A singularity in dimensional regularization

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Dimensional regularization [1] is at present the only regularization which is suitable for practical multi-loop calculations in the Standard $SU(3) \times SU(2) \times U(1)$ Model of strong and electroweak interactions, see e.g. [2]. Hence to study its features is of importance. In the present paper we consider a dimensionally regularized two-loop integral which value depends on the order of calculation's steps. The integral is

(1)
$$I = \int \frac{d^D p d^D k}{(p+k)^{2\alpha}(p+q)^{2\beta} p^{2\gamma}},$$

$$\alpha + \beta + \gamma = D,\ \alpha \neq D/2,$$

where D is the dimension of the momentum space.

Let us first perform integration over the momentum k. One can use the known property of dimensional regularization to nullify massless vacuum integrals (massless tadpoles)

$$\int \frac{d^D k}{(p+k)^{2\alpha}} = 0, \quad \alpha \neq D/2. \tag{2}$$

Hence one obtains the value $I = 0$.
One can change the order of integrations in (1) and integrate first over p.

Here we use the so called uniqueness relation for the triangle diagram [3]

(3)
$$\int \frac{d^D p}{(p+k)^{2\alpha}(p+q)^{2\beta}p^{2\gamma}} =$$

$$\pi^{D/2}\frac{\Gamma(D/2-\alpha)\Gamma(D/2-\beta)\Gamma(\alpha+\beta-D/2)}{\Gamma(D-\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)}$$

$$\frac{1}{(k-q)^{2(\alpha+\beta-D/2)}k^{2(D/2-\beta)}q^{2(D/2-\alpha)}},$$

$$\alpha+\beta+\gamma = D.$$

This relation is obtained by the inversion of momenta $p_\mu = p'_\mu / p'^2$ (and the same for k and q) after which the integrand has only two propagators and integration is easily performed. If $\alpha + \beta + \gamma \neq D$ then the expression for the triangle diagram (3) is much more complicated [4].

Now we can perform integration of the expression (3) over k to obtain the second value for the integral (1)

$$I = \pi^D \frac{\Gamma(D/2-\alpha)\Gamma(\alpha-D/2)}{\Gamma(\alpha)\Gamma(D-\alpha)}. \tag{4}$$

The dependence of the value of I on the order of integrations over momenta k and p appears on the surface $\alpha+\beta+\gamma = D$. For $\alpha+\beta+\gamma \neq D$ one obtains $I = 0$ in both cases.

Let us show this. We apply the Feynman reprezentation to obtain

$$I = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dx\, x \int_0^1 dy\, (xy)^{\alpha-1} \left[x(1-y)\right]^{\beta-1} (1-x)^{\gamma-1}$$

(5)

$$\int \frac{d^D p d^D k}{\left[xy(p+k)^2 + x(1-y)(p+q)^2 + (1-x)p^2\right]^{\alpha+\beta+\gamma}}.$$

Performing integration over p one gets

$$I = \pi^{D/2} \frac{\Gamma(\alpha+\beta+\gamma-D/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dx\, x \int_0^1 dy\, (xy)^{\alpha-1} \left[x(1-y)\right]^{\beta-1}$$

(6)

$$\int \frac{d^D k}{\left[(xyk + x(1-y)q)^2 - xyk^2 - x(1-y)q^2\right]^{\alpha+\beta+\gamma-D/2}}.$$

Integration over k gives

(7) $$I = \pi^D \frac{\Gamma(\alpha+\beta+\gamma-D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dx\; x^{D/2-\gamma-1}(1-x)^{D-\alpha-\beta-1}$$

$$\int_0^1 dy\; y^{\alpha-1-D/2}(1-y)^{D-\alpha-\gamma-1}(1-xy)^{\alpha+\beta+\gamma-3D/2}\frac{1}{q^{2(\alpha+\beta+\gamma-D)}}.$$

Performing now integrations over y and x we obtain

$$I = \pi^D \frac{\Gamma(\alpha+\beta+\gamma-D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \frac{\Gamma(\alpha-D/2)\Gamma(D-\alpha-\gamma)}{\Gamma(D/2-\gamma)} \frac{1}{q^{2(\alpha+\beta+\gamma-D)}}$$

(8)

$$\int_0^1 dx\ x^{D/2-\gamma-1}(1-x)^{D-\alpha-\beta-1}$$

$${}_2F_1(3D/2-\alpha-\beta-\gamma, \alpha-D/2; D/2-\gamma; x) =$$

$$\pi^D \frac{\Gamma(\alpha+\beta+\gamma-D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \frac{\Gamma(D-\alpha-\gamma)\Gamma(D-\alpha-\beta)}{\Gamma(3D/2-\alpha-\beta-\gamma)} \frac{1}{q^{2(\alpha+\beta+\gamma-D)}}$$

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha-D/2+n)}{n!}.$$

The sum is known

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha - D/2 + n)}{n!} = \Gamma(\alpha - D/2)\delta_K(\alpha - D/2), \tag{9}$$

where δ_K is the Kronecker delta-function:
$\delta_K(x) = 0, x \neq 0; \delta_K(0) = 1$.
Thus one obtains I=0 for $\alpha \neq D/2, \alpha + \beta + \gamma \neq D$ independently on the order of integrations.

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