Subtleties in the quasi-classical `tunneling' calculation of Hawking radiation*

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Overview

- Hawking calculates temperature from looking at the asymptotic vacua
- Hawking radiation comes from tunneling?
- Semi-classical method and its problems
- Solution to the problems
- Connection to Black Hole thermodynamics

Hawking radiation from tunneling?

- **Short answer: Not in the usual sense**
  - **The particle momentum is never imaginary**
    - **So not a barrier penetration problem**
    - **More like an `over-the-barrier' problem (Landau & Lifshitz vol. 3)**
    - **imaginary part comes from continuing the path around the pole in p**
  - **Cauchy problem is ill-defined.**

    Particles have to propagate acausally from behind the horizon.

    (however, the horizon is considered to move inward)
- **Semi-classical Anzatz**

$$\varphi \propto e^{-i\frac{S}{\hbar}} \qquad S = Et + S_0(q)$$

The original `tunneling' calculations

- **The Klein-Gordon equation for a scalar field φ of mass m in a curved spacetime is given by**

$$\left[-\frac{\hbar^2}{\sqrt{-g}}\partial_\mu g^{\mu\nu}\sqrt{-g}\,\partial_\nu + m^2\right]\varphi = 0$$

- **Plugging in the semi-classical anzatz gives, to zeroth order, the Hamilton-Jacobi equation for S**

$$g^{\mu\nu}(\partial_\mu S)(\partial_\nu S) + m^2 = 0$$

- **Solving this for S gives the `tunneling' rate and the inverse Hawking temperature** β

$$\Gamma \propto e^{-2\,\mathrm{Im}\int p_r\,dr} \equiv e^{-\beta E}$$

Problems appear…

- **Parikh and Wilczek [prl 85, 5042 (2000)] showed that this formula gives the correct Hawking temperature for a Schwarzschild black hole *in Painleve coordinates*.**

- **We showed [E.T. Akhmedov, et. al, *PLB*, 642, 124 (2006); ibid. *IJMPA*, 22, 1705 (2007)] that their formula is coordinate dependent.**
  - **Gives twice the correct Hawking temperature in other coordinate systems!**
    - **The so-called `factor of two problem'**

- **On the other hand, B.D. Chowdhury [hep-th/0605197] noticed that the the formula is not canonically invariant. The correct formula should be instead:**

$$\Gamma = |\langle x_f | x_i \rangle|^2 = \langle x_i | x_f \rangle \langle x_f | x_i \rangle \propto e^{-\mathrm{Im}\oint p_r dr}$$

- **But with the new formula there is a factor of two problem in all coordinates, including Painleve!**

Problem solved

- We then showed [V. Akhmedova, et. al, arXiv:0804.----; E.T. Akhmedov, et. al, *arXiv:0805*.----] that since the amplitude is really:

$$\Gamma \propto e^{\mathrm{Im}\Delta S} = e^{\mathrm{Im}\left(E\,\Delta t - \oint p_r\,dr\right)}$$

- There is a previously un-noticed temporal component.
  - One can see using Kruskal coordinates that the closed path around the pole has a time contribution to the imaginary part (unlike QM):

$$t \rightarrow t - 4\pi i M$$

- So it is not the usual QM tunneling problem. The inverse temperature is therefore

$$\beta = 4\pi M + \mathrm{Im}\oint p_r\,dr$$

First Law of Black Hole Thermodynamics

- We can now relate this result to Black Hole thermodynamics [T. Pilling, PLB 660, 402 (2008)].
  - The first law of BH thermodyamics is:

$$dM = T\, dS$$

- Where S is the Bekenstein-Hawking entropy, equal to ¼ the area of the event horizon. Thus we can write

$$\beta = \frac{dS}{dM} = 2\pi r_H \frac{dr_H}{dM}$$

- The horizon radius can be written (in Schwarzschild coords) as

$$r_H = \frac{i}{2\pi} \oint \frac{dr}{g_{00}}$$

Temperature from BH Thermodynamics

- Transforming this integral to arbitrary coordinates allows us to write the inverse temperature as

$$\beta = i \frac{dr_H}{dM} \oint \frac{\sqrt{-g}}{g_{00}} dr$$

- Take another look at our quasi-classical result:

$$\beta = 4\pi M + \mathrm{Im} \oint p_r \, dr$$

- For an arbitrary 2-d metric (-,+,+,+) the (outgoing/ingoing) canonical momentum is

$$p_r = -\frac{1}{g_{00}} \left[g_{01} E \pm \sqrt{-g} \left(E^2 + g_{00} m^2 \right) \right]$$

Quasi-classical result revisited

- putting this into our quasi-classical expression for the inverse temperature:

$$\beta = 4\pi M - \frac{1}{E}\,\mathrm{Im}\oint \frac{1}{g_{00}}\left[g_{01}E \pm \sqrt{-g}\left(E^2 + g_{00}m^2\right)\right] dr$$

- Push the pole into the upper half plane and integrate around it

$$\frac{1}{g'_{00}(r_H)} \lim_{\varepsilon \to 0} \mathrm{Im}\left[\int_{r_i}^{r_f} \frac{g_{01} + \sqrt{-g}}{r - r_H - i\varepsilon}\, dr + \int_{r_f}^{r_i} \frac{g_{01} - \sqrt{-g}}{r - r_H - i\varepsilon}\, dr\right]$$

- Notice that the off-diagonal terms cancel. This is why it also gives a facor of two for Painleve.

Comparing with BH Thermodynamics

- Canceling the off-diagonal terms we can write the inverse temperature as

$$\beta = 4\pi M + i \oint \frac{\sqrt{-g}}{g_{00}} dr$$

- Recall the BH thermodynamics result:

$$\beta = i \frac{dr_H}{dM} \oint \frac{\sqrt{-g}}{g_{00}} dr$$

- This works in all stationary metrics!
  - If M=0 and rH = 0, then dr/dM = 1 and the contribution is purely spatial (Unruh radiation)
  - Notice that since dr/dM = 2 for Schwarzschild we see by comparison that the temporal and the spatial parts contribute equally to the temperature.

Conclusion and Implications

- We see that BH thermodynamics and Quasi-classical `tunneling', with the new temporal contribution, give the same temperatures.

- The connection between them shows that one can compute the temperature in any of the following methods:

  - Twice the spatial part
    - the factor of two problem.
  - Temporal part plus spatial part
    - which is the `corrected' quasi-classical method.
  - Twice the temporal part
    - as originally found by Hartle and Hawking in 1976 by noticing the periodicity in euclidean time.
  - Using the simple thermodynamics formula