# Higher Spin Fields in Siegel Space, Currents, Quantization and Theta Functions

**O.A. Gelfond and M.A. Vasiliev**

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Lebedev Institute, Moscow

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# Plan

- Introduction: $4d$ massless fields in ten dimensions
- Unfolded equations and quantization
- Riemann theta functions as solutions of massless field equations
- Conserved currents and fluxes in spinning directions
- Conclusions

# Introduction

$X^{AB} = X^{BA}$ $(A, B, \ldots = 1, \ldots M)$, $\quad d = 4 : M = 4$ Fronsdal 1985

## Equations of Motion

MV 2001

### KG-like

$$\left( \frac{\partial^2}{\partial X^{AB} \partial X^{CD}} - \frac{\partial^2}{\partial X^{AC} \partial X^{BD}} \right) b(X) = 0$$

### Dirac-like

$$\frac{\partial}{\partial X^{AB}} f_C(X) - \frac{\partial}{\partial X^{AC}} f_B(X) = 0$$

# Unfolded Equations

$$\left(\frac{\partial}{\partial X^{AB}} + \mu \frac{\partial^2}{\partial Y^A \partial Y^B}\right) C(Y|X) = 0$$

## Dynamical fields

$$b(X) = C(0|X), \qquad f_A(X) = \frac{\partial}{\partial Y^A} C(Y|X)|_{Y=0} \,.$$

## Auxiliary fields

$$C_{A_1 \ldots A_n}(X) = \frac{\partial^n}{\partial Y^{A_1} \ldots \partial Y^{A_n}} C(Y|X)|_{Y=0} \qquad n > 1.$$

# HS symmetries

$$\delta C(Y|X) = \epsilon \exp j_A h^A \exp\left(j_A Y^A - \mu X^{AB} j_A j_B\right) C(Y^B + h^B - 2\mu X^{BC} j_C|X) \,.$$

# $\mathfrak{sp}(8)$ symmetries

$$P_{AB}C(Y|X) = \frac{\partial^2}{\partial h^A \partial h^B}\epsilon^{-1}\delta C(Y|X)|_{h=j=0} = -\frac{\partial}{\partial X^{AB}}C(Y|X),$$

$$L_A{}^B C(Y|X) = \left(\frac{\partial^2}{\partial h^A \partial j_B} + \frac{M}{2}\delta_A{}^B\right)\epsilon^{-1}\delta C(Y|X)|_{h=j=0} =$$
$$\left(Y^B\frac{\partial}{\partial Y^A} + 2X^{BC}\frac{\partial}{\partial X^{CA}} + \frac{M}{2}\delta_A{}^B\right)C(Y|X),$$

$$K^{AB}C(Y|X) = \frac{\partial^2}{\partial j_A \partial j_B}\epsilon^{-1}\delta C(Y|X)|_{h^A=j_A=0} =$$
$$(Y^B Y^A - 2Y^A X^{BC}\frac{\partial}{\partial Y^C} - 2Y^B X^{AC}\frac{\partial}{\partial Y^C}$$
$$-2X^{AB} - 4X^{BC}X^{AD}\frac{\partial}{\partial X^{CD}})\;C(Y|X)\,.$$

# Quantization

**General solutions of Equations of Motion**

$$b(X)=\frac{1}{\pi^{\frac{M}{2}}}\int d^M\xi\,(b^{+}(\xi)\,\mathrm{exp}\{i\xi_A\xi_B X^{AB}\}+b^{-}(\xi)\,\mathrm{exp}\{-\,i\xi_A\xi_B X^{AB}\})\,,$$

$$f_A(X)=\frac{1}{\pi^{\frac{M}{2}}}\int d^M\xi\,\xi_A(f^{+}(\xi)\,\mathrm{exp}\{i\xi_A\xi_B X^{AB}\}+f^{-}(\xi)\,\mathrm{exp}\{-\,i\xi_A\xi_B X^{A}$$

**For even $M$, $b^{\pm}(\xi)$ is even and $f^{\pm}(\xi)$ is odd**

# Definite Frequency Unfolded Equations

$$\left(\frac{\partial}{\partial X^{AB}} \pm i\,h\frac{\partial^2}{\partial Y^A \partial Y^B}\right) C^{\pm}(Y|X) = 0$$

**distinguish between the positive– and negative–frequencies:**

$$C^{\pm}(Y|X) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi\, c^{\pm}(\xi) \exp \pm i(h\, \xi_A \xi_B X^{AB} + Y^B \xi_B)\,,$$

$$c^{-}(\xi) = \overline{c^{+}(\xi)}\,, \qquad C^{-}(Y|X) = \overline{C^{+}(Y|X)}\,, \qquad c^{\pm}(\xi) = b^{\pm}(\xi) + f^{\pm}(\xi$$

$$[\hat{c}^{\pm}(\xi_1), \hat{c}^{\pm}(\xi_2)] = 0\,, \qquad [\hat{c}^{-}(\xi_1), \hat{c}^{+}(\xi_2)] = \delta(\xi_1 - \xi_2)\,.$$

# Unfolding versus Quantization

# Siegel Space

**Complex coordinates**

$$\mathcal{Z}^{AB} = X^{AB} + i\,\mathbf{X}^{AB} \equiv \Re\mathcal{Z}^{AB} + i\Im\mathcal{Z}^{AB}.$$

**The real part of $\mathcal{Z}^{AB}$ is $X^{AB}$ ,**

**the imaginary part $\mathbf{X}^{AB} = \Im\mathcal{Z}^{AB}$ is positive definite:**

**Upper Siegel half-space $\mathfrak{H}_M$ .**

**The variables $\mathcal{Y}^A = Y^A + i\,\mathbf{Y}^A$ extend Siegel space to Fock-Siegel space.**

$$C^{+}(\mathcal{Y}|\mathcal{Z}) = \int d^M\xi \; c^{+}(\xi) \exp i(h\xi_A\xi_B\mathcal{Z}^{AB} + \xi_A\mathcal{Y}^A)$$

**is holomorphic in** $\mathcal{Z}^{AB}$ and $\mathcal{Y}^A$ **in the upper Fock-Siegel space** $\mathfrak{H}_M \times \mathbb{R}^M$ **for not too bad** $c^{+}(\xi)$.

$$C^{-}(\overline{\mathcal{Y}}|\overline{\mathcal{Z}}) = \int d^M\xi \; c^{-}(\xi) \exp -i(h\xi_A\xi_B\overline{\mathcal{Z}}^{AB} + \xi_A\overline{\mathcal{Y}}^A)\,.$$

**is antiholomorphic in** $\mathcal{Z}^{AB}$ **and** $\mathcal{Y}^A$

$$\overline{C^{+}(\mathcal{Y}|\mathcal{Z})} = C^{-}(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$$

$C^{+}(\mathcal{Y}|\mathcal{Z})$ **and** $C^{-}(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})$ **satisfy the**

**Definite Frequency Unfolded Equations**

$$\left(\frac{\partial}{\partial \mathcal{Z}^{AB}} + i\,h\frac{\partial^2}{\partial \mathcal{Y}^A \partial \mathcal{Y}^B}\right) C^{+}(\mathcal{Y}|\mathcal{Z}) = 0$$

$$\left(\frac{\partial}{\partial \overline{\mathcal{Z}}^{AB}} - i\,h\frac{\partial^2}{\partial \overline{\mathcal{Y}}^A \partial \overline{\mathcal{Y}}^B}\right) C^{-}(\overline{\mathcal{Y}}|\overline{\mathcal{Z}}) = 0\,,$$

**that uplift the massless field equations for (negative)positive frequencies to the full Fock-Siegel space. The (anti)holomor properties reconstruct $C^{\pm}$ in terms of their boundary values $C^{\pm}(Y|X)$ at $\mathcal{M}_M \times \mathbb{R}^M$.**

# Periodic solutions of HS equations

**A positive-frequency solution of the Unfolded Equations periodic under**

$$\mathcal{Y}^A \to \mathcal{Y}^A + n^A\,, \qquad n^A \in \mathbb{Z}^M$$

**has the form**

$$C^+(\mathcal{Y}|\mathcal{Z}) = \sum_{n^A \in \mathbb{Z}^M} c_n^+ \exp i(h\mathcal{Z}^{AB}(2\pi n_A)(2\pi n_B) + 2\pi n_C \mathcal{Y}^C).$$

# Theta functions as solutions of HS equations

**For** $c_n^+ = 1\,,\; h = \frac{1}{4}\pi^{-1}$

$$C^+(\mathcal{Y}|\mathcal{Z}) = \theta(\mathcal{Y},\,\mathcal{Z}) = \sum_{n^A\in\mathbb{Z}^M} \exp i\pi(\mathcal{Z}^{AB}n_A n_B + 2n_A\mathcal{Y}^A)\,.$$

**Complexified space-time coordinates $\mathcal{Z}^{AB}$ : theta function period matrix**

$$\theta(\mathcal{Y}+m\mathcal{Z},\,\mathcal{Z}) = \exp\left(-i\pi\mathcal{Z}^{AB}m_A m_B - 2i\pi m_A\mathcal{Y}^A\right)\theta(\mathcal{Y},\,\mathcal{Z})\,, \qquad m_A$$

**HS theory and the theory of theta functions are based on the $Sp(2M)$ symmetry and its Weyl-Heisenberg extension which is the HS symmetry**

## Theta functions with characteristics

$$\theta[{}^{a}_{b}](\mathcal{Y},\,\mathcal{Z})=\exp\left(i\pi\mathcal{Z}^{AB}a_Aa_B+2i\pi a_A\mathcal{Y}^A+2i\pi a_Ab^A\right)\quad\theta(\mathcal{Y}+\mathcal{Z}a+b,\mathcal{Z})$$

$$(\,b^A\,,a_A\in\mathbb{R}^M)$$

that also solve Unfolded Equations, result from the action of the HS symmetry

$$\delta\theta(\mathcal{Y}|\mathcal{Z})=\theta(\mathcal{Y}^B+k^B-2\mu\mathcal{Z}^{BC}j_C|\mathcal{Z})\;\exp\left(j_Ah^A+j_A\mathcal{Y}^A-\mu\mathcal{Z}^{AB}j_Aj_B\right)$$

with $j_A=2i\pi a_A$, $k^B=b^B$ and $\mu=\frac{i}{4\pi}$ on the theta functions.

## Theta functions: most symmetric solutions of HS field equations

$\Gamma_{1,2}\in Sp(2M|\mathbb{Z})$ is a leftover symmetry

Reduction of the theta-function solution to Minkowski space gives solutions of $4d$ massless field equations

# Higher spin currents in Minkowski space

The infinite set of conformal HS symmetries suggests the existence of the corresponding conserved HS currents.

**Closed three–form** Gelfond, Skvortsov, MV (2006)

$$\Omega_3(\eta, C^k, C^l) = dx_{\alpha\alpha'} \wedge dx^{\alpha\gamma'} \wedge dx^{\gamma\alpha'} w_\gamma w_{\gamma'} \eta(w,u) C^k(Y|x) C^l(iY|x)|_Y$$

$$w_\alpha = \frac{\partial}{\partial Y^\alpha}, \qquad w_{\alpha'} = \frac{\partial}{\partial Y^{\alpha'}}, \qquad u^\alpha = x^{\alpha\alpha'} \frac{\partial}{\partial Y^{\alpha'}}, \qquad u^{\alpha'} = x^{\alpha\alpha'} \frac{\partial}{\partial Y^\alpha}.$$

**provided that** $C^k(Y|x)$ **satisfy** $4d$ **Unfolded Equations**

$$\frac{\partial}{\partial x^{\alpha\beta'}} C^k(Y|x) + \frac{\partial^2}{\partial Y^\alpha \partial Y^{\beta'}} C^k(Y|x) = 0$$

# Bilinear currents in $\mathcal{M}_M$

$2M-$form

$$\varpi^{2M}(g) = \Big(d\,\mathcal{W}_A \wedge (\,\mathcal{W}_B d\,\mathcal{Z}^{AB} - d\,\mathcal{Y}^A)\Big)^M \; g(\mathcal{W},\,\mathcal{Y}\,|\,\mathcal{Z})$$

is closed provided that $g(\mathcal{W},\mathcal{Y}|\mathcal{Z})$ is holomorphic and satisfies the

Current Equations:

$$\Big(\frac{\partial}{\partial\mathcal{Z}^{AB}} + \;\mathcal{W}_{(A}\frac{\partial}{\partial\mathcal{Y}^{\,B)}}\Big)\,g(\mathcal{W},\,\mathcal{Y}\,|\,\mathcal{Z}) = 0\,.$$

Regular solutions of the current equations form a commutative algebra $R$

$$\eta(\mathcal{W},\,\mathcal{Y}\,|\,\mathcal{Z}) = \varepsilon(\mathcal{W}_A,\,\mathcal{Y}^C \,-\, \mathcal{Z}^{CB}\,\mathcal{W}_B)$$

with arbitrary regular $\varepsilon(\mathcal{W},\mathcal{Y})$.

Singular solutions $\mathcal{S}$ form a $R$-module, *i.e.,* although it may not be possible to multiply singular solutions with themselves, they can be multiplied by regular ones.

The closed form $\varpi$ gives rise to nontrivial conserved currents for

$$g(\mathcal{W},\,\mathcal{Y}\,|\mathcal{Z}) = \eta(\mathcal{W},\,\mathcal{Y}\,|\mathcal{Z}) f(\mathcal{W},\,\mathcal{Y}\,|\mathcal{Z})\,,$$

where $\eta(\mathcal{W},\,\mathcal{Y}\,|\mathcal{Z}) \in R$ and

$$f(\mathcal{W},\,\mathcal{Y}\,,|\mathcal{Z}) \;=\; (2\pi)^{-M/2} \int\limits_{\Sigma^M(\mathcal{U})} d^M\mathcal{U}\, \exp\left(-i\mathcal{W}_C\mathcal{U}^C\right) T(\mathcal{U},\,\mathcal{Y}\,|\mathcal{Z})\,,$$

The Current Equations for $f(\mathcal{W},\,\mathcal{Y}\,|\mathcal{Z})$ translate to the following rank 2 Unfolded Equations for the generalized stress tensor

$$\left\{\frac{\partial}{\partial\mathcal{Z}^{AB}} - ih\,\frac{\partial}{\partial\mathcal{Y}^{(A}}\frac{\partial}{\partial\mathcal{U}^{B)}}\right\} T(\mathcal{U},\,\mathcal{Y}|\mathcal{Z}) = 0.$$

This is solved by

$$T(\mathcal{U},\mathcal{Y}|\mathcal{Z}) = \mathcal{C}^{+}(\mathcal{U}-\mathcal{Y}|\mathcal{Z})\mathcal{C}^{-}(\mathcal{U}+\mathcal{Y}|\mathcal{Z})$$

provided that $\mathcal{C}^{+}(\mathcal{U}|\mathcal{Z})$ and $\mathcal{C}^{-}(\mathcal{U}|\mathcal{Z})$ satisfy the equations

$$\left(\frac{\partial}{\partial\mathcal{Z}^{AB}} \pm i\,h\frac{\partial^2}{\partial\mathcal{U}^A\partial\mathcal{U}^B}\right)\mathcal{C}^{\pm}(\mathcal{U}|\mathcal{Z}) = 0\,.$$

## From $\mathcal{M}_4$ to Minkowski space

### In terms of two-component complex spinors

$$\mathcal{Z}^{AB} = (\mathcal{Z}^{\alpha\beta}, \mathcal{Z}^{\alpha\alpha'}, \mathcal{Z}^{\alpha'\beta'}) = (X^{\alpha\beta} + i\mathbf{X}^{\alpha\beta}, X^{\alpha\alpha'} + i\mathbf{X}^{\alpha\alpha'}, X^{\alpha'\beta'} + i\mathbf{X}^{\alpha'\beta'}),$$

$$\mathcal{Y}^A = (\mathcal{Y}^\alpha, \mathcal{Y}^{\alpha'}), \;\; \mathcal{W}^A = (\mathcal{W}^\alpha, \mathcal{W}^{\alpha'}), \;\; \mathcal{U}^A = (\mathcal{U}^\alpha, \mathcal{U}^{\alpha'}).$$

$$\overline{X^{\alpha\beta}} = X^{\alpha'\beta'}, \;\; \overline{X^{\alpha\beta'}} = X^{\beta\alpha'}, \qquad \overline{\mathbf{X}^{\alpha\beta}} = \mathbf{X}^{\alpha'\beta'}, \;\; \overline{\mathbf{X}^{\alpha\beta'}} = \mathbf{X}^{\beta\alpha'}.$$

**Let the integration cycle $\mathbf{\Sigma}^8$ be**

$$\mathbf{\Sigma}^8 = \sigma^3(\mathcal{Z}^{\alpha\beta'}) \times \sigma^1(\mathcal{Z}^{\alpha\beta}, \mathcal{Z}^{\alpha'\beta'}, \mathcal{Y}) \times R^4(\mathcal{W}),$$

**where $\sigma^3(\mathcal{Z}^{\alpha\beta'})$ is a three-dimensional surface in the complexified Minkows space and $\sigma^1(\mathcal{Z}^{\alpha\beta}, \mathcal{Z}^{\alpha'\beta'}, \mathcal{Y})$ is a one-dimensional cycle in the complexified spinning space.**

**To reduce integration over $\sigma^3(\mathcal{Z}^{\alpha\beta'}) \times \sigma^1$ to non-zero integration over a space surface $\sigma^3(\mathcal{Z}^{\alpha\beta'})$ in Minkowski space the current should have a singularity inside $\sigma^1$.**

**An elementary calculation then shows**

$$Q = \int_{\Sigma^8} d^4\mathcal{W} \wedge d\mathcal{Z}_{\alpha\gamma'} \wedge d\mathcal{Z}^{\alpha\beta'} \wedge d\mathcal{Z}^{\beta\gamma'} \wedge d\Lambda(\mathcal{W},\mathcal{Y}\,|\mathcal{Z})\, g(\mathcal{W},\mathcal{Y}\,|\mathcal{Z})\;\; \mathcal{W}_\beta\, \mathcal{W}_{\beta'}\,,$$

**where**

$$\Lambda(\mathcal{W},\,\mathcal{Y}\,|\mathcal{Z}) = \Big(\mathcal{W}_\mu \mathcal{W}_\nu\, \mathcal{Z}^{\mu\nu} - \mathcal{W}_{\mu'}\mathcal{W}_{\nu'}\, \mathcal{Z}^{\mu'\nu'} + \mathcal{W}_\mu\, \mathcal{Y}^\mu - \mathcal{W}_{\mu'}\, \mathcal{Y}^{\mu'}\Big)$$

**That** $\Lambda(\mathcal{W},\mathcal{Y}\,|\mathcal{Z})$ **solves the current equation and is independent of** $\mathcal{Z}^{\alpha\beta'}$. **allows to introduce a singularity free of the Minkowski coordinates** $\mathcal{Z}^{\alpha\beta'}$.

$$g(\mathcal{W},\mathcal{Y}|\mathcal{Z}) = \Lambda^{-1}(\mathcal{W},\mathcal{Y}|\mathcal{Z})\,\eta(\mathcal{W},\mathcal{Y}|\mathcal{Z})\,f(\mathcal{W},\mathcal{Y}|\mathcal{Z}),$$

**where** $\eta(\mathcal{W},\,\mathcal{Y}\,|\mathcal{Z})$ **is a polynomial parameter, while** $f(\mathcal{W},\,\mathcal{Y}\,|\mathcal{Z})$ **is Fourier transform of the stress tensor** $T_{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{U},\,\mathcal{Y}\,|\mathcal{Z})$.

The idea is to choose a cycle $\sigma^1$ so that

$$Q = \int_{\Sigma^8} d^4\mathcal{W} \wedge d\mathcal{Z}_{\alpha\gamma'} \wedge d\mathcal{Z}^{\alpha\beta'} \wedge d\mathcal{Z}^{\beta\gamma'} \wedge \frac{d\Lambda}{\Lambda} \mathcal{W}_\beta \mathcal{W}_{\beta'}\, \eta\; f$$

gives the residue in $\Lambda(\mathcal{W},\mathcal{Y}|\mathcal{Z})$ leading to

$$Q \sim \int_{\mathbb{R}^4\times\sigma^3} d^4\mathcal{W} \wedge d\mathcal{Z}_{\alpha\gamma'} \wedge d\mathcal{Z}^{\alpha\beta'} \wedge d\mathcal{Z}^{\beta\gamma'} \mathcal{W}_\beta \mathcal{W}_{\beta'}\; \eta\; f\,|_{\Lambda=0}.$$

This gives the conserved charge in Minkowski space

$$\int_{\sigma^3} d^3X^{\beta\beta'} \frac{\partial^2}{\partial U^\beta \partial U^{\beta'}} \eta\left(\frac{\partial}{\partial U^C}, -X^{AB}\frac{\partial}{\partial U^B}\right) T(U,0|X)|_{U=0}$$

The doubling of variables $\mathcal{W},\mathcal{Y}$ allows us to introduce singularities (fluxes) in the complexified spinning variables $\mathcal{W},\mathcal{Y},\mathcal{Z}^{\alpha\beta},\mathcal{Z}^{\alpha'\beta'}$ needed to reproduce HS currents in Minkowski space-time .

# Conclusions

**Unfolded dynamics and quantum mechanics**

**Unfolded HS dynamics and theta functions**

**HS currents in Minkowski space are supported by fluxes in the spinning space**

**Application: $\mathcal{D}$–functions and integral evolution formula**