QUARKS 2008

CRYPTO-HERMITICITY AND CRYPTO-SUPERSYMMETRY OF NON-ANTICOMMUTATIVE HAMILTONIANS

[E.A. Ivanov + A.S., JHEP, 07; A.S., PRD 08]

Noncommutative theories: $[x_\mu, x_\nu] = \omega_{\mu\nu}$

Star Product:

$$f(x)g(x) \to f(x)\star g(x) = \exp\left\{-\frac{\omega_{\mu\nu}}{2}\frac{\partial^2}{\partial x_\mu \partial y_\nu}\right\} f(x)g(y)\Big|_{x=y}$$

then $\quad x_\mu \star x_\nu - x_\nu \star x_\mu = \omega_{\mu\nu}$

- NC Lagrangians

$$\lambda \int \phi^3(x)\, d^n x \;\to \lambda \int \phi \star \phi \star \phi\, d^n x$$

- Superspace: $(x_\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$, grassmanian $\theta_\alpha, \bar{\theta}^{\dot{\alpha}}$.
- superfields: $\Phi(x_\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$
- SUSY Lagrangians: $S = \int d^4x d^2\theta d^2\bar{\theta} F(\Phi_i)$

NAC theories: $\{\theta, \theta\} \neq 0$

Example: NAC Wess-Zumino model
(Seiberg, 03)

- deformation:

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} \ , \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = 0$$

- star product:

$$X(\theta)\star Y(\theta) \;=\; \exp\left\{-\frac{C^{\alpha\beta}}{2}\frac{\partial^2}{\partial\theta_1^\alpha\partial\theta_2^\beta}\right\}X(\theta_1)Y(\theta_2)\Big|_{1=2}$$

- Lagrangian:

$$\mathcal{L} \;=\; \int d^4\theta\,\bar{\Phi}\star\Phi + \left[\int d^2\theta\left(\frac{m\Phi\star\Phi}{2} + \frac{\lambda\Phi\star\Phi\star\Phi}{3}\right) + \mathrm{c.c}\right]$$

- only the last term is deformed

$$\frac{\lambda}{3}\int d^2\theta\,\Phi^3 \;\to\; \int d^2\theta\,\Phi * \Phi * \Phi = \lambda F\phi^2 - \frac{\lambda}{3}\,\det\|C\|F^3\ .$$

- Lorentz invariant, but complex

Does it mean anything in real time ?

YES !

- The Hamiltonian is Hermitian in disguise (crypto-Hermitian).

Crypto-Hermitian Hamiltonians

Bronzan et al, Amati et al, 76 (Regge field theory); (Gasymov, 80 ($V(x) = e^{ix}$); Calicetti et al, 80 ($V(x) = x^2 + i\beta x^3$) C.Bender + S.Boettcher, 98; A.Mostafazadeh, 05)

The most trivial example

$$H = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 1, \lambda_2 = 2$.

Eigenvectors:

$$x^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- Not orthogonal. However, one can choose

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

such that $x^{(i)} M x^{(j)} = \delta^{ij}$. Then $H^\dagger = MHM^{-1}$.

- One can represent $M = A^T A$ with

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Then

$$\tilde{H} = \ AHA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

is manifestly Hermitian and the eigenvectors

$$\tilde{x}^{(1)} = Ax^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \tilde{x}^{(2)} = Ax^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are manifestly orthonormal !

Example:

$$H = \frac{p^2 + x^2}{2} + igx = \frac{p^2 + x'^2}{2} + \frac{g^2}{2}$$

with $x' = x + ig = e^{-gp} x e^{gp}$. Then $\tilde{H} = e^{-gp} H e^{gp}$ is Hermitian.

Example:

$$H = \frac{p^2 + x^2}{2} + igx^3 \ .$$

Take

$$R = g\left(\frac{2}{3}p^3 + x^2 p\right) - g^3\left(\frac{64}{15}p^5 + \frac{20}{3}p^3 x^2 + 4px^4 - 6p\right) + O(g^5)\ .$$

Then

$$\tilde{H} = e^R H e^{-R} = \frac{p^2 + x^2}{2} + g^2\left(3p^2 x^2 + \frac{3x^4}{2} - \frac{1}{2}\right) + O(g^4)$$

is Hermitian.

- New coord. and momenta:

$$p' = e^R p e^{-R} = p + 2igxp + g^2(2p^3 - px^2) + \ldots$$

$$x' = e^R x e^{-R} = x - ig(x^2 + 2p^2) - g^2(x^3 - 2xp^2) + \ldots\ .$$

- $\{p', x'\}_{P.B.} \neq 1$ starting from the terms $\sim g^4$.

Not a canonical transformation !

- Weyl symbols:

$$\begin{aligned} xp \quad &\to \quad (\hat{p}x + x\hat{p})/2, \\ x^2 p \quad &\to \quad (x^2\hat{p} + \hat{p}x^2 + x\hat{p}x)/3\ , \ldots \end{aligned}$$

NAC SQM

(Aldrovandi + Schaposnik, 06)

$$S \; = \; -\int dt\, d\bar{\theta} d\theta \left[\frac{1}{2} (D \star X) \star (\bar{D} \star X) + V_\star(X) \right],$$

- X - real supervariable
- $\star$ corresponds to deformations

$$\{\theta, \theta\} = C, \quad \{\bar{\theta}, \bar{\theta}\} = \bar{C}, \quad \{\theta, \bar{\theta}\} = \tilde{C}$$

- $V_\star(X) = \sum_n c_n (X \star \ldots \star X)_n$.
- Lifting deformations $\rightarrow$ Witten's SQM in chiral basis,

$$t = \tau - i\theta\bar{\theta}, D = \frac{\partial}{\partial \theta} - 2i\bar{\theta}\frac{\partial}{\partial t}, \quad \bar{D} = -\frac{\partial}{\partial \bar{\theta}}$$

Lagrangian in components.

$$L \;=\; -i\dot{x}F - \frac{\partial \tilde{V}(x,F)}{\partial x}F + \frac{1}{2}F^2 + i\bar{\psi}\dot{\psi} + \frac{\partial^2 \tilde{V}(x,F)}{\partial x^2}\bar{\psi}\psi\,,$$

with

$$\tilde{V}(x,F) \;=\; \int_{-1/2}^{1/2} d\xi\, V(x+\xi c F)\;, \qquad\qquad c^2 = \tilde{C}^2 - C\bar{C}\;.$$

(cf. Alvarez-Gaume + Vazquez-Mozo, 05)

- Lagrangian and Hamiltonian are complex.
- Supercharges:

$$Q = \frac{\partial}{\partial \theta}, \quad \bar{Q} = -\frac{\partial}{\partial \bar{\theta}} - 2i\theta\frac{\partial}{\partial t}$$

- $\star$ product commutes with Q, but not with $\bar{Q}$.

- Only one obvious Nöther supercharge

$Q = \psi p$

- But

$$\bar{Q} = \bar{\psi}\left(p + 2i\frac{\partial \tilde{V}}{\partial x}\right).$$

also commutes with the Hamiltonian !

- Algebra

$$Q^2 = \bar{Q}^2 = 0, \; \{Q, \bar{Q}\} = 2H$$

holds

Crypto-Hermiticity of H

Let $V(X) = \lambda X^3/3$. Then $\tilde{V}(x, F) = \lambda x^3/3 + \lambda c^2 x F^2/12$ and

$$H = p^2/2 + i\lambda p x^2 - i\beta p^3 - 2\lambda x \bar{\psi}\psi \ ,$$ with $\beta = \lambda c^2/12$

- Rotate it with

$$R = -\lambda x^3/3 + \beta x p^2 - 2\lambda\beta x^2 \bar{\psi}\psi - 2\beta^2 p^2 \bar{\psi}\psi + O(\lambda^3, \beta^3, \lambda^2\beta, \lambda\beta^2) \ .$$

- The conjugated Hamiltonian

$$\tilde{H} = e^R H e^{-R} = p^2/2 - 2\lambda x \bar{\psi}\psi + [\lambda^2 x^4 + 3\beta^2 p^4]/2 + \lambda\beta/2 + \dots$$

is Hermitian.

- rotated supercharges:

$$\tilde{Q} = e^{\hat{R}} Q e^{-\hat{R}} = \psi[p - i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + \beta^2 p^3 + \ldots]\,,$$
$$\bar{\tilde{Q}} = e^{R} \bar{Q} e^{-R} = \bar{\psi}[p + i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + \beta^2 p^3 + \ldots]\,.$$

are adjoint to each other.

NAC WZ SQM

- reduced Hamiltonian

$$H = \bar{\pi}\pi + \bar{\phi}\phi + g\phi^2\bar{\phi} + \bar{g}\bar{\phi}^2\phi + g\bar{g}\bar{\phi}^2\phi^2 - (1+2g\phi)\psi_1\psi_2 - (1+2\bar{g}\bar{\phi})\bar{\psi}_2\bar{\psi}_1 + \beta(\bar{\phi}+\bar{g}\bar{\phi}^2)^3$$

- supercharges:

$$Q_\alpha = \pi\psi_\alpha + i\epsilon_{\alpha\gamma}\bar{\psi}_\gamma(\bar{\phi}+\bar{g}\bar{\phi}^2)\,,$$
$$\bar{Q}_\beta = \bar{\pi}\bar{\psi}_\beta - i\epsilon_{\beta\delta}\psi_\delta(\phi+g\phi^2) + ?..$$

- This Hamiltonian is crypto-Hermitian.
- No rotation operator was explicitly constructed,

but reality of the spectrum was checked.

SPECTRUM

- A vacuum state with zero energy
- A degenerate quartet of first excited states with

$$\Delta E_1 = -\frac{155}{36}\beta g^3 \ .$$

- This suggests that SUSY holds

NO WONDER !

- In undeformed case, the state $|\Psi\rangle$ annihilated by $\bar{Q}_\alpha$ exists.
- $Q_1|\Psi\rangle$, $Q_2|\Psi\rangle$, and $Q_1 Q_2|\Psi\rangle$ represent three other members of the quartet.
- After deformation, $|\Psi\rangle \to |\tilde{\Psi}\rangle$.
- $Q_1|\tilde{\Psi}\rangle$, $Q_2|\tilde{\Psi}\rangle$, and $Q_1 Q_2|\tilde{\Psi}\rangle$ still have the same energy.