First Order String Theory: Nonlinear Background Equations

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Motivations: serious problems with string theory in nontrivial backgrounds

- Non-linear two-dimensional sigma-model

$$\int_\Sigma d^2z\ G_{\mu\nu}(X)\partial X^\mu \bar{\partial} X^\nu + \ldots$$

- Dependence of quantization upon the choice of target-space co-ordinates;

Why the first-order theory instead?

- Sigma-model: typical expansion around the point where $G_{\mu\nu} = \delta_{\mu\nu}$, far from singular backgrounds;

- No off-shell formulation, ($k^2 = 0$ for the photons and gravitons).

One of alternatives: to consider the *first*-order action

$$S_0 = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z(p_i\bar{\partial}X^i + p_{\bar{i}}\partial X^{\bar{i}}) \qquad (1)$$

with the OPE

$$X^i(z)p_j(z') \sim \frac{\alpha'\delta^i_j}{z-z'} + \ldots \qquad (2)$$

Here $i, j = 1, \ldots, D/2$ and $\bar{i}, \bar{j} = 1, \ldots, D/2$.

An analog of the Hamiltonian formalism,
but *requires* some (local !) choice of the target-space complex structure.

- Allows functions $G^{\mu\nu}(X)$ as perturbations, in contrast to sigma-model, where they do not exist off-shell!

- Free action corresponds to $G^{\mu\nu}(X) \to 0$, i.e. to a singular background!

Already some problems in the flat space - "unphysical" mode counting
($\bar{\partial} X^i = 0$ - (holomorphic functions $H^0(\Sigma)$), while
$\bar{\partial} p_i = 0$ - (holomorphic one-forms $H^1(\Sigma)$)),
and extra poles in the correlation functions if
$g_\Sigma > 1$;
however the *gaussian* path integral can be always computed.

Generic perturbation of the singular first-order background by

$$S_{\text{int}} = \int_\Sigma d^2z \Big(g^{i\bar{j}} p_i p_{\bar{j}} + \bar{\mu}_i^{\bar{j}} \partial X^i p_{\bar{j}} + + \mu_{\bar{i}}^j \bar{\partial} X^{\bar{i}} p_j + b_{i\bar{j}} \partial X^i \bar{\partial} X^{\bar{j}} \Big) \tag{3}$$

with $g^{i\bar{j}} = g^{i\bar{j}}(X)$ etc - almost arbitrary (but "transversal" $\partial_i g^{i\bar{j}} = 0$) functions.

The complete set of primary operators

$$\begin{aligned} O_g &= g^{i\bar{j}}(X) p_i p_{\bar{j}} \\ O_b &= b_{i\bar{j}}(X) \partial X^i \bar{\partial} X^{\bar{j}} \end{aligned} \tag{4}$$

and (the real operator)

$$\begin{aligned} \Phi = O_\mu + O_{\bar{\mu}} = \\ = \bar{\mu}_i^{\bar{j}}(X) \partial X^i p_{\bar{j}} + \mu_{\bar{i}}^{j}(X) \bar{\partial} X^{\bar{i}} p_j \end{aligned} \tag{5}$$

Nonvanishing components of the Zamolodchikov metric: $\langle O_g O_b \rangle$ and $\langle \Phi \Phi \rangle$.

Relation to the "physical" background fields

$$
\begin{aligned}
G_{k\bar{l}} &= g_{\bar{i}j}\bar{\mu}^{\bar{i}}_{k}\mu^{j}_{\bar{l}} + g_{k\bar{l}} - b_{k\bar{l}} \\
B_{k\bar{l}} &= g_{\bar{i}j}\bar{\mu}^{\bar{i}}_{k}\mu^{j}_{\bar{l}} - g_{k\bar{l}} - b_{k\bar{l}} \\
G_{ki} &= -g_{i\bar{j}}\bar{\mu}^{\bar{j}}_{k} - g_{k\bar{j}}\bar{\mu}^{\bar{j}}_{i} \\
B_{ki} &= g_{k\bar{j}}\bar{\mu}^{\bar{j}}_{i} - g_{i\bar{j}}\bar{\mu}^{\bar{j}}_{k} \\
\phi &= \log\sqrt{g}
\end{aligned}
\tag{6}
$$

i.e. to $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ from the bosonic string spectrum.

First four relations are classical, while the dilaton

$$\phi \propto \log \sqrt{g}$$

arises due to *anomaly*, e.g.

$$\int DpD\bar{p}\ e^{-S_g[X,\bar{X},p,\bar{p}]} \sim \\ \sim e^{-\mathcal{S}[X,\bar{X}]+\frac{1}{2\pi}\int_\Sigma d^2z\sqrt{h}R\log\sqrt{g}} \tag{7}$$

when coming back to the second-order theory.

The source of anomaly: holomorphic (target-space!) reparameterization $\delta X^i = \epsilon v^i(X)$, $\delta p_i = -\epsilon p_j \frac{\partial v^j}{\partial X^i}$with the anomalous current $p_i v^i(X)$

$$\bar{\partial} \left\langle p_i v^i(X) \right\rangle = \frac{1}{2\pi} R \ \partial_i v^i(X) = \frac{1}{2\pi} R \ \mathcal{L}_v \log \Omega \tag{8}$$

where $\Omega \propto \bigwedge_{i=1}^{D/2} dX^i$ is the holomorphic top-form.

Different measures in the path integrals, or

$$\begin{aligned} \#(p) - \#(X) = \mathrm{dim} H^1(\Sigma) - \mathrm{dim} H^0(\Sigma) = \\ = g_\Sigma - 1 \propto \int_\Sigma \sqrt{h} R \end{aligned} \tag{9}$$

Algebra: the charges

$$\begin{aligned} n_v &= \frac{1}{2\pi i\alpha'}\oint_{S^1} dz\ v^i(X)p_i \\ r_\omega &= \frac{1}{2\pi i\alpha'}\oint_{S^1} dz\ \omega_i(X)\partial X^i \end{aligned} \tag{10}$$

commute as

$$[n_{v_1}, n_{v_2}] = n_{[v_2,v_1]} + \alpha' r_{\omega(v_1,v_2)} \tag{11}$$

$$[r_\omega, n_v] = r_{\mathcal{L}_v\omega} \tag{12}$$

$$[r_{\omega_1}, r_{\omega_2}] = 0 \tag{13}$$

with extension

$$\omega_n(v_1, v_2) = \tfrac{1}{2}(\partial_k v_2^l \partial_n \partial_l v_1^k - \partial_n \partial_k v_1^l \partial_l v_2^k)$$

and Lie derivative

$$(\mathcal{L}_v \omega)_k = v^i \partial_i \omega_k + \omega_i \partial_k v^i$$

This is an α'-deformation of the semi-direct product!

OPE of two vertex operators

$$O_g(z_1)O_g(z_2) \ \sim \ \frac{\delta_\epsilon g^{k\bar{l}} \ p_k p_{\bar{l}}}{|z_1 - z_2|^2} + \ldots \tag{14}$$

where the beta-function

$$\begin{gathered}\delta_\epsilon g^{k\bar{l}} \sim \alpha'(g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}} - \partial_i g^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}}) + \\ +O(\alpha')\end{gathered} \tag{15}$$

Background equation

$$g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}} - \partial_i g^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}} = 0. \tag{16}$$

- Nonlinear equation (!) already at the first nonvanishing order;

- Corresponds to expansion around a singular background;

- Replaces the *linearized* gauge-fixed Einstein equation of conventional sigma-model.

Upon $\partial_i g^{i\bar{j}} = 0$ it can be rewritten as the system

$$\begin{gathered} R_{\mu\nu} = -\frac{1}{4} H_{\mu\lambda\rho} H_{\nu}^{\lambda\rho} + 2\nabla_\mu \nabla_\nu \phi, \\ \nabla_\mu H^{\mu\nu\rho} - 2(\nabla_\lambda \phi) H^{\lambda\nu\rho} = 0, \\ 4(\nabla_\mu \phi)^2 - 4\nabla_\mu \nabla^\mu \phi + \\ + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0 \end{gathered} \tag{17}$$

of the Einstein equations for the physical fields G, B and ϕ.

Conformal invariance: in BRST-language

$$
\begin{aligned}
Q\Psi &= 0, \\
m_2(\Psi, \Psi) &= 0
\end{aligned}
\tag{18}
$$

with

$$
Q = Q_{BRST} = \oint_{S^1} dz\,(cT + :bc\partial c:) \tag{19}
$$

and $[Q, cn_v] = 0$, $[Q, cr_\omega] = 0$. Bilinear operation m_2 extracts the coefficients from OPE of two vertex operators, as was done above.

Generally

$$Q\Psi + m_2(\Psi,\Psi) + m_3(\Psi,\Psi,\Psi) + \ldots = 0 \tag{20}$$

with some *polyvertex* fields $m_n(\Psi,\ldots,\Psi)$.

How to get the polyvertex contributions (higher nonlinear terms for the beta-functions)?

Consider the 3-point function

$$\begin{gathered}\langle O_g(x)\Phi(y)\int_\Sigma d^2z\Phi(z)\rangle = \\ = 2\frac{g^{i\bar{j}}\beta_{i\bar{j}}}{|x-y|^2}\int_\Sigma \frac{d^2z}{\pi}\frac{1}{|x-z|^2|y-z|^2}\end{gathered} \tag{21}$$

where

$$g^{i\bar{j}}\beta_{i\bar{j}} = g^{k\bar{k}}\partial_{[i}\bar{\mu}^{\bar{j}}_{k]}\partial_{[\bar{k}}\mu^{i}_{\bar{j}]} \tag{22}$$

The logarithmic divergence

$$\int_{|z-x,y|>\epsilon} \frac{d^2z}{\pi} \frac{1}{|x-z|^2|y-z|^2} \simeq \\ \simeq \frac{2}{|x-y|^2} \log \frac{|x-y|^2}{\epsilon^2} \tag{23}$$

gives contribution to the beta-function

$$\delta b_{k\bar{k}} = B_{k\bar{k}} = \beta_{k\bar{k}} + O(\mu^3) + O(\mu^4) \equiv \\ \equiv B^{(2)}_{k\bar{k}} + B^{(3)}_{k\bar{k}} + O(|\mu|^4) \tag{24}$$

which looks like the *squared* Kodaira-Spencer equations.

To extract the $O(\mu^3)$ contribution compute the 4-point function

$$\int_{\Sigma^{\otimes 2}} d^2z d^2w \langle O_g(x)\Phi(y)\Phi(z)\Phi(w)\rangle = \\ = \frac{2}{|x-y|^2}\int \frac{d^2z}{\pi}\frac{1}{|x-z|^2|y-z|^2}\int d^2\zeta f(\zeta) \tag{25}$$

with the double cross-ratio

$$\zeta = \frac{(x-w)(y-z)}{(x-z)(y-w)} \tag{26}$$

Explicit computation (free theory with the action

$$S_0 = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z(p_i\bar{\partial}X^i + p_{\bar{i}}\partial X^{\bar{i}})$$

and integration by parts over the zero mode $\int d^D X F[g,\mu;\partial g,\partial\mu;\ldots]$) leads to

$$\int d^2\zeta f(\zeta) = \partial_{[l}\bar{\mu}^{\bar{k}}_{i]}\mu^k_{[\bar{k}}\partial_k\mu^l_{\bar{j}]} + c.c. = B^{(3)}_{i\bar{j}} \qquad (27)$$

i.e. exactly the $O(\mu^3)$ contribution to the squared Kodaira-Spencer equation.

Checked and corrected by the background field computation

$$
\begin{aligned}
g^{i\bar{j}} B_{i\bar{j}} &= g^{i\bar{j}} F^{l}_{\bar{k}\bar{j}} \bar{F}^{\bar{l}}_{ki} M^{k}_{l} \bar{M}^{\bar{k}}_{\bar{l}} = \\
&= g^{i\bar{j}} \left(B^{(2)}_{i\bar{j}} + B^{(3)}_{i\bar{j}} \right) + O(|\mu|^4)
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
\bar{F}^{\bar{j}}_{ik} &\equiv \partial_{[i} \bar{\mu}^{\bar{j}}_{k]} - \bar{\mu}^{\bar{l}}_{[i} \partial_{\bar{l}} \bar{\mu}^{\bar{j}}_{k]} \\
M &= (1 - \mu\bar{\mu})^{-1}
\end{aligned}
\tag{29}
$$

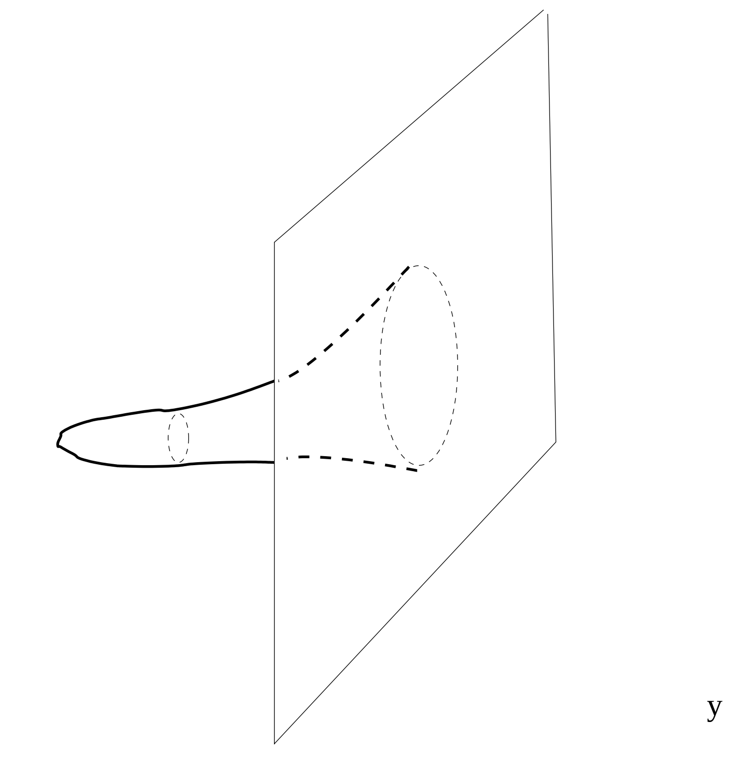
Properties of the first-order theory:

- Disappearance of the on-shell condition (as a linear equation on vertex operator);

- Disappearance of the higher-spin fields or Regge descendants from the theory (anomalous dimensions cannot compensate the dimensional polynomials of the derivatives of the co-ordinate fields);

- Certain “relation” to the high-spin gauge theories: the tower of massive high spin states dissapear, but still with the same central charge $c = D$, but “functionally many” light primary fields;

- Also reminds a little the AdS-like gauge/string duality.



being similar to the (open string) phenomenon on the D-brane near the AdS throat, where metric is infinite (singular background!).

Problems and applications:

- Global properties of the first-order systems;

- Relation with bosonization of the WZW theories, or the first-order systems on flag manifolds;

- Bosonization:

$$p \sim e^{-u} \partial \xi \sim i \partial v e^{-u+iv}$$
$$X \sim e^{u} \eta \sim e^{u-iv} \tag{30}$$

Still no canonical way of constructing multiloop measure for the 10d superstring!

- Relation with the Berkovits formulation of ten-dimensional superstring;

  First-order systems: the "pure spinors"

  $$Q_m(X) = \gamma^m_{\alpha\beta} X^\alpha X^\beta = 0$$

  i.e. set of quadrics $\{Q_m\}$ in $\mathbb{P}^{2^{D/2}-1}$

Conclusions

- First-order CFT as expansion in the singular background;

- Background equations of motion: immediately nonlinear, some interesting properties;

- Polyvertex structures and beta-functions: the Kodaira-Spencer equation from the multipoint correlation functions.