Cancellations of ultraviolet divergences in supergravity

Pierre Vanhove

IPhT - CEA/Saclay

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based on

- arXiv:0802.0868 and arXiv:0805.3682 with N.E.J.Bjerrum-Bohr
- hep-th/0611273 and hep-th/0610299 with M.B. Green, J.G.Russo

Outline

1. UV behaviour of gravity amplitudes
2. One-loop gravity amplitudes in $\mathcal{N} = 8$ supergravity
3. Conclusion & Outlook

UV behaviour of gravity amplitudes

Gravity describes the interactions of a massless spin 2 particle with a dimensionfull coupling constant

$$[\kappa^2_{(D)}] = (\mathit{length})^{D-2}$$

A L-loop 4-point *pure* gravity amplitude in dimensions D has the mass

$$[\mathfrak{M}^{(4)}_L] = \mathrm{mass}^{(D-2)L+2}$$

UV behaviour of pure gravity amplitudes

- At one-loop $L = 1$ the amplitude is diverging with for counter-term

$$\mathfrak{M}_1^{(4)} \sim \frac{1}{\epsilon}\,[\alpha R^2_{mnpq} + \beta R^2_{mn} + \gamma R^2], \qquad D = 4 - 2\epsilon$$

  - In 4d $R^2_{mnpq} \sim 4R^2_{mn} - R^2$, and for *pure* gravity $R_{mn} = 0 = R$ so the divergence is zero *on-shell* 't Hooft/Veltman

- At two-loop $L = 2$ Sagnotti/Goroff; van de Ven

$$\mathfrak{M}_2^{(4)} \sim \frac{1}{\epsilon}\left(\kappa^2_{(4)} R_{mnpq}\right)^3$$

- At L-loop order a new counter-term arises

$$\mathfrak{M}_L^{(4)} \sim \frac{1}{\epsilon^{L-1}}\left(\kappa^2_{(4)}\, R_{mnpq}\right)^{L+1}$$

The theory is not renormalisable

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UV behaviour of supergravity amplitudes

$\mathcal{N} = 8$ supergravity

- The amplitude factorizes a R^4 and more powers of derivatives

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L-6-2\beta_L}\, D^{2\beta_L} R^4$$

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$$D \geq D_c = 2 + \frac{6 + 2\beta_L}{L}$$

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- Various arguments from string duality and explicit computations indicate that Green,Russo,Vanhove

$$\beta_L = L \text{ for } L \geq 2$$

UV behaviour of multiloop amplitudes

When $\beta_L = L$ the amplitude behaves has Green,Russo,Vanhove

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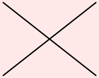
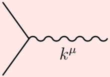
$$[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-4)L-6}\, D^{2L} R^4$$

- Non-renormalisation theorems:
  $D^{2g} R^4$ are not renormalized after genus-g in string theory
  - confirmed by explicit computation to $g \leq 5$ Berkovits
- The UV dependence is *the same as for $\mathcal{N} = 4$ SYM amplitudes*
- If true for all L then the theory is perturbatively finite in 4d

$$D \geq D_c = 4 + \frac{6}{L}$$

Structure of gravity amplitudes

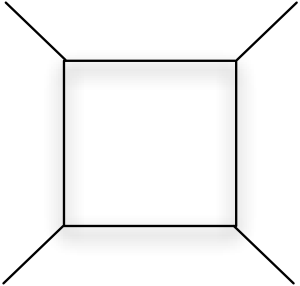
$$[\mathfrak{M}_L^{(4)}] = \text{mass}^{(D-4)L-6}\, D^{2L} R^4 \text{ indicates "effective" interactions}$$

 : φ^4 or 

: φ^2coupled to electromagnetism

$\mathcal{N} = 8$ Supergravity amplitudes are expected to be expandable on the *same* basis of integral functions as $\mathcal{N} = 4$ SYM

UV divergences at higher-loop

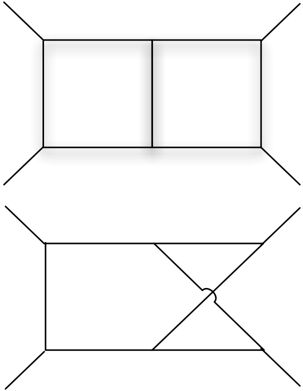


One loop amplitude is given by φ^3 scalar box amplitude with $\beta_1 = 0$

$$\mathfrak{M}_{4;1}^{(D)} = R^4\,[I_{\rm box}(s,t) + I_{\rm box}(s,u) + I_{\rm box}(t,u)]$$

Green,Schwarz,Brink

UV divergences at higher-loop

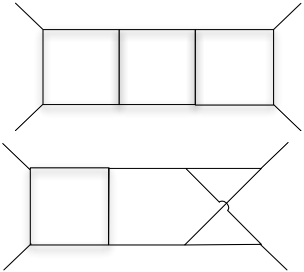


Two-loop amplitude is given by the φ^3 scalar planar and non-planar doublebox amplitude with $\beta_2 = 2$

$$\begin{aligned} \mathfrak{M}_{4;1}^{(D)} &= D^4 R^4 \Big[I^{P}_{\text{doublebox}}(s) + I^{NP}_{\text{doublebox}}(s) \\ &+ (t-, u - \text{channels}) \Big] \end{aligned}$$

Bern et al.; D'Hoker, Phong
Berkovits; Berkovits, Mafra

UV divergences at higher-loop



higher-loop φ^3 scalar *ladder* contributions behaves as $\beta_L = 2(L-1)$

$$\mathfrak{M}^{(D)}_{4;1} = D^{4L-2} R^4 \Big[I^{P}_{\mathrm{ladder}}(s) + I^{NP}_{\mathrm{ladder}}(s) + (t-, u-\mathrm{channels}) \Big]$$

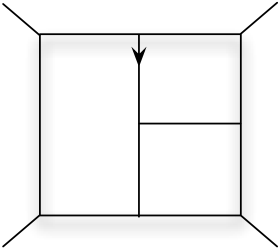
which is too much converging for being the leading $\mathcal{N}=8$ UV behaviour

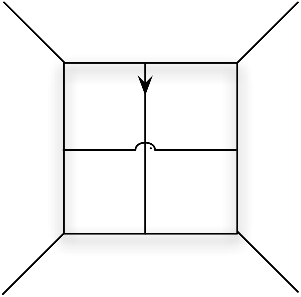
Berkovits

Bern, Dixon, Roiban

Bern, Carrasco, Dixon, Johansson, Kosower, Roiban

UV divergences at higher-loop





higher-loop have non-ladder topologies which *individually* do *not* behaves as $\beta_L = L$

$$= D^4 R^4 \int d^D\ell_1 d^D\ell_2 d^D\ell_3 \, \frac{(\ell_1 \cdot k_1)(\ell_2 \cdot k_2)}{(\ell_1 - k_1)^2 \cdots}$$

only the sum of all these contributions satisfy the rule $\beta_L = L$

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On-shell Supersymmetry and counter-terms

(On-shell) Extended Supersymmetry improves UV behaviour

Grisaru, Deser, Stelle, Howe, Lindström, Kallosh, ...

- $\mathcal{N} \geq 1$ susy forbids R^3 counterterms: supergravity is finite at $L = 2$
- for $\mathcal{N} \leq 4$ susy a *first possible* counter-terms at $L = 3$ loop order

$$S^{(3)}_{\mathcal{N}\leq 4} = \int d^4x \int d^{4\mathcal{N}}\theta\, E\,(\varphi + \cdots + \theta^N R)^4 = \int d^4x \sqrt{g}\,(\kappa^2_{(4)} R)^4$$

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On-shell Supersymmetry and counter-terms

$\mathcal{N} = 8$ sugra is invariant under local $SU(8)_R$ and global E_7

spin/dimension	2	3/2	1	1/2	0
superfield	$W_{\alpha\beta\gamma\delta}$	$W^i_{\alpha\beta\gamma}$	$W^{ij}_{\alpha\beta}$	χ^{ijk}_α	φ^{ijkl}
$SU(8)_R$ rep.	1	8	28	56	70
Loop	11	10	9	8	7

- $SU(8) \times E_7$ invariance only use superfield of dimension $\geq 1/2$.

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- $SU(8) \times E_7$ invariance only use superfield of dimension $\geq 1/2$.
- A first possible counter-term is a eight-loop one Kallosh

$$\int d^4x d^{32}\theta E \, (\chi^{\alpha}_{ijk}\bar{\chi}^{ijk}_{\alpha})^2 = \int d^4x \sqrt{g}\, \kappa^{18}_{(4)} D^{10} R^4$$

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- $SU(8) \times E_7$ invariance only use superfield of dimension $\geq 1/2$.
- Conformal Invariance and $SO(1,9)$ Lorentz invariance implies that the *first possible* counter-term is the nine-loop contributions

Green, Russo, Vanhove

$$\int d^4x d^{32}\theta\, E\, W^4 \sim \int d^4x \sqrt{g}\, \kappa^{20}_{(4)} D^{12} R^4$$

On-shell Supersymmetry and counter-terms

After some order in derivative supersymmetry runs out of steam and no more powers of momenta is factorized out of the amplitude leading to a formula

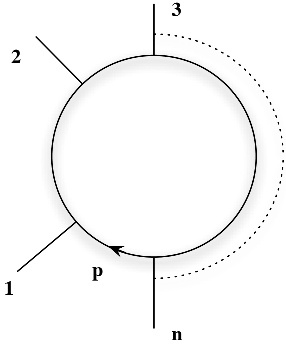
$$D \geq D_c = 2 + \frac{\beta_L}{L}; \quad 6 \leq \beta \leq 18$$

Which gives a possible first divergence

$$3 \leq L \leq 9, \qquad D = 4$$

The absence of 3-loop divergences in 4d in the four-graviton amplitude found by Bern et al. means that $\mathcal{N} \geq 5$ on-shell supersymmetries are at work

One-loop amplitudes



- For $\mathcal{N}=4$ SYM integrals $n-4$ powers of loop momenta

$$\mathfrak{A}_{n;1} \sim \int d^D\ell \, \frac{\ell^{n-4}}{(\ell-k_1)^2 \cdots (\ell-k_1-\cdots-k_n)^2}$$

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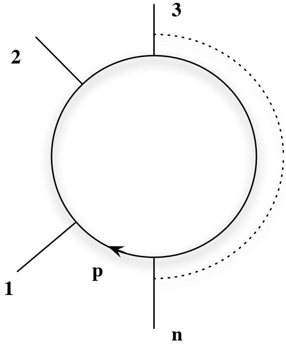

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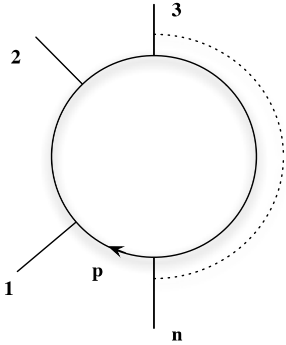
- For $\mathcal{N}=8$ amplitude $2n-8$ are expected

$$\mathfrak{M}_{n;1} \sim \int d^D\ell \, \frac{\ell^{2n-8}}{(\ell-k_1)^2\cdots(\ell-k_1-\cdots-k_n)^2}$$

One-loop amplitudes



- For $\mathcal{N} = 4$ SYM integrals $n-4$ powers of loop momenta

$$\mathfrak{A}_{n;1} \sim \int d^D\ell \, \frac{\ell^{n-4}}{(\ell - k_1)^2 \cdots (\ell - k_1 - \cdots - k_n)^2}$$

- but only $n-4$ are found

Bjerrum-Bohr et al., Bern et al.

$$\mathfrak{M}_{n;1} \sim \int d^D\ell \, \frac{\ell^{n-4}}{(\ell - k_1)^2 \cdots (\ell - k_1 - \cdots - k_n)^2}$$

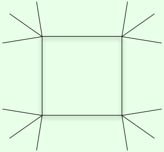
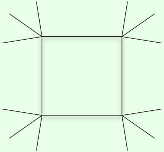
Extra cancellation of $n-4$ powers of loop momentum beyond the supersymmetric ones

The no triangle property in $\mathcal{N}=8$ Bjerrum-Bohr, Vanhove

- Gravity does not have color factor
  - summation over all the permutations at one-loop
  - Sum over all the planar and nonplanar diagrams at higher loop order
- Gauge invariance $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu$

$$\begin{aligned}\mathfrak{M}_{n;1} &\sim (\varepsilon_{\mu\nu}\, k)^8\, (\varepsilon_{\mu\nu}\, k)^{n-4} \int d^D\ell\, \frac{(\ell\cdot \bar{h})^{n-4}}{(\ell-k_1)^2 \cdots (\ell - k_{1\cdots n})^2} \\ &\sim \frac{(\varepsilon_{\mu\nu}\, k)^n\, (k_i\cdot k_j)^4}{(k_i\cdot k_j)^{n/2}} \int d^D\ell\, \frac{1}{(\ell-K_1)^2\cdots(\ell-K_{1\dots 4})^2}\end{aligned}$$

The $\mathcal{N}=8$ are given by scalar boxes like $\mathcal{N}=4$ SYM

$\mathfrak{M}_{n;1} = \mathcal{O}_8$  , $\mathfrak{A}_{n;1} = \mathcal{O}_4$ 

Gravity theories with less or not supersymmetry

Since the **extra** cancellations are not due to supersymmetry they occur as well in theories with less or no supersymmetry

- $\mathcal{N} = 4$ would have a critical dimension for UV divergence

$$D \geq D_c = 3 + \frac{6}{L}$$

- Pure gravity One-loop amplitude would be at most logarithmically diverging like QCD

Bjerrum-Bohr, Vanhove

Summary & Outlook

We have explained that colorless gauge theory like gravity exhibit important cancellations in on-shell amplitudes. Bjerrum-Bohr,Vanhove

- Consequences at higher-loop order
  - Needed to go beyond the six genus limit non-renormalisation in pure spinor string theory
  - These cancellation are needed for the $\beta_L = L$ rule to all loop order.
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