The (T)DBM equation as a twisted loop Toda system: Generalizations and soliton solutions*

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The equations

- **The (T)DBM equation**

$$\partial_+ \partial_- \phi = -m^2 \left[\exp(-2\phi) - \exp(\phi) \right]$$

Tzitzéica, 1908
Dodd & Bullough, 1977
Mikhailov, 1981

Geometry of hyperbolic surfaces and soliton theory

- **The Toda equation**

$$\partial_+ \left(\gamma^{-1} \partial_- \gamma \right) = \left[c_-, \gamma^{-1} c_+ \gamma \right]$$

$$\gamma = \begin{pmatrix} \Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_p \end{pmatrix}, \qquad \Gamma_\alpha : \mathbb{R}^2 \to \mathrm{GL}_{n_\alpha}(\mathbb{C})$$

$$n_1 + n_2 + \ldots + n_p = n$$

$$c_+ = \begin{pmatrix} 0 & C_{+1} & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & C_{+(p-1)} \\ C_{+0} & & & & 0 \end{pmatrix}, \qquad \partial_- C_{+\alpha} = 0$$

$$c_- = \begin{pmatrix} 0 & & & & C_{-0} \\ C_{-1} & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & C_{-(p-1)} & 0 \end{pmatrix}, \qquad \partial_+ C_{-\alpha} = 0$$

Twisted loop Lie algebras and loop groups

$$\sigma \in S^1, \qquad \varepsilon_M = \mathrm{e}^{2\pi \mathrm{i}/M}, \qquad M := pL, \quad L \to 1$$

$$\mathscr{L}_{A,M}(\mathfrak{gl}_n(\mathbb{C})) \ : \ \xi(\varepsilon_M \sigma) = A(\xi(\sigma)), \quad A^M = \mathrm{id}_{\mathfrak{gl}_n(\mathbb{C})}$$

$$\mathscr{L}_{a,M}(\mathrm{GL}_n(\mathbb{C})) \ : \ \chi(\varepsilon_M \sigma) = a(\chi(\sigma)), \quad a^M = \mathrm{id}_{\mathrm{GL}_n(\mathbb{C})}$$

The exponential law

$$C^\infty(\mathscr{M}, C^\infty(\mathscr{N}, \mathscr{P})) = C^\infty(\mathscr{M} \times \mathscr{N}, \mathscr{P})$$

where $\mathscr{M}$, $\mathscr{N}$, $\mathscr{P}$ are finite-dimensional manifolds, and $\mathscr{N}$ is compact, is useful for a consistent formulation of the Toda equations associated with the loop groups in terms of finite-dimensional manifolds.

Abelian loop Toda systems

- **I. Inner automorphisms,** $p = n$

$$A(x) = hxh^{-1}, \qquad x \in \mathfrak{gl}_n(\mathbb{C}), \qquad h_{ij} = \varepsilon_n^{n-i+1}\delta_{ij}$$

$$a(g) = hgh^{-1}, \qquad g \in \mathrm{GL}_n(\mathbb{C}), \qquad \Gamma_\alpha : \mathbb{R}^2 \to \mathbb{C}^\times$$

$$\partial_+\left(\Gamma_1^{-1}\,\partial_-\Gamma_1\right) = -m^2(\Gamma_1^{-1}\Gamma_2 - \Gamma_p^{-1}\Gamma_1),$$
$$\partial_+\left(\Gamma_2^{-1}\,\partial_-\Gamma_2\right) = -m^2(\Gamma_2^{-1}\Gamma_3 - \Gamma_1^{-1}\Gamma_2),$$
$$\vdots$$
$$\partial_+\left(\Gamma_{n-1}^{-1}\,\partial_-\Gamma_{n-1}\right) = -m^2(\Gamma_{n-1}^{-1}\Gamma_n - \Gamma_{n-2}^{-1}\Gamma_{n-1}),$$
$$\partial_+\left(\Gamma_n^{-1}\,\partial_-\Gamma_n\right) = -m^2(\Gamma_n^{-1}\Gamma_1 - \Gamma_{n-1}^{-1}\Gamma_n)$$

- **II. Outer automorphisms,** $n = 2s-1,\ s \geq 2,\ p = n$

$$A(x) = -h({B^{-1}}^t x B)h^{-1}, \qquad x \in \mathfrak{gl}_n(\mathbb{C}) \qquad h_{ij} = \varepsilon_{4s-2}^{2s-i}\delta_{ij}$$

$$a(g) = h({B^{-1}}^t g^{-1} B)h^{-1}, \qquad g \in \mathrm{GL}_n(\mathbb{C})$$

$$B = \left(\begin{array}{c|ccccccc} 1 & & & & & & & \\ \hline & & & & & & & 1 \\ & & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & \cdot^{\cdot^{\cdot}} & & & & \\ & -1 & & & & & & \end{array}\right)$$

$$\Gamma_1 = 1, \qquad \Gamma_{2s-\alpha+1} = \Gamma_\alpha^{-1}, \quad \alpha = 2, \ldots, s$$

$$C_{\pm 0} = C_{\pm 1} = C_{\pm s} = m,\ C_{\pm\alpha} = -C_{\pm(2s-\alpha)} = m,\ \alpha = 2, \ldots, s-1$$

$$\partial_+(\Gamma_2^{-1}\partial_-\Gamma_2) = -m^2(\Gamma_2^{-1}\Gamma_3 - \Gamma_2),$$
$$\partial_+(\Gamma_3^{-1}\partial_-\Gamma_3) = -m^2(\Gamma_3^{-1}\Gamma_4 - \Gamma_2^{-1}\Gamma_3),$$
$$\vdots$$
$$\partial_+(\Gamma_{s-1}^{-1}\partial_-\Gamma_{s-1}) = -m^2(\Gamma_{s-1}^{-1}\Gamma_s - \Gamma_{s-2}^{-1}\Gamma_{s-1}),$$
$$\partial_+(\Gamma_s^{-1}\partial_-\Gamma_s) = -m^2(\Gamma_s^{-2} - \Gamma_{s-1}^{-1}\Gamma_s)$$

For $s = 2$, denoting Γ_2 by Γ, we have

$$\partial_+(\Gamma^{-1}\partial_-\Gamma) = -m^2(\Gamma^{-2} - \Gamma)$$

Setting $\Gamma = \exp(\phi)$ we obtain the *(T)DBM equation*.

III. Outer automorphisms, $n = 2s,\ s \geq 2,\ p = n-1$

$$A(x) = -h(B^{-1\,t}xB)h^{-1}, \qquad x \in \mathfrak{gl}_n(\mathbb{C})$$

$$a(g) = h(B^{-1\,t}g^{-1}B)h^{-1}, \qquad g \in \mathrm{GL}_n(\mathbb{C})$$

$$h_{11} = \varepsilon_{4s-2}^{2s-1} = -1, \qquad h_{ii} = \varepsilon_{4s-2}^{2s-i+1}, \qquad i = 2, \ldots, n$$

$$B = \left(\begin{array}{cc|cccccc} & 1 & & & & & & \\ 1 & & & & & & & \\ \hline & & & & & & & 1 \\ & & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & \cdot^{\cdot^{\cdot}} & & & & \\ & & -1 & & & & & \end{array}\right)$$

$$J_2^{-1}\,{}^t\Gamma_1\, J_2 = \Gamma_1^{-1}, \qquad \Gamma_{2s-\alpha+1} = \Gamma_\alpha^{-1}$$

$$C_{-\alpha} = m, \qquad C_{+\alpha} = m, \qquad \alpha = 2, \ldots, s$$

$$(C_{-1})_{11} = (C_{-1})_{12} = m/\sqrt{2}, \qquad (C_{+1})_{11} = (C_{+1})_{21} = m/\sqrt{2},$$

$$\partial_+(\Gamma_1^{-1}\partial_-\Gamma_1) = -\,(m^2/2)(\Gamma_1^{-1} - \Gamma_1)\Gamma_2, \qquad [\Gamma_1 \equiv (\Gamma_1)_{11}]$$

$$\partial_+(\Gamma_2^{-1}\partial_-\Gamma_2) = -\,m^2\Gamma_2^{-1}\Gamma_3 + (m^2/2)(\Gamma_1^{-1} + \Gamma_1)\Gamma_2,$$

$$\partial_+(\Gamma_3^{-1}\partial_-\Gamma_3) = -\,m^2(\Gamma_3^{-1}\Gamma_4 - \Gamma_2^{-1}\Gamma_3),$$

$$\vdots$$

$$\partial_+(\Gamma_{s-1}^{-1}\partial_-\Gamma_{s-1}) = -\,m^2(\Gamma_{s-1}^{-1}\Gamma_s - \Gamma_{s-2}^{-1}\Gamma_{s-1}),$$

$$\partial_+(\Gamma_s^{-1}\partial_-\Gamma_s) = -\,m^2(\Gamma_s^{-2} - \Gamma_{s-1}^{-1}\Gamma_s)$$

Another form of writing the Toda equations associated with affine Lie groups:

$$\Gamma_\alpha \to \Delta_\alpha \sim \Delta_0^{\{d\}} \prod_\beta^{\{\alpha\}} \Gamma_\beta$$

leads to

$$\partial_+ \left(\Delta_\alpha^{-1} \partial_- \Delta_\alpha\right) = -m^2 \prod_{\beta=0}^{n-1} \Delta_\beta^{-a_{\alpha\beta}}$$

in case of $A_{n-1}^{(1)}$, the same form with $n \to s$ in case of $A_{2s-2}^{(2)}$, and with $n \to s+1$ for $A_{2s-1}^{(2)}$. Here $(a_{\alpha\beta})$ is the corresponding generalized Cartan matrix.

The rational dressing

- **Constructing dressing mappings**

  Since $[c_-, c_+] = 0$, we can start with the “bare” solution $\gamma = I_n$ and “dress” it by a mapping

$$\Psi : \mathbb{R}^2 \to \mathscr{L}_{a,M}(\mathrm{GL}_n(\mathbb{C})) \Longrightarrow \psi : \mathbb{R}^2 \times S^1 \to \mathrm{GL}_n(\mathbb{C})$$

In case of the twisted loop groups we take *rational mappings* subject to

$$\psi(z, \varepsilon_{4s-2}\sigma) = h\,B^{-1}\,{}^t\psi^{-1}(z,\sigma)\,B\,h^{-1},$$

extend it analytically from S^1 to the whole Riemann sphere, and prove that $\gamma = \psi(z,\infty)$ satisfies the Toda equation. The procedure is required to be consistent with the *grading* and *gauge-fixing* conditions.

General solution

$$\Gamma_1 = I_{n_1} - \sum_{i,j=1}^{r} \mu_i^{2s-1}\, \widetilde{u}_{i,1}\, (\widetilde{R}_1^{-1})_{ij}\, {}^t\widetilde{u}_{j,1}\, J_{n_1},$$

$$\Gamma_\alpha = 1 + \sum_{i,j=1}^{r} \widetilde{u}_{i,\alpha}\, (\widetilde{R}_\alpha^{-1})_{ij}\, \widetilde{u}_{j,2s+1-\alpha}, \qquad \alpha = 2, \ldots, s,$$

$$\Gamma_\alpha = 1 - \sum_{i,j=1}^{r} \widetilde{u}_{i,\alpha}\, (\widetilde{R}_\alpha^{-1})_{ij}\, \widetilde{u}_{j,2s+1-\alpha}, \qquad \alpha = s+1, \ldots, 2s-1$$

$$\widetilde{u}_{i,\alpha} = \mu_i^{\alpha} u_{i,\alpha}, \qquad \partial_- u_i = -\mu_i^{-1} c_- u_i, \qquad \partial_+ u_i = -\mu_i c_+ u_i$$

Here the r complex numbers μ_i are the pole positions of the meromorphic dressing mappings.

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The $r \times r$ matrices $\widetilde{R}_\alpha$ are defined explicitly as

$$(\widetilde{R}_1)_{ij} = \frac{1}{\mu_i^{2s-1} + \mu_j^{2s-1}} \Bigg(\mu_i^{2s-1} ({}^t\widetilde{u}_{i,1} J_{n_1} \widetilde{u}_{j,1}) \mu_j^{2s-1}$$
$$- \mu_j^{2s-1} \sum_{\beta=2}^{s} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} + \mu_j^{2s-1} \sum_{\beta=s+1}^{2s-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} \Bigg)$$

for $\alpha = 1$. Useful (symmetry) properties

$$(\widetilde{R}_1)_{ij} = -(\widetilde{R}_2)_{ji}, \qquad (\widetilde{R}_\alpha)_{ij} = (\widetilde{R}_{2s+2-\alpha})_{ji}$$

$$(\widetilde{R}_\alpha)_{ij} = \frac{1}{\mu_i^{2s-1} + \mu_j^{2s-1}} \Bigg(-\mu_i^{2s-1} ({}^t\widetilde{u}_{i,1} J_{n_1} \widetilde{u}_{j,1}) \mu_j^{2s-1} + \mu_j^{2s-1} \sum_{\beta=2}^{\alpha-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} - \mu_i^{2s-1} \sum_{\beta=\alpha}^{s} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} + \mu_i^{2s-1} \sum_{\beta=s+1}^{2s-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} \Bigg)$$

for $\alpha = 2, \ldots, s$. Useful (shift symmetry) properties

$$(\widetilde{R}_{\alpha+1})_{ij} = (\widetilde{R}_\alpha)_{ij} + \widetilde{u}_{i,2s+1-\alpha} \widetilde{u}_{j,\alpha}$$

$$(\widetilde{R}_\alpha)_{ij} = \frac{1}{\mu_i^{2s-1} + \mu_j^{2s-1}} \Bigg(-\mu_i^{2s-1} ({}^t\widetilde{u}_{i,1} J_{n_1} \widetilde{u}_{j,1}) \mu_j^{2s-1} + \mu_j^{2s-1} \sum_{\beta=2}^{s} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} - \mu_j^{2s-1} \sum_{\beta=s+1}^{\alpha-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} + \mu_i^{2s-1} \sum_{\beta=\alpha}^{2s-1} \widetilde{u}_{i,2s+1-\beta} \widetilde{u}_{j,\beta} \Bigg)$$

for $\alpha = s+1, \ldots, 2s-1$. Useful (shift symmetry) properties

$$(\widetilde{R}_{\alpha+1})_{ij} = (\widetilde{R}_\alpha)_{ij} - \widetilde{u}_{i,2s+1-\alpha} \widetilde{u}_{j,\alpha}$$

These mappings satisfy

$$J_{n_1}^{-1}\,{}^t\Gamma_1\,J_{n_1} = \Gamma_1^{-1}, \qquad \Gamma_{2s+1-\alpha} = \Gamma_\alpha^{-1}, \qquad \alpha = 2,\ldots,2s-1$$

It was shown

$$\Gamma_\alpha = \frac{\det \widetilde{R}_{\alpha+1}}{\det \widetilde{R}_\alpha}, \qquad \alpha = 2,\ldots,2s-1$$

For type II we have $n_1 = 1$ and

$$\Gamma_1 = \frac{\det \widetilde{R}_2}{\det \widetilde{R}_1} = (-1)^r$$

For type III $n_1 = 2$ and Γ_1 is a mapping of $\mathbb{R}^2$ to $\mathrm{SO}_2(\mathbb{C})$.

Soliton solutions

By an N-soliton solution we mean a solution depending on N linear combinations of independent variables.

- **Type II system** $(n = p = 2s-1)$

$$\Gamma_\alpha = \frac{\det T_{\alpha+1}}{\det T_\alpha}, \qquad T_\alpha = D^{-1}(f,f)\widetilde{R}'_\alpha$$

$$(\widetilde{R}'_\alpha)_{ij} = D_{ij}(f,f) + \varepsilon_{4s-2}^{2\rho_i(\alpha-1)} \mathrm{e}^{\widetilde{Z}_i} D_{ij}(\widetilde{f},f) + D_{ij}(f,\widetilde{f}) \mathrm{e}^{\widetilde{Z}_j} \varepsilon_{4s-2}^{-2\rho_j(\alpha-1)} + \varepsilon_{4s-2}^{2\rho_i(\alpha-1)} \mathrm{e}^{\widetilde{Z}_i} D_{ij}(\widetilde{f},\widetilde{f}) \mathrm{e}^{\widetilde{Z}_j} \varepsilon_{4s-2}^{-2\rho_j(\alpha-1)}$$

$$D_{ij}(f,g) = \frac{f_i}{f_i+g_j}, \qquad \widetilde{Z}_i = m\kappa_{\rho_i}(\zeta_i^{-1}z^- + \zeta_i z^+) + \ldots$$

$$\kappa_{\rho_i} = 2\sin\theta_{\rho_i}, \quad \theta_{\rho_i} = \frac{\pi\rho_i}{2s-1}, \qquad f_i = \varepsilon_{4s-2}^{\rho_i}\zeta_i, \quad \widetilde{f}_i = \varepsilon_{4s-2}^{-\rho_i}\zeta_i, \qquad \zeta_i \propto \mu_i$$

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For one-soliton solution we set $r = 1$ (here T_α are ordinary functions)

$$T_{\alpha+1} = 1 + 2\frac{\cos(2\alpha-1)\theta_\rho}{\cos\theta_\rho}\mathrm{e}^{\widetilde{Z}} + \mathrm{e}^{2\widetilde{Z}}$$

For two-soliton solutions we set $r = 2$

$$\begin{aligned}
\det T_{\alpha+1} \quad = \quad & 1+2\frac{\cos(2\alpha-1)\theta_{\rho_1}}{\cos\theta_{\rho_1}}\mathrm{e}^{\widetilde{Z}_1}+2\frac{\cos(2\alpha-1)\theta_{\rho_2}}{\cos\theta_{\rho_2}}\mathrm{e}^{\widetilde{Z}_2}+\mathrm{e}^{2\widetilde{Z}_1}+\mathrm{e}^{2\widetilde{Z}_2} \\
& +\left(2\eta_{12}^{+}\frac{\cos(2\alpha-1)(\theta_{\rho_1}-\theta_{\rho_2})}{\cos\theta_{\rho_1}\cos\theta_{\rho_2}}+2\eta_{12}^{-}\frac{\cos(2\alpha-1)(\theta_{\rho_1}+\theta_{\rho_2})}{\cos\theta_{\rho_1}\cos\theta_{\rho_2}}\right)\mathrm{e}^{\widetilde{Z}_1+\widetilde{Z}_2} \\
& +2\eta_{12}^{+}\eta_{12}^{-}\left(\frac{\cos(2\alpha-1)\theta_{\rho_1}}{\cos\theta_{\rho_1}}\mathrm{e}^{\widetilde{Z}_1+2\widetilde{Z}_2}+\frac{\cos(2\alpha-1)\theta_{\rho_2}}{\cos\theta_{\rho_2}}\mathrm{e}^{2\widetilde{Z}_1+\widetilde{Z}_2}\right) \\
& +\left(\eta_{12}^{+}\eta_{12}^{-}\right)^2\mathrm{e}^{2(\widetilde{Z}_1+\widetilde{Z}_2)}
\end{aligned}$$

where the quantities

$$\eta_{12}^{+} = \frac{(\zeta_1\zeta_2^{-1} + \zeta_2\zeta_1^{-1}) + 2\cos(\theta_{\rho_1} + \theta_{\rho_2})}{(\zeta_1\zeta_2^{-1} + \zeta_2\zeta_1^{-1}) + 2\cos(\theta_{\rho_1} - \theta_{\rho_2})}$$

and

$$\eta_{12}^{-} = \frac{(\zeta_1\zeta_2^{-1} + \zeta_2\zeta_1^{-1}) - 2\cos(\theta_{\rho_1} - \theta_{\rho_2})}{(\zeta_1\zeta_2^{-1} + \zeta_2\zeta_1^{-1}) - 2\cos(\theta_{\rho_1} + \theta_{\rho_2})}$$

make sense as $soliton\ interaction$ factors.

The same could be reached along the lines of the Hirota's approach. The functions $\det T_{\alpha+1}$ of the rational dressing formalism coincide with the functions τ_α of the Hirota's method.

Nirov & Razumov, 2004–2008

Setting $s=2$, so that $n=p=3$ we describe the (T)DBM equation. Here $\Gamma_1=(-1)^r$, $\Gamma_2=\Gamma_3^{-1}\equiv\Gamma$. Besides, $\theta_\rho=\pi\rho/3$, where ρ takes values 0, ±1 and ±2.

In particular, the one-soliton solution can be written in the form

$$\Gamma=\frac{1-4\,\mathrm{e}^{\widetilde{Z}}+\mathrm{e}^{2\widetilde{Z}}}{(1+\mathrm{e}^{\widetilde{Z}})^2}$$

For the two-soliton solution one should respectively simplify the expression of $\det T_{\alpha+1}$ given above for $r=2$.

Geometrically, these soliton solutions describe stationary and moving affine spheres as surfaces immersed in $\mathbb{R}^3$.

Rogers & Schief, 1994–1999

Type III system $(n = p + 1 = 2s)$

In this case we have two different classes of soliton solutions.

One is the same as in the odd-dimensional case, save that here $\Gamma_1 = (-1) J_2^r$, also to be transformed to I_2.

Another (new) class of soliton solutions is constructed by taking into account the null-eigenspace of c_- and c_+.

$$\Gamma_1 = I_2 + \sum_{i,j=1}^{r} v_i \, (\widetilde{R}_1'^{-1})_{ij} \, {}^t v_j \, J_2$$

where v_i are 2-dimensional column vectors with components

$$v_{i,0} = \frac{1}{\sqrt{2}} (1 + \exp(-Z_i')), \qquad v_{i,1} = -\frac{1}{\sqrt{2}} (1 - \exp(-Z_i'))$$

and

$$\Gamma_\alpha = \frac{\det T_{\alpha+1}}{\det T_\alpha}, \qquad \alpha = 2, \ldots, s$$

Here we have used the notations

$$(\widetilde{R}'_1)_{ij} = D_{ij}(\zeta^{2s-1}, \zeta^{2s-1}) - \mathrm{e}^{-Z'_i} D_{ij}(\zeta, \zeta) \mathrm{e}^{-Z'_j}$$

$$(\widetilde{R}'_\alpha)_{ij} = D_{ij}(\zeta^{2s-1}, \zeta^{2s-1}) - (-1)^\alpha \zeta_i^{2s-\alpha} \mathrm{e}^{-Z'_i} D_{ij}(\zeta, \zeta) \mathrm{e}^{-Z'_j} \zeta_j^{\alpha-2s}$$

for the matrix elements of the $r \times r$ matrices $\widetilde{R}'_1$ and $\widetilde{R}'_\alpha$, with

$$T_\alpha = D^{-1}(\zeta^{2s-1}, \zeta^{2s-1}) \widetilde{R}'_\alpha$$

and now

$$Z'_i = m(\zeta_i z^- + \zeta_i z^+) - \ldots$$

Setting $r = 1$ we obtain a novel one-soliton solution

$$\Gamma_1 = \begin{pmatrix} 0 & \Gamma \\ \Gamma^{-1} & 0 \end{pmatrix}, \qquad \Gamma = \frac{1 + \exp(-Z')}{1 - \exp(-Z')}$$

and

$$\Gamma_\alpha = \frac{1 + (-1)^\alpha \exp(-2Z')}{1 - (-1)^\alpha \exp(-2Z')}, \qquad \alpha = 2, \ldots, s$$

Apart from $\Gamma_{2s+1-\alpha} = \Gamma_\alpha^{-1}$ here we also have $\Gamma_{\alpha+1} = \Gamma_\alpha^{-1}$.

Setting $r = 2$ we obtain new two-soliton solutions

$$\Gamma_1 = \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma^{-1} \end{pmatrix}, \qquad \Gamma = \frac{1 + \mathrm{e}^{-\widetilde{Z}_1} - \mathrm{e}^{-\widetilde{Z}_2} - \eta_{12}\, \mathrm{e}^{-(\widetilde{Z}_1 + \widetilde{Z}_2)}}{1 - \mathrm{e}^{-\widetilde{Z}_1} + \mathrm{e}^{-\widetilde{Z}_2} - \eta_{12}\, \mathrm{e}^{-(\widetilde{Z}_1 + \widetilde{Z}_2)}}$$

where the “solitons-interaction” factor and the exponents are

$$\eta_{12} = \frac{\zeta_1 - \zeta_2}{\zeta_1 + \zeta_2} \cdot \frac{\zeta_1^{2s-1} - \zeta_2^{2s-1}}{\zeta_1^{2s-1} + \zeta_2^{2s-1}}, \qquad \widetilde{Z}_i = Z_i' - \delta'$$

for some constant δ'. To define Γ_α for $\alpha = 2, \dots, s$ we have

$$\det T_{\alpha+1} = 1 + (-1)^\alpha \, (\mathrm{e}^{-2\widetilde{Z}_1} + \mathrm{e}^{-2\widetilde{Z}_2}) - 4\,(-1)^\alpha \, \frac{\zeta_1^\alpha \zeta_2^{2s-\alpha} + \zeta_2^\alpha \zeta_1^{2s-\alpha}}{(\zeta_1 + \zeta_2)(\zeta_1^{2s-1} + \zeta_2^{2s-1})} \, \mathrm{e}^{-(\widetilde{Z}_1 + \widetilde{Z}_2)} + \eta_{12}^2 \, \mathrm{e}^{-2(\widetilde{Z}_1 + \widetilde{Z}_2)}$$

*) $(\zeta_1 \leftrightarrow \zeta_2) \Longrightarrow (\Gamma \leftrightarrow \Gamma^{-1})$

Concluding remarks

- Our construction is based on the classification of Toda equations associated with loop groups of complex classical Lie groups whose Lie algebras are endowed with *integrable $\mathbb{Z}$-gradations* with finite-dimensional grading subspaces (Nirov & Razumov, 2005–2007).
- Although initially there were no group and algebra defining conditions in the case of general linear groups, certain restrictions were to be imposed on the mappings entering the loop Toda equation just due to the specific structure of the outer automorphisms generating the $\mathbb{Z}$-gradations.
- As a consequence, unlike the untwisted case, the pole positions of the dressing meromorphic mappings turned out to be related in the twisted loop group case.