BRST approach to Lagrangian Construction for Massive Higher Spin Fields

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The talk is based on

I.L. Buchbinder, V.A. Krykhtin

Gauge invariant Lagrangian construction for massive bosonic higher spin fields in D dimentions, Nucl. Phys. B727 (2005) 536

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Gauge invariant Lagrangian construction for massive higher spin fermionic fields, Phys. Lett. B641 (2006) 386.

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Gauge invariant Lagrangian formulation of higher massive bosonic field theory on AdS space, Nucl. Phys. B762 (2007) 344.

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BRST approach to Lagrangian construction for fermionic higher spin fields in AdS space, Nucl. Phys. B787 (2007) 211.

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Gauge invariant Lagrangian construction for massive bosonic mixed symmetry higher spin fields, Phys. Lett. B656 (2007) 253.

Plan of the talk

1. Brief description of the procedure

2. Lagrangian construction for the bosonic fields in Minkowski space

3. Comments on Lagrangian construction for the fermionic fields in Minkowski space

4. Comments on Lagrangian construction for the fields in AdS space

5. Comments on Lagrangian construction for the fields corresponding to arbitrary Young tableau

BRST-BFV approach to gauge theories

Lagrangian is known

$\Rightarrow$ We find Hamiltonian and constraints T_a, $[T_a, T_b] = f^c_{ab} T_c$

$\Rightarrow$ BRST-BFV charge is introduced according to the rule

$$Q = \eta^a T_a + \frac{1}{2} \eta^b \eta^a f^c_{ab} \mathcal{P}_c, \qquad Q^2 = 0,$$

where η^a and $\mathcal{P}_a$ are canonically conjugate ghost variables.

$\Rightarrow$ After quantization the BRST-BFV charge becomes a Hermitian operator acting in extended space of states including ghost operators. *The physical states in the extended space are defined by the equation*

$$Q|\Psi\rangle = 0.$$

Due to the nilpotency of the BRST-BFV operator, $Q^2 = 0$, *the physical states are defined up to transformation*

$$|\Psi'\rangle = |\Psi\rangle + Q|\Lambda\rangle$$

which is treated as a gauge transformation.

BRST-BFV approach to construction of Lagrangians

Application of BRST-BFV construction in the higher spin field theory looks inverse to above quantization problem.

Lagrangian is unknown.
The initial point is equations defining the irreducible representations of Poincare or AdS groups with definite spin and mass

$\Rightarrow$ we construct BRST-BFV charge $\Rightarrow$ we find Lagrangian

BRST-BFV approach to construction of Lagrangians

Generic procedure looks as follows.

The equations defining the irreducible representation are treated as the operators of first class constraints in some auxiliary Fock space.

However a part of these constraints are non-Hermitian operators and in order to construct a Hermitian BRST-BFV operator we have to involve the operators which are Hermitian conjugate to the initial constraints and which are not constraints.

Then for closing the algebra to the complete set of operators we must add some more operators which are not constraints as well.

Because of presence of such operators (nonconstraints) the standard BRST-BFV construction can not be applied it its literal form.

However, as we will see, this problem can be solved.

Masive bosonic field in Minkowski space

The totally symmetric tensor field $\Phi_{\mu_1 \cdots \mu_s}$, describing the irreducible spin-s massive representation of the Poincare group must satisfy the following constraints

$$(\partial^2 + m^2)\Phi_{\mu_1 \cdots \mu_s} = 0, \quad \partial^{\mu_1}\Phi_{\mu_1 \mu_2 \cdots \mu_s} = 0, \quad \eta^{\mu_1 \mu_2}\Phi_{\mu_1 \cdots \mu_s} = 0. \quad (1)$$

In order to avoid manipulation with a number of indices it is convenient to introduce Fock space generated by creation and annihilation operators a_μ^+, a_μ ($\mu = 0, 1, 2, \ldots, d-1$) satisfying the commutation relations

$$\left[a_\mu, a_\nu^+\right] = -\eta_{\mu\nu}, \qquad \eta_{\mu\nu} = (+, -, \ldots, -).$$

Then we define the operators ($p_\mu = -i\frac{\partial}{\partial x^\mu}$)

$$l_0 = -p^2 + m^2, \qquad l_1 = a^\mu p_\mu, \qquad l_2 = \frac{1}{2}a^\mu a_\mu.$$

These operators act on states in the Fock space

$$|\Phi\rangle \;=\; \sum_{s=0}^{\infty} \Phi_{\mu_1\cdots\mu_s}(x)a^{\mu_1+}\ldots a^{\mu_s+}|0\rangle \tag{2}$$

which describe all integer spin fields simultaneously if the following constraints on the states take place

$$l_0|\Phi\rangle = 0, \qquad l_1|\Phi\rangle = 0, \qquad l_2|\Phi\rangle = 0. \tag{3}$$

If constraints (3) are fulfilled for the general state (2) then equations (1) are fulfilled for each component $\Phi_{\mu_1\cdots\mu_s}(x)$ in (2) and hence the relations (3) describe all free massive higher spin bosonic fields simultaneously.

In order to have real Lagrangian the BRST operator must be a Hermitian operator.

It means that the set of operators underlying the BRST operator must
1) be invariant under Hermitian conjugation
2) form an algebra $[l_i, l_j] \sim l_k$.

In the case under consideration the constraint l_0 is Hermitian, $l_0^+ = l_0$, however the constraints l_1, l_2 are not Hermitian.

We extend the set of the constraints l_0, l_1, l_2 adding two new operators

$$l_1^+ = a^{\mu+} p_\mu, \qquad l_2^+ = \frac{1}{2} a^{\mu+} a_\mu^+. \tag{4}$$

As a result, the set of operators l_0, l_1, l_2, l_1^+, l_2^+ is invariant under Hermitian conjugation.

We want to point out that operators l_1^+, l_2^+ are not constraints on the space of ket-vectors (2) since they may not annihilate the physical states. Taking Hermitian conjugation of (3) we see that l_1^+, l_2^+ (together with l_0) are constraints on the space of bra-vectors

$$\langle\Phi|l_0 = \langle\Phi|l_1^+ = \langle\Phi|l_2^+ = 0. \qquad (5)$$

The set of the operators l_0, l_1, l_1^+, l_2, l_2^+ does not form an algebra.

To get an algebra we add to the above set of operators all operators generated by the commutators of l_0, l_1, l_1^+, l_2, l_2^+. Doing such a way we obtain two new operators

$$m^2 \qquad \text{and} \qquad g_0 = -a_\mu^+ a^\mu + \frac{d}{2}. \qquad (6)$$

Now the set of opertors

$$l_0, \quad l_1, \quad l_1^+, \quad l_2, \quad l_2^+, \quad g_0, \quad m^2$$

is invariant under Hermitian conjugation and form an algebra.

Let us emphasize once again that operators l_1^+, l_2^+ are not constraints on the space of ket-vectors. The constraints in space of ket-vectors are l_0, l_1, l_2 and they are the first class constraints in this space. Analogously, the constraints in space of bra-vectors are l_0, l_1^+, l_2^+ and they also are the first class constraints but only in this space, not in space of ket-vectors.

Since the operator m^2 is obtained from the commutator

$$[l_1, l_1^+] = l_0 - m^2, \tag{7}$$

where l_1 is a constraint in the space of ket-vectors and l_1^+ is a constraint in the space of bra-vectors, then m^2 can not be regarded as a constraint neither in the ket-vector space nor in the bra-vector space.

Analogously the operator g_0 is obtained from the commutator

$$[l_2, l_2^+] = g_0, \tag{8}$$

where l_2 is a constraint in the space of ket-vectors and l_2^+ is a constraint in the space of bra-vectors. Therefore g_0 can not also be regarded as a constraint neither in the ket-vector space nor in the bra-vector space.

One can show that a straightforward use of BRST-BFV construction as if all the operators l_0, l_1, l_2, l_1^+, l_2^+, g_0, m^2 are the first class constraints doesn't lead to the proper equations (3). This happens because among the above hermitian operators there are operators which are not constraints (g_0 and m^2 in the case under consideration) and they bring two more equations (in addition to (3)) onto the physical field (2).

Thus in order to reproduce equations of motion

$$l_0|\Phi\rangle = 0, \qquad l_1|\Phi\rangle = 0, \qquad l_2|\Phi\rangle = 0$$

we must somehow get rid of the supplementary equations generated by Hermitian operators g_0 and m^2.

The method of avoiding the supplementary equations consists in constructing new enlarged expressions for the operators of the algebra, so that *the operators which are not constraints will not give any supplementary equations on the physical field.*

Let us act as follows.

We enlarge the representation space of the operator algebra by introducing additional (new) creation and annihilation operators and enlarge expressions for the operators

$$l_i \longrightarrow L_i = l_i + l'_i, \qquad\qquad l_i = \{l_0, l_1, l_1^+, l_2, l_2^+, g_0, m^2\}$$

The enlarged operators must satisfy two conditions:

1) They must form an algebra $[L_i, L_j] \sim L_k$;
2) The operators which can't be regarded as constraints must be zero or contain arbitrary parameters whose values will be defined later from the condition of reproducing the correct equations of motion.

In the case of higher spin fields in Minkowski space the algebra of the operators is a Lie algebra

$$[l_i, l_j] = f_{ij}^k \, l_k. \tag{9}$$

Since we suppose to construct the additional parts of the operators l'_j in terms of new creation and annihilation operators and constants of the theory then they shall be commute with the initial operators l_i

$$[l_i, l'_j] = 0.$$

Therefore we have

$$[L_i, L_j] \;=\; [l_i, l_j] + [l'_i, l'_j] = f_{ij}^k L_k - f_{ij}^k l'_k + [l'_i, l'_j]$$

and in order to provide $[L_i, L_j] \sim L_k$ one must assume that

$$[l'_i, l'_j] = f_{ij}^k \, l'_k.$$

There exists the method which allows us to construct explicit expressions for the operators in terms of creation and annihilation operators on the base of their algebra.

C. Burdik, Realizations Of The Real Simple Lie Algebras: The Method Of Construction, J. Phys A: Math. Gen 18 (1985) 3101;
C. Burdik, A. Pashnev, M. Tsulaia, Auxiliary representations of Lie algebras and the BRST constructions, Mod. Phys. Lett. A15 (2000) 281;
C. Burdik, O. Navratil, A. Pashnev, On the Fock Space Realizations of Nonlinear Algebras Describing the High Spin Fields in AdS Spaces, [hep-th/0206027].

Thus the problem of constructing of the additional parts for the nonlinear algebra (13) can be solved.

Explicit form of the additional parts

$$m'^2 = -m^2, \qquad g'_0 = b_1^+ b_1 + \tfrac{1}{2} + 2b_2^+ b_2 + h,$$

$$l_1'^+ = m b_1^+, \qquad l'_1 = m b_1, \qquad l'_0 = 0,$$

$$l_2'^+ = -\tfrac{1}{2} b_1^{+2} + b_2^+, \qquad l'_2 = -\tfrac{1}{2} b_1^2 + (b_2^+ b_2 + h) b_2.$$

$$[\, b_1, b_1^+] = [\, b_2, b_2^+] = 1.$$

$$M^2 = m^2 + m'^2 = 0 \qquad G_0 = g_0 + g'_0 = h + \ldots$$

Operators l'_2 and $l_2'^+$ are not mutualy conjugate. This makes the BRST operator nonhermitian. In order to restore hermitian properties of the operators we change the scalar product so that

$$\langle \Psi_1 | K_h l'_2 | \Psi_2 \rangle = \langle \Psi_2 | K_h l_2'^+ | \Psi_1 \rangle^*$$

Here K_h is an invertible operator acting as the unit operator in the entire Fock space, but for the sector controlled by the auxiliary creation and annihilation operators used at constructing the additional parts.

This scalar product provides the reality of the Lagrangian.

The BRST-BVF operator is constructed in the usual way.
Now one need to define the arbitrary parameters.
For this we assume that the state vectors $|\Psi\rangle$ in the extended Fock space, including the ghost fields, is independent of the ghosts corresponding to the operators which are not constraints. Let us denote these ghost as η_G, $\mathcal{P}_G$ corresponding to the extended operator $G_0 = g_0 + g_0'$. Thus $\mathcal{P}_G|\Psi\rangle = 0$.
Let us extract the dependence of the BRST-BFV operator on the ghosts η_G, $\mathcal{P}_G$

$$Q' \;=\; Q + \eta_G(\sigma + h) - \eta_2^+\eta_2\mathcal{P}_G$$

where $\sigma + h = G_0 +$ *ghost fields*, with h being the arbitrary parameter to be defined. After this the equation on the physical states in the BRST-BFV approach $Q'|\Psi\rangle = 0$ yields two equations

$$Q|\Psi\rangle = 0, \qquad (\sigma + h)|\Psi\rangle = 0. \tag{10}$$

From the last equation in (10) we find the possible values of h whereas the first equation is equation on the physical state.

The possible values of h

$$-h \;=\; n+\frac{d-5}{2}, \qquad n=0,1,2,\dots\ .$$

The numbers n are related with the spin s of the corresponding eigenvectors as $s=n$.

Let us denote the eigenvectors of the operator σ corresponding to the eigenvalues $n+\frac{d-5}{2}$ as $|\Psi\rangle_n$

$$\sigma|\Psi\rangle_n \;=\; \left(n+\frac{d-5}{2}\right)|\Psi\rangle_n.$$

The solutions to the system of equations (10) are enumerated by $n=0,1,2,\dots$ and satisfy the equations

$$Q_s|\Psi\rangle_s \;=\; 0, \tag{11}$$

where in the BRST operator we substituted $s+\frac{d-5}{2}$ instead of $-h$.

This equation of motion (11) can be obtained from the Lagrangian

$$-\mathcal{L}_s = \int d\eta_0 \ {}_s\langle\Psi|K_s Q_s|\Psi\rangle_s. \tag{12}$$

These Lagrangian (12) and equation of motion (11) are invariant under the reducible gauge transformation

$$\delta|\Psi\rangle_s = Q_s|\Lambda\rangle_s, \qquad gh(|\Lambda\rangle_s) = -1,$$
$$\delta|\Lambda\rangle_s = Q_s|\Omega\rangle_s, \qquad gh(|\Omega\rangle_s) = -2.$$

Since it is not possible to write the state vector with ghost number -3 the order of reducibility of the theory is one.

Lagrangian construction for the fermionic fields

The Lagrangian construction for the fermionic higher spin theories have *two specific differences* compared to the bosonic ones and demands some comments.
One of the specific features consists in that we have the fermionic operators in the algebra of constraints and corresponding them the bosonic ghosts. We can write these ghosts in any power in the Fock space states. Therefore the states in the Fock space can have any ghost number. *As a result the resulting theory will be a gauge theory where the order of reducibility grows with the spin of the field.*

Another specific features is that in the fermionic theory we must obtain Lagrangian which is linear in derivatives. But if we try to construct Lagrangian similar to the bosonic case (12) we obtain Lagrangian which has the second order in derivatives. *To overcome this problem one first partially fixes the gauge and partially solves some field equations.* Then the obtained equations are still Lagrangian and thus we can derive the correct Lagrangian.

Lagrangian construction for the fields in AdS

The main difference of the Lagrangian construction in AdS space is that the algebra generated by the constraints is nonlinear, but it has a special structure

$$[l_i, l_j] \; = \; f_{ij}^{k} l_k + f_{ij}^{km} l_k l_m, \qquad (13)$$

where f_{ij}^{k}, f_{ij}^{km} are constants. The constants f_{ij}^{km} are proportional to the scalar curvature and disappear in the flat limit.

This has two consequences.

1) The algebra of the enlaged operators is changed in comparison with the algebra of the initial operators (13)

$$[L_i, L_j] \; = \; f_{ij}^{k} L_k - (f_{ij}^{km} + f_{ij}^{mk}) l'_m L_k + f_{ij}^{km} L_k L_m,$$

with the additional parts satisfying the algebra additional parts

$$[l'_i, l'_j] \; = \; f_{ij}^{k} l'_k - f_{ij}^{km} l'_m l'_k.$$

2) The BRST-BFV operator is defined unambiguously.

The construction of BRST-BFV operator is based on following general principles:

1. The BRST operator Q' is Hermitian, $Q'^{+} = Q'$, and nilpotent, $Q'^2 = 0$.

2. The BRST operator Q' is built using a set of first class constraints. In the case under consideration the operators L_0, L_1, L_1^+, L_2, L_2^+, G_0 are used as a set of such constraints.

3. The BRST operator Q' satisfies the special initial condition

$$Q'\Big|_{\mathcal{P}=0} \;=\; \eta_0 L_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_2 + \eta_2 L_2^+ + \eta_G G_0.$$

Straightforward calculation of the commutators allows us to find the algebra of the enlarged operators. In particular we get the following commutation relations

$$\begin{aligned}
[L_1, L_0] &= -6rL_1 - 8rl_1'^{+}L_2 - 8rl_2'L_1^{+} - 4rl_1'G_0 - 4rg_0'L_1 \\
&\qquad +8rL_1^{+}L_2 + 4rG_0L_1, \\
[L_0, L_1^{+}] &= -6rL_1^{+} - 8rl_2'^{+}L_1 - 8rl_1'L_2^{+} - 4rl_1'^{+}G_0 - 4rg_0'L_1^{+} \\
&\qquad + 8rL_2^{+}L_1 + 4rL_1^{+}G_0.
\end{aligned}$$

All possible ways to order the operators in the right hand sides are described in terms of arbitrary parameters ξ_1, ξ_2, ξ_3, ξ_4. The arbitrariness in the BRST operator stipulated by the parameters ξ_i is resulted in arbitrariness of introducing the auxiliary fields in the Lagrangians and hence does not affect the dynamics of the basic field.

After that the construction of the Lagrangians for the fields in AdS space goes along the same way as for fields in Minkowsky space.

Fields corresponding to an arbitrary Young tableau

Let us consider the Lagrangian construction for the fields with index symmetry corresponding to Young tableau with 2 rows ($s_1 \geq s_2$)

$$\Phi_{\mu_1\cdots\mu_{s_1},\nu_1\cdots\nu_{s_2}}(x) \quad \longleftrightarrow \quad \begin{array}{|c|c|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \cdots & \cdots & \mu_{s_1} \\ \hline \nu_1 & \nu_2 & \cdots & \nu_{s_2} \\ \cline{1-4} \end{array}$$

The tensor field is symmetric with respect to permutation of each type of the indices $\Phi_{\mu_1\cdots\mu_{s_1},\nu_1\cdots\nu_{s_2}}(x) = \Phi_{(\mu_1\cdots\mu_{s_1}),(\nu_1\cdots\nu_{s_2})}(x)$ and in addition must satisfy the following equations

$$(\partial^2 + m^2)\Phi_{\mu_1\cdots\mu_{s_1},\nu_1\cdots\nu_{s_2}}(x) = 0, \tag{14}$$

$$\partial^{\mu_1}\Phi_{\mu_1\cdots\mu_{s_1},\nu_1\cdots\nu_{s_2}}(x) = \partial^{\nu_1}\Phi_{\mu_1\cdots\mu_{s_1},\nu_1\cdots\nu_{s_2}}(x) = 0, \tag{15}$$

$$\eta^{\mu_1\mu_2}\Phi_{\mu_1\mu_2\cdots\mu_{s_1},\nu_1\cdots\nu_{s_2}} = \eta^{\nu_1\nu_2}\Phi_{\mu_1\cdots\mu_{s_1},\nu_1\nu_2\cdots\nu_{s_2}} = \\ = \eta^{\mu_1\nu_2}\Phi_{\mu_1\cdots\mu_{s_1},\nu_1\cdots\nu_{s_2}} = 0, \tag{16}$$

$$\Phi_{(\mu_1\cdots\mu_{s_1},\nu_1)\cdots\nu_{s_2}}(x) = 0. \tag{17}$$

Then we define Fock space generated by creation and annihilation operators

$$[a_i^{\mu}, a_j^{+\nu}] = -\eta^{\mu\nu}\delta_{ij}, \qquad \eta^{\mu\nu} = diag(+,-,-,\cdots,-) \qquad i,j = 1,2. \tag{18}$$

The number of pairs of creation and annihilation operators one should introduce is determined by the number of rows in the Young tableau corresponding to the symmetry of the tensor field. Thus we introduce two pairs of such operators. An arbitrary state vector in this Fock space has the form

$$|\Phi\rangle \;=\; \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty} \Phi_{\mu_1\cdots\mu_{s_1},\nu_1\cdots\nu_{s_2}}(x)\; a_1^{+\mu_1}\cdots a_1^{+\mu_{s_1}} a_2^{+\nu_1}\cdots a_2^{+\nu_{s_2}}|0\rangle.$$

To get the equations (14)–(17) on the coefficient functions we introduce the following operators

$$l_0 = -p^\mu p_\mu + m^2, \quad l_i = a_i^\mu p_\mu, \quad l_{ij} = \frac{1}{2} a_i^\mu a_{j\mu} \quad g_{12} = -a_1^{+\mu} a_{2\mu} \qquad (19)$$

where $p_\mu = -i\partial_\mu$.

One can check that restrictions (14)–(17) are equivalent to

$$l_0|\Phi\rangle = 0\,, \qquad l_i|\Phi\rangle = 0\,, \qquad l_{ij}|\Phi\rangle = 0\,, \qquad g_{12}|\Phi\rangle = 0 \qquad (20)$$

respectively.

Now we can generalize this construction to the fields corresponding to k-row Young tableau. For this purpose one should introduce Fock space generated by k pairs of creation and annihilation operators (18), where $i,j = 1,2,\ldots,k$, and then introduce operators (19), but now with $i,j = 1,2,\ldots,k$. Operator g_{12} is generalized to operator $g_{ij} = -a_i^{+\mu} a_{j\mu}$ where $i < j$.

After this the Lagrangian construction can be carried out as usual.

Summary

We have considered the basic principles of gauge invariant Lagrangian construction for massive higher spin fields.

This method can be applied to any free higher spin field model in Minkowski and AdS spaces.

The construction is also applied to tensor higher spin fields with index symmetry corresponding to a multirow Young tableau.

No off-shell constraints on the fields and gauge parameters are imposed.

The Lagrangians obtained possess a reducible gauge invariance and for the fermionic fields the order of reducibility grows with value of the spin.