion and Friedmann equations

In the spatially flat Friedmann–Robertson–Walker Universe we get the following equation of motion for the space homogeneous scalar field ϕ

$$(\xi^2\mathcal{D}+1)e^{-2\mathcal{D}}\phi = c\phi. \tag{26}$$

Let us make an assumption, that $\phi(t)$ and $H(t)$ satisfy the following equation

$$\mathcal{D}\phi = -\alpha^2\phi, \tag{27}$$

where α is a root of eq. (17).

In this case eq. (26) is solved. Using formulas (20) and (21), we rewrite system (9) in the following form:

$$\left\{ \begin{array}{l} 3H^2 = \dfrac{\eta\alpha}{2m_p^2}\left(\dot{\phi}^2 - \alpha^2\phi^2 + \Lambda'\right), \\ \\ \dot{H} = -\dfrac{\eta\alpha}{2m_p^2}\dot{\phi}^2. \end{array} \right. \qquad (28)$$

So, our assumption allows to transform a system with a nonlocal scalar field into a system with a local one.

In the same way we obtain systems with two or more local fields. If

$$\phi(t) = \sum_{n=1}^{N} \phi_n(t), \qquad (29)$$

where $H(t)$ and $\phi_n(t)$ is a solution of (27) with $\alpha = \alpha_n$ and all α_n $(n = 1..N)$ are different roots of (17), then system (9) transforms into the following system with N scalar fields:

$$\left\{ \begin{array}{l} 3H^2 = \dfrac{1}{2m_p^2}\left(\displaystyle\sum_{n=1}^{N} \eta_{\alpha_n}\left(\dot{\phi}_n^2 - \alpha_n^2\phi_n^2\right) + \Lambda'\right), \\ \dot{H} = -\dfrac{1}{2m_p^2}\left(\displaystyle\sum_{n=1}^{N} \eta_{\alpha_n}\dot{\phi}_n^2\right). \end{array} \right. \qquad (30)$$

In the case of two real roots $\alpha_1 > 0$ and $\alpha_2 > \alpha_1$:

$$\left\{ \begin{array}{l} 3H^2 = \dfrac{1}{2m_p^2}\left(\eta_{\alpha_1}\left(\dot{\phi}_1^2 - \alpha_1^2\phi_1^2\right) + \eta_{\alpha_2}\left(\dot{\phi}_2^2 - \alpha_2^2\phi_2^2\right) + \Lambda'\right), \\ \dot{H} = -\dfrac{1}{2m_p^2}\left(\eta_{\alpha_1}\dot{\phi}_1^2 + \eta_{\alpha_2}\dot{\phi}_2^2\right), \end{array} \right.$$

we have obtained that $\eta_{\alpha_1} < 0$ and $\eta_{\alpha_2} > 0$. Therefore the corresponding two-field model is a quintom one.

4.2 Exact Solution in the case $N = 1$

At present time one of the possible scenarios of the Universe evolution considers the Universe to be a D3-brane (3 spatial and one time variable) embedded in higher-dimensional space-time. This D-brane is unstable and does evolve to the stable state. A phantom scalar field is an open string theory tachyon. According to the Sen's conjecture this tachyon describes brane decay, at which a slow transition in a stable vacuum takes place.

We assume that the phantom field $\phi(t)$ smoothly rolls from the unstable perturbative vacuum ($\phi = 0$) to a nonperturbative one $\phi = A_0 \neq 0$ and stops there. In other words we seek a king-like solution.

At $c = 1$ one of solutions of eq. (17) is $\alpha = 0$ and we obtain the following system

$$\begin{cases} 3H^2 = \dfrac{\eta\alpha}{2m_p^2}\left(\dot{\phi}^2 + \Lambda'\right), \\ \dot{H} = -\dfrac{\eta\alpha}{2m_p^2}\dot{\phi}^2. \end{cases} \tag{31}$$

If $\Lambda' > 0$, then there exist the following real solution:

$$H_1(t) = \sqrt{\frac{\Lambda'}{6m_p^2}}\tanh\left(\sqrt{\frac{3\Lambda'}{2m_p^2}}(t-t_0)\right), \tag{32}$$

$$\phi(t) = \pm\sqrt{\frac{2m_p^2}{3(2-\xi^2)}}\arctan\left(\sinh\left(\sqrt{\frac{3\Lambda'}{2m_p^2}}(t-t_0)\right)\right) + C_2,$$

where t_0 is an arbitrary real constant.

The Hubble parameter $H(t)$ is a monotonically increasing function, so $w < -1$.

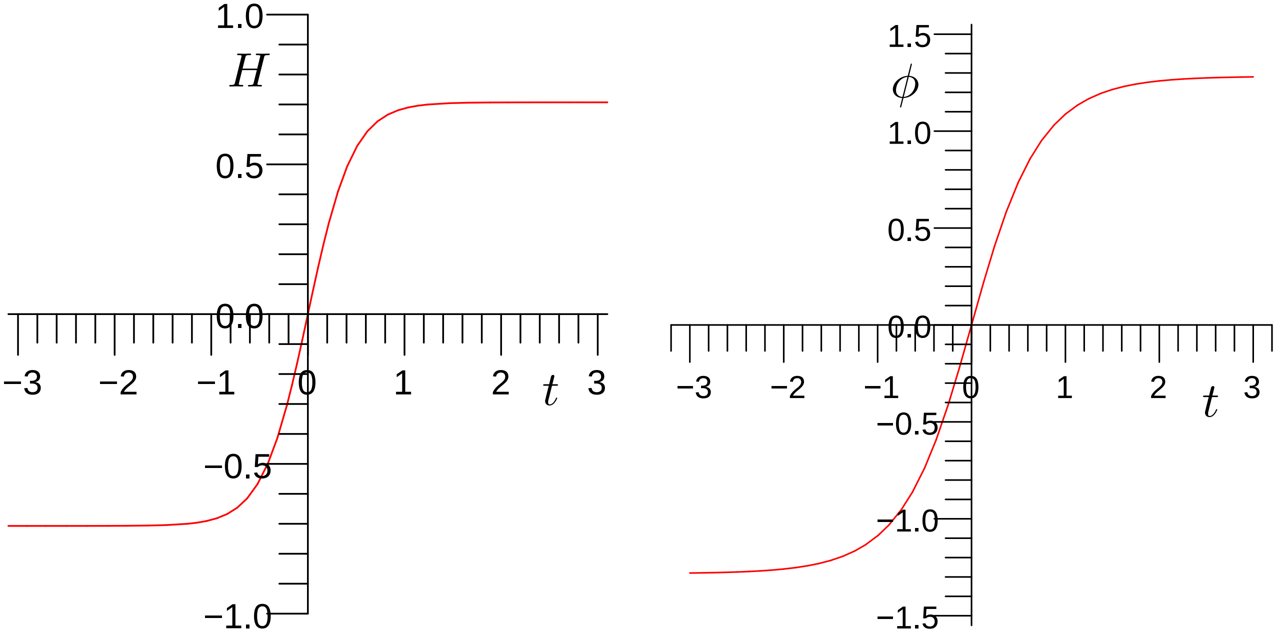

Figure 1: The functions $H_1(t)$ (right) and $\phi_1(t)$ (left) at $\Lambda' = 3$, $m_p^2 = 1$, $\xi^2 = 1$, $t_0 = 0$ and $C_2 = 0$.

Note that we have found two-parameter set of exact solutions at any $\Lambda' > 0$. In other words, we have found the general solution.

It is interesting that type of solution essentially depends on sign of Λ'. In the case $\Lambda' < 0$ we obtain the following general solution:

$$H(t) = -\sqrt{\frac{-\Lambda'}{6m_p^2}}\tan\left(\sqrt{-\frac{3\Lambda'}{2m_p^2}}(t-t_0)\right), \qquad (33)$$

$$\phi(t) = \pm\sqrt{\frac{8(\xi^2-2)}{3m_p^2}}\operatorname{arctanh}\left(\frac{\cos\left(\sqrt{\frac{-3\Lambda'}{2m_p^2}}(t-t_0)\right)-1}{\sin\left(\sqrt{\frac{-3\Lambda'}{2m_p^2}}(t-t_0)\right)}\right)+C_2.$$

5 Conclusions

We have studied the SFT inspired linear nonlocal model. This model is characterized by two positive parameters: ξ^2 and c and have obtained:

1. Roots of the characteristic equation do not depend on the form of the metric.

2. In an arbitrary metric the energy-momentum tensor for an arbitrary N-mode solution is a sum of the energy-momentum tensors for the corresponding one-mode solutions.

3. In the FRW spatially flat metric the pressure for a one-mode solution corresponding to a real root can be positive or negative. Namely, for $c \leq 1$ the one mode pressure is positive and for $c > 1$ it could be negative or positive.

5. Our linear model with one nonlocal scalar field generates an infinite number of local models. Hence, special solutions for nonlocal model in the FRW metric can be obtained.

6. We have constructed an exact kink-like solution, which correspond to monotonically increasing Universe with phantom dark energy.