Nonlocal observables in statistical mechanics and logarithmic conformal field theory

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1 Two dimensional conformal field theory and critical phenomena

A typical example is the Ising model. The critical point is described by a two dimensional conformal field theory.

We are interested in local observables, for example corellations of spin variables.

The guideline in constructing a conformal field theory is a requirement of closed associative operator algebra.

The standard procedure of constructing a conformal field theory is as follows.
We start from a chiral algebra of symmetry (Virasoro, superconformal, Kac–Mody algebra).
Then we find a class of representations closed under fusion. This guarantees a closed associative operator algebra.

These allow us in principle to calculate all correlation functions of any local observables.

We can ask the question what happen if we include *nonlocal* observables into the theory.

For example we can be interested in probability that two Ising spins belong to the same claster. In this way we obtain a model that contains local and nonlocal observables.

There are also models that do not contain nontrivial local observables. A typical example of such a model is the two dimensional percolation.

The problem in the percolation theory is a calculation of the probability that there are several clasters between different points.

Another one model in which only nonlocal observables are worth of investigating is the model of branched polymers. At the same time branched polymers give the self–organized critical state in the abelian sand–pile model.

2 Logarithmic conformal field theory

We can investigate universality classes of critical phenomena with nonlocal observables.

As a guideline we keep the requirement of a closed associative operator algebra.

In this approach the universaliti classes of critical phenomena with nonlocal observables are described by so called *logarithmic conformal field theory.*

The terminology appears from the fact that a conformal field theory corresponding to critical phenomena with nonlocal observables contains logarithms in correlation functions.

Appearing of logarithms in correlation functions is equivalent to appearing of nontrivial Jordan cells in dilatation operator L_0.

The name "logarithmic" and first examples was suggested in

V. Gurarie, *Logarithmic Operators in Conformal Field Theory*, Nucl.Phys. B410 (1993) 535-549.

A significant application of logarithmic conformal field theory was

J. Cardy, *Conformal Invariance and Percolation*, arXiv: math-ph/0103018

where crossing probabilities in scailing limit of two dimensional percolation were calculated.

Until the recent time there was no a systematic way to construct a logarithmic CFTs.

We suggested a method of constructing a LCFT based on a quantum group approach.

J. Fjelstad, J. Fuchs, S. Hwang, A.M. Semikhatov, and I.Yu. Tipunin, *Logarithmic conformal field theories via logarithmic deformations*, Nucl. Phys. B633 (2002) 379–413 [hep-th/0201091]

J. Fuchs, S. Hwang, A.M. Semikhatov, and I.Yu. Tipunin, *Nonsemisimple fusion algebras and the Verlinde formula*, Commun. Math. Phys. 247 (2004) 713–742 [hep-th/0306274]

B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, Commun.Math.Phys. 265 (2006) 47-93, hep-th/0504093

B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Logarithmic extensions of minimal models: characters and modular transformations*, Nucl.Phys. B757 (2006) 303-343, hep-th/0606196

As an application of the method a set of LCFTs was constructed. The models are numerated by pair of coprime integers (p, p'), $1 \leq p < p'$. The models with $2 \leq p < p'$ are logarithmic extensions of the Virasoro minimal models. Logarithmic conformal field models $(1, p)$ are also interesting, however they do not correspond to minimal models.

All these models have large symmetry algebra $W_{p,p'}$ which contains the Virasoro algebra.

In

P.A. Pearce, J. Rasmussen, J.-B. Zuber, *Logarithmic Minimal Models*, J.Stat.Mech. 0611 (2006) P017, hep-th/0607232

there was suggested a class of lattice models $\mathcal{LM}(p, p')$ where p and p' are coprime integers with $1 \leq p < p'$. These models contain critical percolation and critical branched polymers as particular cases $\mathcal{LM}(3, 4)$ and $\mathcal{LM}(1, 2)$ respectively.

In the papers

P.A. Pearce, J. Rasmussen, P. Ruelle, *Integrable Boundary Conditions and W-Extended Fusion in the Logarithmic Minimal Models LM(1,p)*, arXiv:0803.0785

J. Rasmussen, P.A. Pearce, *W-Extended Fusion Algebra of Critical Percolation*, arXiv:0804.4335

J. Rasmussen, *W-Extended Logarithmic Minimal Models*, arXiv:0805.2991

it was shown that the scailing limit of $\mathcal{LM}(p, p')$ coincide with (p, p') logarithmic conformal field theories.

3 $(1, p)$ LCFTs

We fix integer $p \geq 2$

$$\alpha_+ = \sqrt{2p}, \qquad \alpha_- = -\sqrt{\frac{2}{p}}, \qquad \alpha_+\alpha_- = -2, \qquad \alpha_0 = \alpha_+ + \alpha_- = \sqrt{\frac{2}{p}}(p-1)$$

We start with free scalar field φ with the Lagrangian

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi + \alpha_0 R \varphi$$

The energy momentum tensor is

$$T = \frac{1}{2}\partial\varphi\partial\varphi + \frac{\alpha_0}{2}\partial^2\varphi$$

This T commutes with screening operators

$$F = \frac{1}{2\pi i}\oint dz e^{\alpha_- \varphi(z)}, \qquad e = \frac{1}{2\pi i}\oint dz e^{\alpha_+ \varphi(z)}$$

To construct a LCFT we study the centralizer of the screening F.

The centralizer of F is a W algebra generated by two fields $W^{\pm}(z)$ with the conformal dimension $2p - 1$.

This W algebra has very complicate structure and we even cannot write commutators of the algebra in a closed form. For example

$$W^+(z)W^-(w) = \frac{1}{(z-w)^{4p-2}} + \frac{cT}{(z-w)^{4p-4}} + \ldots$$

However, the representation category of this algebra is equivalent to the representation category of the quantum group $\overline{U}_q s\ell(2)$. This quantum group is constructed by Drinfeld double procedure from the screening F.

The Hopf algebra $\overline{U}_q s\ell(2)$ is generated by E, F, and K with the relations

$$E^p = F^p = 0, \quad K^{2p} = 1$$

and the Hopf-algebra structure given by

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F,$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1},$$

where $q = e^{\frac{i\pi}{p}}$.

This algebra has $2p$ irreducible representations $X_s^{\pm}$, $1 \leq s \leq p$.

The W algebra representations that demonstrate nondiagonal L_0 action correspond to projective $\overline{U}_q s\ell(2)$ modules.

Any statement in this LCFT can be reformulated in terms of the quantum group $\overline{U}_q s\ell(2)$.