Two-Loop Bhabha Scattering

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Introduction

- Why two-loop Bhabha?
  - *Crucial for precision physics at e^+e^- colliders*
  - *Test ground for perturbative methods*
  - *Classical problem of perturbative QED*
- Topics discussed
  - *two-loop photonic corrections*
  - *two-loop heavy-flavor corrections*

Luminosity determination at e^+e^- colliders

- High-Energy Small-Angle Scattering *(ILC-GigaZ)*

  required accuracy 0.1 pm *available accuracy* 0.5 pm

- Low-Energy Large-Angle Scattering
  (BABAR, BELLE, BEPC/BES, DAΦNE, KEKB, PEP-II, ...)

  required accuracy 1 pm *available accuracy* 5 pm

- High-Energy Large-Angle Scattering
  (Luminosity spectrum at ILC)

  required accuracy 5 pm *available accuracy* **?**

Born approximation

$$\frac{d\sigma^{(0)}}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-x+x^2}{x}\right)^2 + \mathcal{O}(m_e^2/s), \qquad x = \frac{1-\cos\theta}{2}$$

H.J. Bhabha (1935)

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H.J. Bhabha (1935)

- Phenomenologycally interesting:
  - *High energy region* $s,\ t,\ u \gg m_e^2$
  - *Small angle Bhabha scattering* $t \ll s,\ x \sim 0$
  - *Large angle Bhabha scattering* $t \sim s,\ x \sim 1$

Radiative corrections

$$\boxed{\text{Observable}} = \boxed{\text{Virtual corrections}} + \boxed{\text{Bremsstrahlung}}$$

$$\sigma = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^n \sigma^{(n)}, \quad \sigma^{(1)} = \sigma_v^{(1)} + \sigma_r^{(1)}, \quad \sigma^{(2)} = \sigma_{vv}^{(2)} + \sigma_{rv}^{(2)} + \sigma_{rr}^{(2)}, \ldots$$

Two types of IR divergences

- Soft divergences $\mathrm{d}l/l$, regulated by λ or ϵ. Disappear in soft photon inclusive cross section with the energy cutoff $\mathcal{E}_{cut}$ on the emitted photons

- Collinear divergences $\mathrm{d}\theta/\theta$, regulated by $m_e$or ϵ. Disappear in soft/collinear photon inclusive cross section with the energy and angular cutoff $\mathcal{E}_{cut}$ and θ_{cut} on the emitted photons

Inclusive cross section

1. Keep $m_e \neq 0$
2. Split real radiation into “soft” and “hard” by $\mathcal{E}_{cut} \ll m_e$
3. Compute the virtual+soft real part analytically
4. Compute the hard real part with actual experimental cuts by means of Monte Carlo

Structure of the virtual + soft corrections

- First order

$$
\begin{aligned}
\frac{d\sigma^{(1)}}{d\sigma^{(0)}} &= \delta_1^{(1)} \ln\left(\frac{s}{m_e^2}\right) + \delta_0^{(1)} + \mathcal{O}(m_e^2/s) \\
\delta_1^{(1)} &= 4\ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + \ldots, \qquad \mathcal{E} = \sqrt{s}/2
\end{aligned}
$$

- Second order

$$
\begin{aligned}
\frac{d\sigma^{(2)}}{d\sigma^{(0)}} &= \delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_1^{(2)} \ln\left(\frac{s}{m_e^2}\right) + \delta_0^{(2)} + \mathcal{O}(m_e^2/s) \\
\delta_2^{(2)} &= 8\ln^2\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + 12\ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + \ldots \\
\delta_1^{(2)} &= -16\left[1 + \ln\left(\frac{1-x}{x}\right)\right] \ln^2\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + \ldots
\end{aligned}
$$

Two-loop corrections

- Large angle scattering ➪ arbitrary x
  - *Full massless result for virtual correction*

    Z. Bern, L. Dixon, A. Ghinculov
  - *Logarithmic corrections, leading order in* m_e^2/s

    E.W. Glover, J.B. Tausk, J.J. van der Bij
  - $m_e \neq 0$, *electron loop insertions*

    R. Bonciani, A. Ferroglia, P. Mastrolia, E. Remiddi, J.J. van der Bij
  - *Photonic corrections, leading order in* m_e^2/s

    A. Penin, T. Becher, K. Melnikov
  - *Heavy flavor VP, leading order in* m_f^2/s,

    T. Becher, K. Melnikov; M. Czakon *et al.*
  - *Heavy flavor VP, exact in* m_μ^2/s,

    R. Bonciani, A. Ferroglia, A.A. Penin, M. Czakon *et al.*

Based on

Phys. Rev. Lett. **95**, 010408 (2005)

Nucl. Phys. **B 734**, 185 (2006)

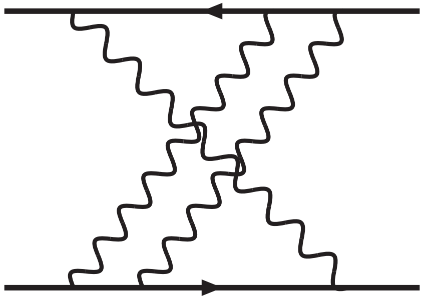
A.A. Penin

Phys. Rev. Lett. **100**, 131601 (2008)

JHEP **0802**, 080 (2008)

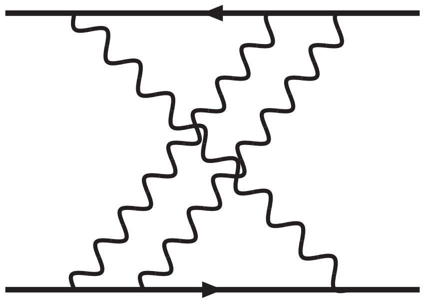
R. Bonciani, A. Ferroglia, A.A. Penin

Photonic corrections



2 loops + 4 legs + 3 scales $s,\ t,\ m_e^2$ ⇨ *no chance?*

Photonic corrections



2 loops + 4 legs + 3 scales $s,\ t,\ m_e^2$ ➪ *no chance?*

Leading order in m_e^2/s ➪ *Infrared matching*

$$\boxed{m_e, \lambda \Rightarrow d = 4 - 2\epsilon}$$

Infrared matching

1. For a given amplitude $\mathcal{A}$ construct an auxilary amplitude $\bar{\mathcal{A}}$ with the same structure of IR singularities

2. Compute the matching term for $\lambda,\ m_e = 0$

$$\delta\mathcal{A} = \left[\mathcal{A}(\epsilon) - \bar{\mathcal{A}}(\epsilon)\right]_{\epsilon\to 0}$$

3. Compute the auxilary amplitude $\bar{\mathcal{A}}$ for $\lambda,\ m_e \to 0$

4. The amplitude $\mathcal{A}$ in the limit $\lambda,\ m_e \to 0$ is given by

$$\mathcal{A}(\lambda, m_e)\Big|_{\lambda,\ m_e\to 0} = \bar{\mathcal{A}}(\lambda, m_e)\Big|_{\lambda,\ m_e\to 0} + \delta\mathcal{A}$$

How to construct $\bar{A}$?

- Exponentiation of IR singularities

  (D.R. Yennie, S.C. Frautschi, H. Suura; A. Mueller; J. Collins; ...)

- Factorization of collinear singularities

  (J. Frenkel, J. Taylor)

- Nonrenormalization of infrared exponents

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➥ The auxilary amplitude

$$\boxed{\bar{\mathcal{A}}^{(2)} = \frac{1}{2}\left(\mathcal{A}^{(1)}\right)^2 + 2\left[\mathcal{F}^{(2)} - \frac{1}{2}\left(\mathcal{F}^{(1)}\right)^2\right]}$$

Result *(page 1 of 2)*

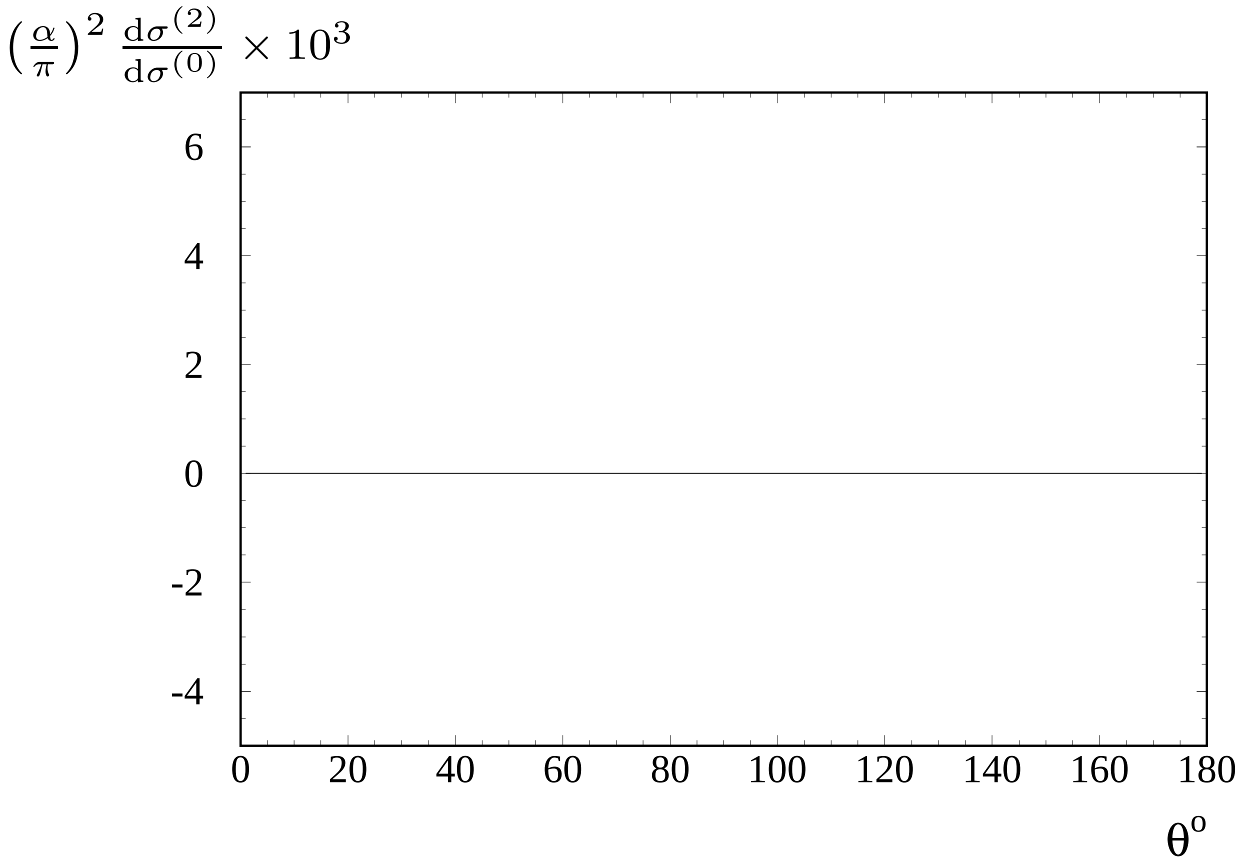
$$\delta_0^{(2)} = 8\mathcal{L}_\varepsilon^2 + \left(1-x+x^2\right)^{-2}\Bigg[\left(\frac{4}{3}-\frac{8}{3}x-x^2+\frac{10}{3}x^3-\frac{8}{3}x^4\right)\pi^2+\left(-12+16x-18x^2+6x^3\right)\ln(x)$$
$$+\left(2x+2x^3\right)\ln(1-x)+\left(-3x+x^2+3x^3-4x^4\right)\ln^2(x)+\left(-8+16x-14x^2+4x^3\right)\ln(x)$$
$$\times\ln(1-x)+\left(4-10x+14x^2-10x^3+4x^4\right)\ln^2(1-x)+\left(1-x+x^2\right)^2\left(16+8\mathrm{Li}_2(x)\right.$$
$$\left.-8\mathrm{Li}_2(1-x)\right)\Bigg]\mathcal{L}_\varepsilon+\frac{27}{2}-2\pi^2\ln(2)+\left(1-x+x^2\right)^{-2}\Bigg(\left(\frac{83}{24}-\frac{125}{24}x+\frac{13}{4}x^2+\frac{19}{24}x^3-\frac{25}{24}x^4\right)$$
$$\times\pi^2+\left(-9+\frac{43}{2}x-34x^2+22x^3-9x^4\right)\zeta(3)+\left(-\frac{11}{90}-\frac{5}{24}x+\frac{29}{180}x^2+\frac{23}{180}x^3-\frac{49}{480}x^4\right)\pi^4$$
$$+\left[-\frac{93}{8}+\frac{231}{16}x-\frac{279}{16}x^2+\frac{93}{16}x^3+\left(-\frac{3}{2}+\frac{13}{4}x-\frac{7}{12}x^2-\frac{11}{8}x^3\right)\pi^2+\left(12-12x+8x^2\right.\right.$$
$$\left.\left.-x^3\right)\zeta(3)\right]\ln(x)+\left[\frac{9}{2}-\frac{43}{8}x+\frac{17}{8}x^2+\frac{29}{8}x^3-\frac{9}{2}x^4+\left(\frac{x}{4}+\frac{x^2}{2}+\frac{5}{24}x^3+\frac{19}{48}x^4\right)\pi^2\right]\ln^2(x)$$
$$+\left(\frac{67}{24}x-\frac{5}{4}x^2-\frac{2}{3}x^3\right)\ln^3(x)+\left(\frac{7}{48}x+\frac{5}{96}x^2-\frac{x^3}{12}+\frac{43}{96}x^4\right)\ln^4(x)+\Bigg\{3x+3x^3+\left(\frac{7}{6}x\right.$$
$$\left.-\frac{73}{24}x^2+\frac{15}{8}x^3\right)\pi^2+\left(-6+6x-x^2-4x^3\right)\zeta(3)+\left[-8+\frac{21}{2}x-\frac{45}{4}x^2+x^4+\left(1-\frac{x}{6}+\frac{x^2}{12}\right.\right.$$
$$\left.\left.-\frac{x^3}{3}-\frac{x^4}{8}\right)\pi^2\right]\ln(x)+\left(6-11x+\frac{35}{4}x^2-\frac{15}{8}x^3\right)\ln^2(x)+\left(\frac{2}{3}+\frac{x}{12}-\frac{x^3}{3}+\frac{5}{24}x^4\right)\ln^3(x)\Bigg\}$$
$$\times\ln(1-x)+\left[\frac{7}{2}-6x+\frac{45}{4}x^2-6x^3+\frac{7}{2}x^4+\left(-\frac{17}{24}+\frac{7}{6}x-\frac{25}{24}x^2-\frac{13}{48}x^4\right)\pi^2+\left(-3+\frac{23}{4}x\right.\right.$$
$$\left.\left.-\frac{23}{4}x^2+\frac{9}{8}x^3\right)\ln(x)+\left(\frac{7}{2}-\frac{41}{8}x+\frac{31}{8}x^2+\frac{3}{8}x^3-\frac{13}{16}x^4\right)\ln^2(x)\right]\ln^2(1-x)+\left[\frac{3}{8}x+\frac{1}{6}x^2\right.$$
$$\left.+\frac{3}{8}x^3+\left(-4+\frac{29}{6}x-\frac{49}{12}x^2+\frac{5}{6}x^3+\frac{7}{8}x^4\right)\ln(x)\right]\ln^3(1-x)+\left(\frac{1}{32}-\frac{3}{4}x+\frac{71}{48}x^2-\frac{29}{24}x^3\right.$$
$$\left.+\frac{9}{32}x^4\right)\ln^4(1-x)+\Bigg\{8-16x+24x^2-16x^3+8x^4+\left(\frac{7}{3}-3x+\frac{3}{4}x^2+\frac{5}{6}x^3-\frac{2}{3}x^4\right)\pi^2$$

Result (page 2 of 2)

$$
\begin{aligned}
&+\left[-6+\frac{11}{2}x-4x^2+x^3+\left(2-\frac{11}{4}x+\frac{7}{4}x^2+\frac{x^3}{4}-x^4\right)\ln(x)\right]\ln(x)+\left[\frac{3}{2}x-\frac{x^2}{4}+x^3\right.\\
&\left.+\left(-4+9x-\frac{15}{2}x^2+2x^3\right)\ln(x)+\left(-1-\frac{7}{2}x+\frac{25}{4}x^2-5x^3+2x^4\right)\ln(1-x)\right]\ln(1-x)+\Big(2\\
&\left.-4x+6x^2-4x^3+2x^4\Big)\,\mathrm{Li}_2(x)\right\}\mathrm{Li}_2\,(x)+\Bigg\{-8+16x-24x^2+16x^3-8x^4+\left[-\frac{2}{3}+\frac{4}{3}x\right.\\
&\left.+\frac{x^2}{2}-\frac{5}{3}x^3+\frac{2}{3}x^4\right]\pi^2+\left[6-8x+9x^2-3x^3+\left(\frac{3}{2}x-\frac{x^2}{2}-\frac{3}{2}x^3+2x^4\right)\ln(x)\right]\ln(x)+\Bigg[-x\\
&\left.-\frac{x^2}{4}-\frac{x^3}{2}+\left(10-14x+9x^2\right)\ln(x)+\left(-8+11x-\frac{31}{4}x^2+\frac{x^3}{2}+x^4\right)\ln(1-x)\right]\ln(1-x)\\
&\left.+\left(-4+8x-12x^2+8x^3-4x^4\right)\mathrm{Li}_2(x)+\left(2-4x+6x^2-4x^3+2x^4\right)\mathrm{Li}_2(1-x)\right\}\mathrm{Li}_2\,(1-x)\\
&+\left[\frac{5}{2}x-5x^2+2x^3+\left(-4-x+x^2+2x^3-2x^4\right)\ln(x)+(6-6x+x^2+4x^3)\ln(1-x)\right]\mathrm{Li}_3\,(x)\\
&+\left[\frac{x}{2}-\frac{x^3}{2}+(-6+5x+3x^2-5x^3)\ln(x)+\left(6-10x+10x^3-6x^4\right)\ln(1-x)\right]\mathrm{Li}_3\,(1-x)\\
&+\left(-2+\frac{17}{2}x-\frac{17}{2}x^3+2x^4\right)\mathrm{Li}_4\,(x)+\left(7x-\frac{9}{2}x^2-4x^3+6x^4\right)\mathrm{Li}_4\,(1-x)+\Bigg(-6+4x\\
&\left.+\frac{9}{2}x^2-7x^3\Big)\,\mathrm{Li}_4\left(-\frac{x}{1-x}\right)\right)\,,
\end{aligned}
$$

$$
\mathcal{L}_\varepsilon=\left[1-\ln\left(x/(1-x)\right)\right]\ln\left(\mathcal{E}_{cut}/\mathcal{E}\right).
$$

Two-loop photonic corrections to LA Bhabha scattering



$$\sqrt{s} = 1 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$$

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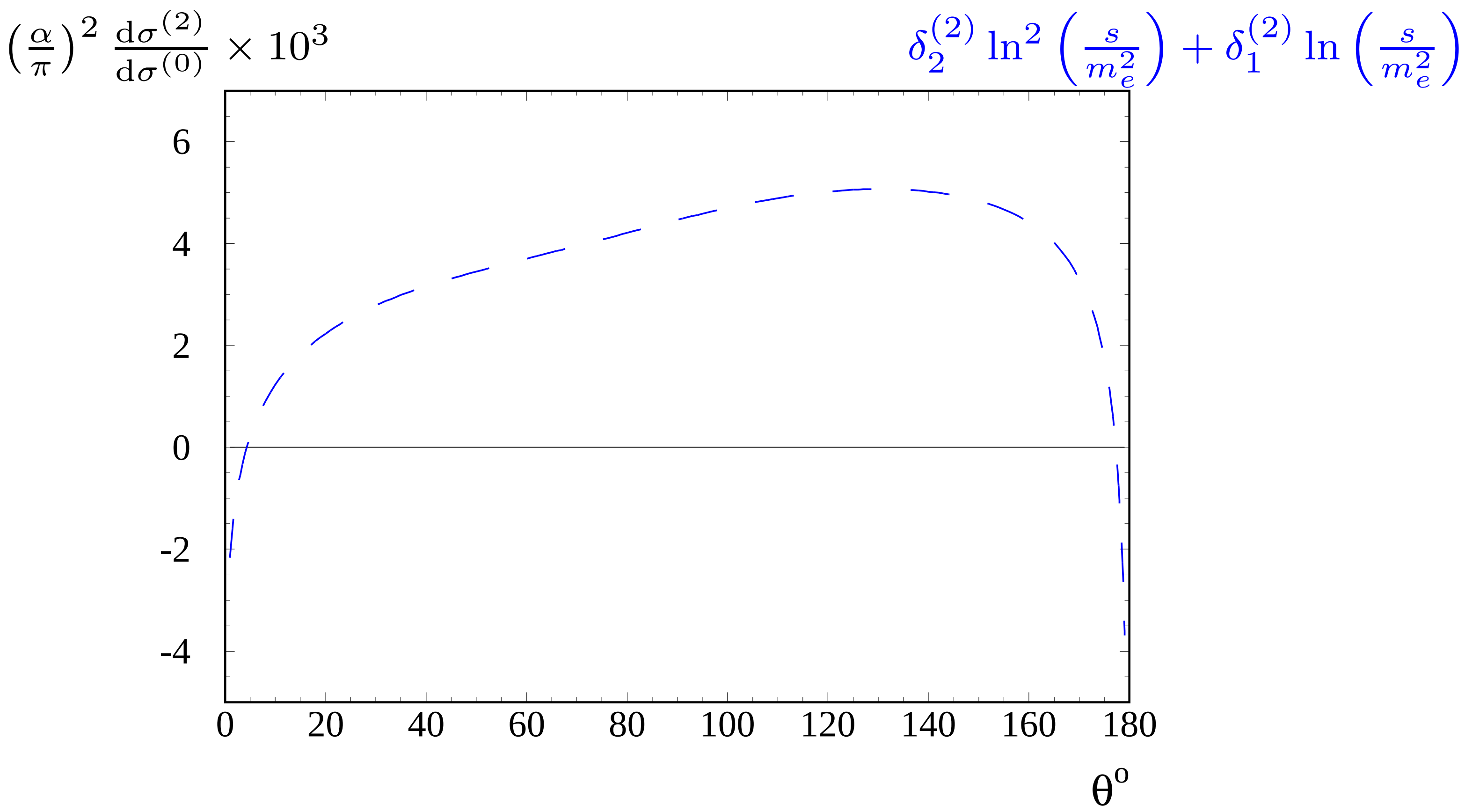

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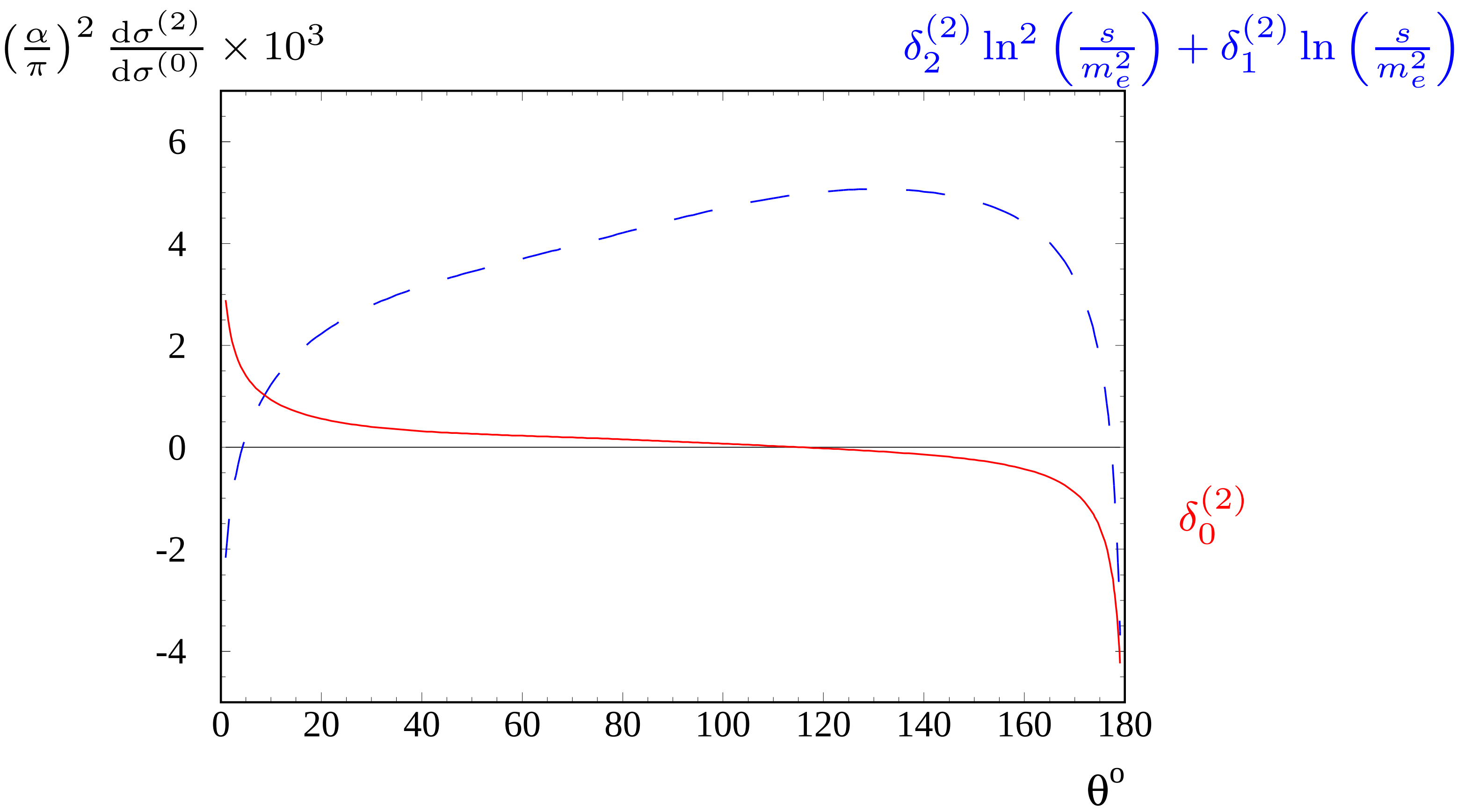

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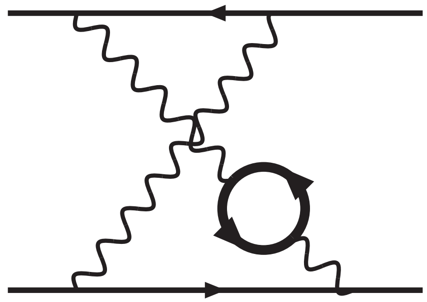
Heavy flavor corrections



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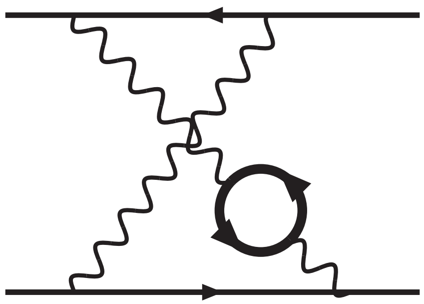


2 loops + 4 legs + 4 scales $s,\ t,\ m_e^2,\ m_f^2$ ⇨ *no chance?*

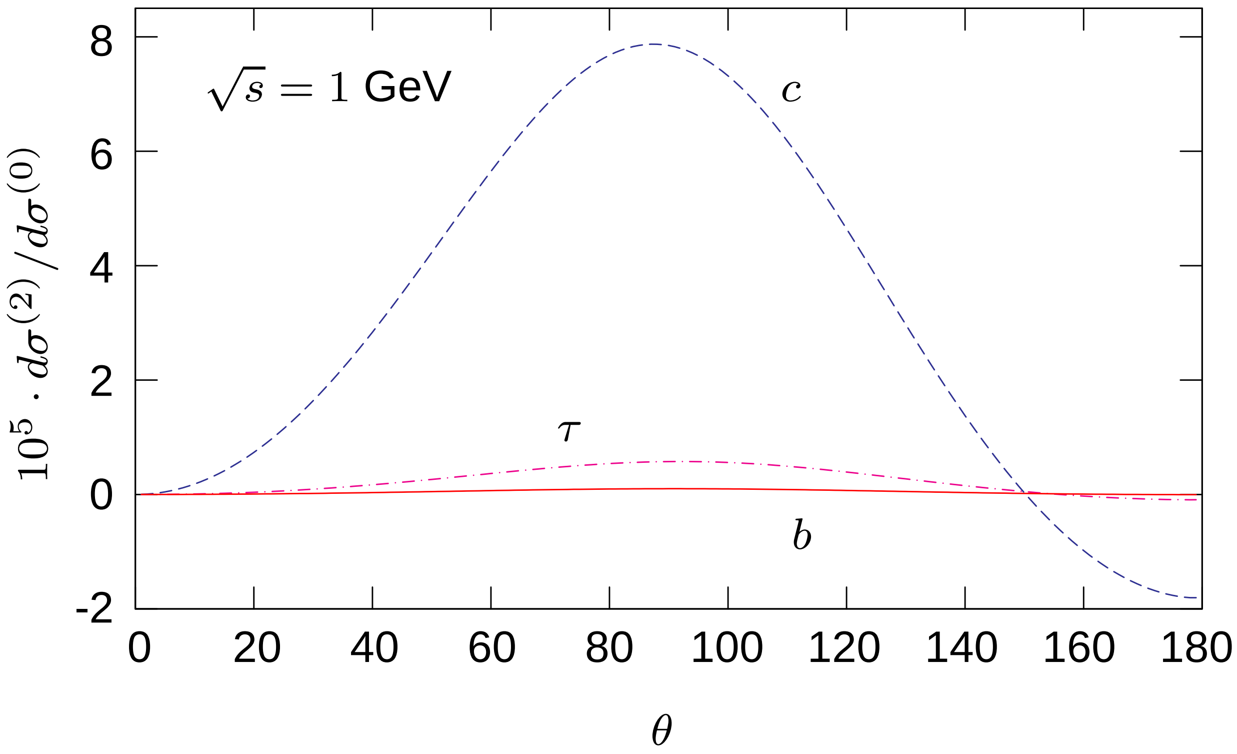
Factorization of collinear divergences

↪ *the sum of 2PI diagrams is finite in the limit* $m_e \to 0$

Heavy flavor corrections

- Calculation
  - $\epsilon \neq 0, \; m_e = 0$
  - *Reduction of FI to MI by Laporta algorithm*
  - *MI by differential equations*
- Result
  - *Generalized harmonic polylogarithms (fully analytical)*

Two-loop HF corrections to LA Bhabha scattering



Summary

- Knowledge of the general infrared structure of QED is extremely helpful in high order calculations
- After long way the two-loop QED corrections to Bhabha scatterng are here
- QED result is already in use by Monte Carlo event generators: *BABAYAGA, LABSMC, SAMBHA*

Summary

- Knowledge of the general infrared structure of QED is extremely helpful in high order calculations
- After long way the two-loop QED corrections to Bhabha scatterng are here
- QED result is already in use by Monte Carlo event generators: *BABAYAGA, LABSMC, SAMBHA*
  - ⇨ *last version of* BABAYAGA *claims* 2 *pm accuracy*

Problems to solve

- MC event generators
  - *Consistent inclusion of two-loop corrections*
  - *One-loop correction to single emission*
  - *Pair emission*