The Leading Singularity Method

at Two Loops

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- In maximally supersymmetric $\mathcal{N} = 4$ super-Yang Mills theory, the problem of computing any one-loop amplitude can be reduced to that of computing tree amplitudes. (1990s–2004 [Bern, Dixon, Kosower; Britto, Cachazo, Feng; . . .])

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$\Rightarrow$ The key to such simplification is that although one-loop amplitudes have many poles and branch cuts with a complicated structure of intersections, they are completely determined by their highest codimension singularities, called the leading singularity.

Singularities at Higher Loops

Higher loop amplitudes have even more complicated singularities and it is not yet completely understood how to exploit the structure of these singularities...

In my talk I will describe a method called the leading singularity method which has three very attractive features: [Cachazo, 2008]

Singularities at Higher Loops

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In my talk I will describe a method called the leading singularity method which has three very attractive features: [Cachazo, 2008]

- It provides a natural basis of integrals to work with
- The coefficients are determined by solving simple linear (!) equations
- The linear equations are almost trivial to write down

Our Target

Much of what I will say in this talk will be more general, but the specific target of our current calculation is the two-loop six-particle MHV amplitude in $\mathcal{N} = 4$ super-Yang Mills theory.

The parity-even part of this amplitude was recently computed by Bern, Dixon, Kosower, Roiban, MS, Vergu, and Volovich, 2008.

The leading singularity method succeeds in reproducing this result, and calculates also the parity-odd part of the amplitude with no additional work.

Background: Calculation of Scattering Amplitudes

Any L-loop scattering amplitude can, in principle, be obtained by summing over all Feynman diagrams:

$$\mathcal{A}^{(L)}(p) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p,\ell) \tag{1}$$

$p =$ external momenta

$\ell =$ loop momenta

However, in practice this is a hopeless exercise due to the enormously large number of Feynman diagrams and their complexity in Yang-Mills theory.

Background: Calculation of Scattering Amplitudes

$$\mathcal{A}^{(L)}(p) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p,\ell) \tag{2}$$

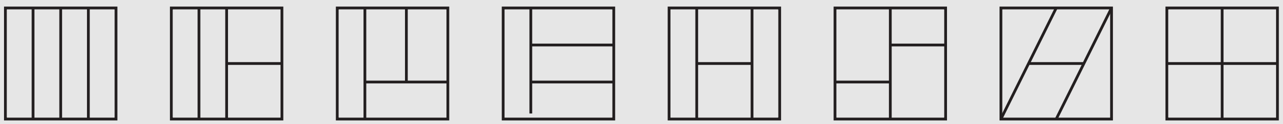
Rather, calculations typically proceed by first finding a representation of the amplitude in terms of a relatively simple basis of integrals $\{I_i\}$:

$$\mathcal{A}^{(L)}(p) = \sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p,\ell) \tag{3}$$

where the coefficients $c_i(p)$ are computed by other means, such as the unitarity-based method [Bern, Dixon, Kosower, 1990s].

Example

For example, unitarity based methods were used to express the four-loop four-particle amplitude in $\mathcal{N} = 4$ super-Yang-Mills as the sum of the following eight scalar integrals:



[Bern, Czakon, Dixon, Kosower, Smirnov, 2006]

A Difficulty

One important difficulty is that there is no known basis of integrals in the general case. Only in some special cases is a basis known:

- one-loop, any number of external particles;

  $\Rightarrow$ scalar box integrals

- and a very plausible conjecture exists for four particles at any number of loops which has emerged from the work of [Bern, Czakon, Dixon, Drummond, Henn, Korchemsky, Kosower, Smirnov, Sokatchev, and others, 2006-2007].

  $\Rightarrow$ dual conformal integrals

Another Difficulty

Even when a basis of integrals can be proven (or guessed!) for a particular amplitude, determining the coefficients can be an enormous computation...

For example, using the unitarity-based method, the coefficients of the two-loop six-particle amplitude come out in a Mathematica file many megabytes large!

A Pedagogical Introduction

to the

Leading Singularity Method

Contour Integrals

The idea is to look at the equation

$$\sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p,\ell) = \int d\ell_1 \cdots d\ell_L \, \sum_j F_j(p,\ell) \tag{4}$$

at the level of the integrand, and instead of integrating over the real ℓ-axis in $\mathbb{C}^{4L}$ we integrate over closed contours $\Gamma \subset \mathbb{C}^{4L}$ to obtain linear equations for the desired coefficients!

$$\sum_i c_i(p) \int_\Gamma I_i(p,\ell) = \int_\Gamma \sum_j F_j(p,\ell) \tag{5}$$

We can require this to be true for any contour Γ.

By choosing many different contours, we get many different linear equations!

Which contours give the most useful equations?

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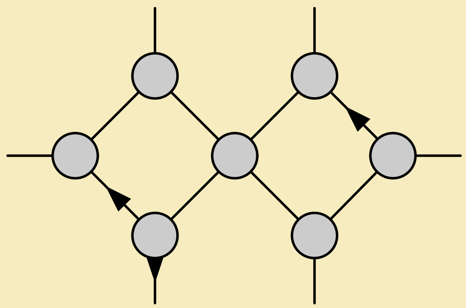
Fortunately it is easy to identify the isolated poles of the integrand:

The poles in Feynman diagrams occur when internal propagators go on-shell.

For the two-loop six-particle amplitude there are five obvious topologies associated with singularities where eight different propagators are going simultaneously on-shell, and hence can be associated with T^8 contours in $\mathbb{C}^8$:



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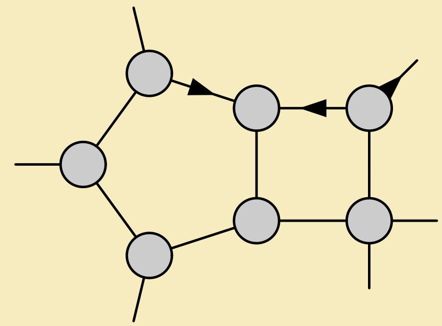

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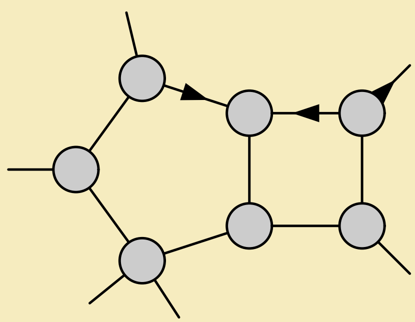

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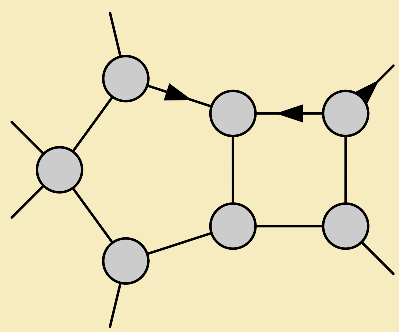

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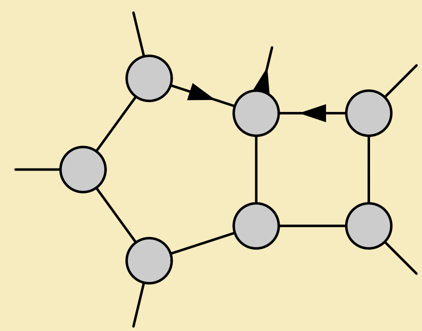

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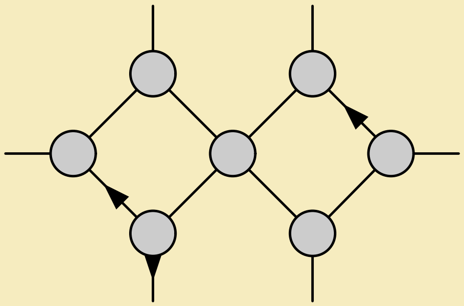
For example, if we look at the first diagram:



(A)

it represents the sum over the subset of all Feynman diagrams which contain all eight of the indicated propagators.

This set of Feynman diagrams has isolated poles at

$$\begin{aligned} S \quad = \quad & \{(\ell_1, \ell_2) \in \mathbb{C}^8 : \ell_1^2 = 0, (\ell_1 + p_1)^2 = 0, (\ell_1 - p_2)^2 = 0, \\ & (\ell_1 - p_2 - p_3)^2 = 0, \ell_2^2 = 0, (\ell_2 - p_4)^2 = 0, \\ & (\ell_2 + p_5)^2 = 0, (\ell_2 + p_5 + p_6)^2 = 0\} \end{aligned}$$

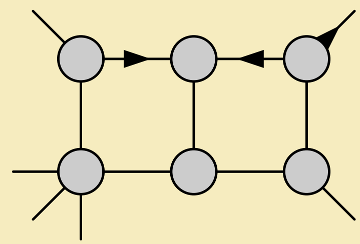
For generic external momenta p_i this consists of four distinct points in $\mathbb{C}^8$.

At each of these four points, the amplitude has an isolated order-8 pole.

To calculate the residue at this pole (i.e., the result of integrating over the corresponding contour Γ) is simple: just take the product of seven on-shell tree-level amplitudes, at each of the grey circles, and evaluate this product at the corresponding solution (ℓ_1, ℓ_2).

There are other, more subtle leading singularities:

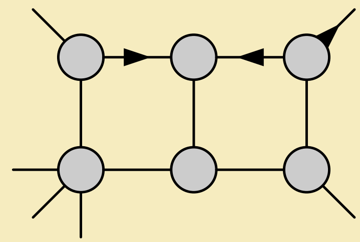
To see how these arise, consider the topology:



(F)

Although it looks like there is only a pole of order 7, not 8, there is another hidden singularity.

To expose it, consider a contour integral which computes the residue at either of the two singularities of the right-hand box:



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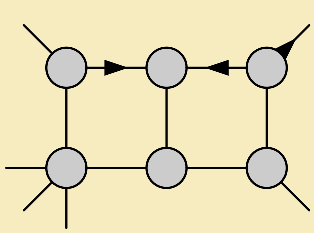
To expose it, consider a contour integral which computes the residue at either of the two singularities of the right-hand box:

$$\int_{\Gamma} d^4\ell_2 \frac{1}{\ell_2^2(\ell_1+k_1)^2(\ell_1+k_1+k_2)^2(\ell_1+\ell_2)^2} = \frac{1}{2}\frac{1}{(k_1+k_2)^2(\ell_2-k_1)^2}$$

where the right-hand side is just the Jacobian evaluated at the location of the singularity. Now this Jacobian has itself another singularity $1/(\ell_2-k_1)^2$.

The conclusion is that there do exist isolated poles of order 8 in such topologies. The residues at these poles can be computed by integrating over appropriate contours Γ.

There are a total of 8 different topologies of this type:



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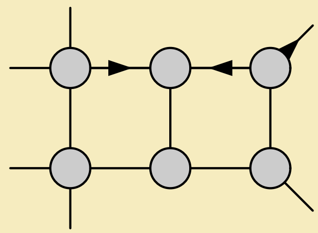

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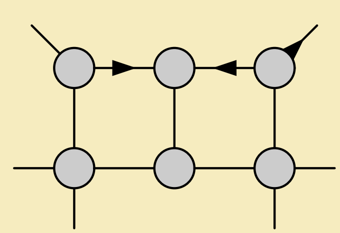

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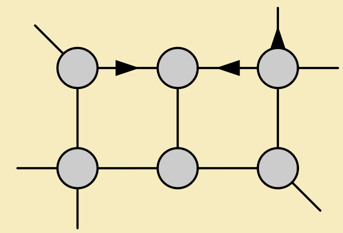

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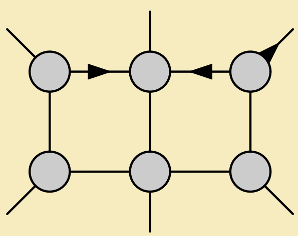

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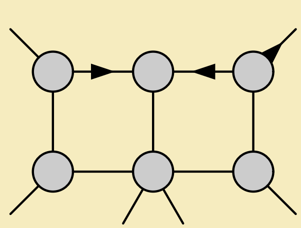

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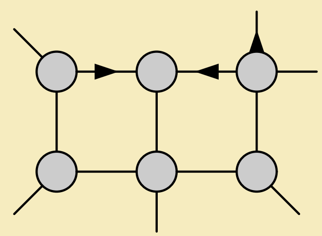

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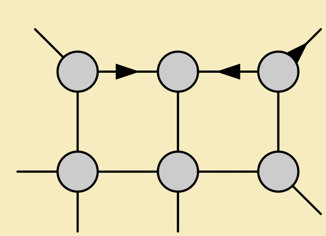

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We begin with a set that just contains the 13 scalar integrals appropriate to the 13 different topologies shown on the previous slides.

It turns out that with just this set of integrals, the linear equations have no solution, so we must add additional integrals to the set $\{I_i\}$

There is a systematic procedure to do this, which ends when one is able to solve all of the equations...

Contructing a Basis of Integrals

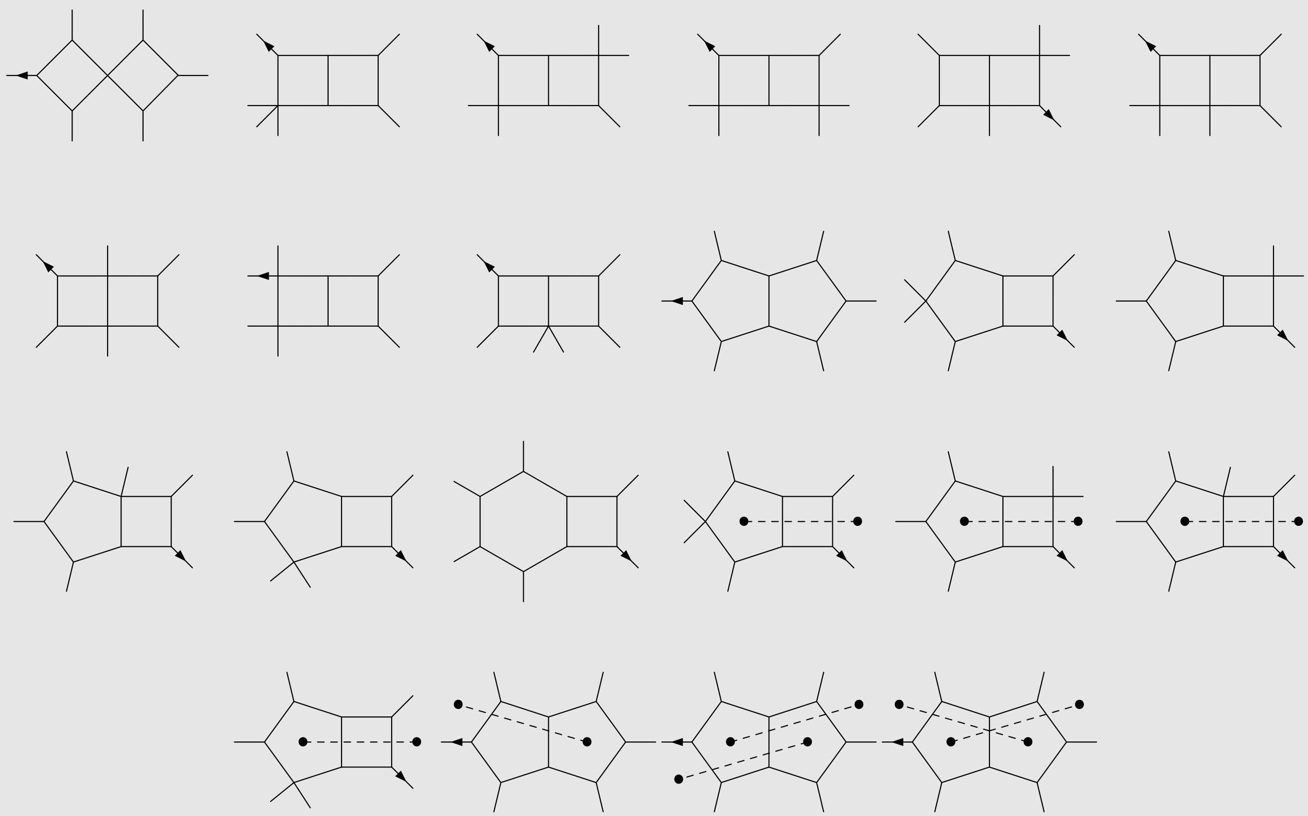
It can happen that when this procedure finishes, one ends up with a set of integrals $\{I_i\}$ that is overcomplete.

This happens because loop integrals for 6 or more external particles can frequently be expressed as linear combinations of other integrals. [van Neerven and Vermaseren, 1984].

If this happens, then the equations do not have a unique solution: given any solution $\{c_i\}$, one can add any set of coefficients $\{\tilde{c}_i\}$ that is actually zero due to a reduction identity.

Result

We find a representation of the 2-loop six-particle MHV amplitude in terms of



(Several of these can actually be set to zero using reduction identities).

Result

The parity-even part of the amplitude agrees with the recent result of [Bern, Dixon, Kosower, Roiban, MS, Vergu, Volovich, 2008].

The parity-odd part of our result is a new result.

Note that the full coefficients, both the parity even and parity odd parts, emerge from solving the same linear equations in the leading singularity method—in fact it is unnatural to separate the two parts, and we have only done this in order to make the comparison and check our results.

The ABDK/BDS Conjecture

One reason for the recent interest in multi-loop amplitudes in $\mathcal{N} = 4$ super-Yang Mills theory is the ABDK/BDS conjecture, which at two-loops takes the form

$$M_n^{(2)}(\epsilon) = \frac{1}{2}(M^{(1)}(\epsilon))^2 - (\zeta(2) + \zeta(3)\epsilon + \zeta(4)\epsilon^2 + \cdots)M^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon)$$

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in dimensional regularization to $D = 4 - 2\epsilon$.

Although this conjecture is now known to be false in general —this was one of the main results of our recent paper [Bern, Dixon, Kosower, Roiban, MS, Vergu, Volovich, 2008], we find that the parity-odd part of the two-loop six-particle amplitude does in fact satisfy ABDK.

(It is in fact reasonable to believe that the parity-odd part always satisfies ABDK/BDS, but this remains unproven.)

Conclusion

The motivation for our work was two-fold

- To unlock previously hidden mathematical richness lurking deep inside multi-loop gluon amplitudes in $\mathcal{N} = 4$ SYM, and
- To exploit that structure to help simplify otherwise formidable computations.

The leading singularity method provides a relatively simple way to find representations of complicated amplitudes in terms of a simple basis of integrals by just solving linear equations.

One final comment is that the helicity information (MHV versus non-MHV) appears only in the inhomogeneous terms (the right-hand side) of the equations.