Recent Developments in the Skyrme model

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The Skyrme model

- The Skyrme model is a nonlinear theory of pions which models the strong interaction between atomic nuclei.
- It is defined in terms of $U(t, \mathbf{x}) \in SU(2)$.

The Skyrme Lagrangian

$$L = \int \left(-\frac{1}{2}\,\mathrm{Tr}\,(R_\mu R^\mu) + \frac{1}{16}\,\mathrm{Tr}\,([R_\mu, R_\nu][R^\mu, R^\nu]) + m^2\,\mathrm{Tr}\,(U - 1_2) \right) d^3x,$$

where $R_\mu = (\partial_\mu U)U^\dagger$, and m is the pion mass.

- The Skyrme Lagrangian has localized finite energy solutions which behave like particles and are known as Skyrmions.

Symmetries of the Skyrme Lagrangian

- Recall: topological charge B (counts number of Skyrmions)
- The Skyrme energy is invariant under

$$\mathbb{E}_3 \times SO(3) \times \mathcal{P}$$

- Here, the parity is given by $\mathcal{P} : U(\mathbf{x}) \mapsto U^\dagger(-\mathbf{x})$.
- In the following, we ignore translations and focus on $SO(3) \times SO(3)$: rotations in space and target space (known as isorotations).
- Rotations are $\mathbf{x} \mapsto R\mathbf{x}$ where $R^T R = 1$.
- Isorotations are given by $U \mapsto AUA^\dagger$, where $A \in SU(2)$.

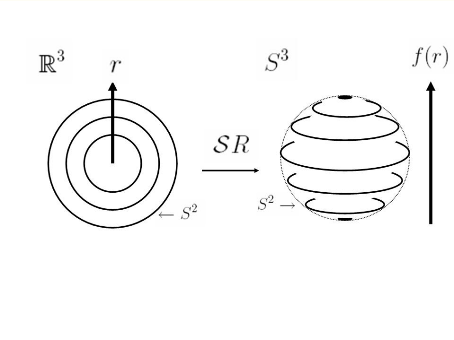
The rational map ansatz (a good approximation)

- Stereographic projection: $S^2 \to \mathbb{C} \cup \infty : (\theta, \phi) \mapsto z = \tan(\frac{\theta}{2})\mathrm{e}^{i\phi}$,

Inverse map:

$$\mathbf{n}_z = \frac{1}{1+|z|^2} \begin{pmatrix} z + \bar{z} \\ i(\bar{z} - z) \\ 1 - |z|^2 \end{pmatrix} .$$

Consider
$R : S^2 \to S^2 : z \mapsto R(z) = \frac{p(z)}{q(z)}$,
where $p(z)$ and $q(z)$ are polynomials.



The ansatz for the Skyrme field is

$$U(r, z) = \exp\left(i f(r)\, \mathbf{n}_{R(z)} \cdot \boldsymbol{\tau}\right) ,$$

- The baryon number is given by

$$B = -\frac{1}{2\pi^2} \int f' \sin^2 f \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{dR}{dz} \right| \right)^2 \frac{2i\, dz\, d\bar{z}}{(1+|z|^2)^2}\, dr\,.$$

- B equals the (polynomial) degree of the rational map.
- The energy is given by

$$E = 4\pi \int_0^\infty \left(r^2 f'^2 + 2B \sin^2 f (f'^2 + 1) + \mathcal{I}\, \frac{\sin^4 f}{r^2} + 2m^2 r^2 (1 - \cos f) \right) dr\,,$$

where

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i\, dz\, d\bar{z}}{(1+|z|^2)^2}\,.$$

- To minimise E one first minimises $\mathcal{I}$ over all maps of degree B. The profile function f is then found by solving the Euler-Lagrange equation for E with B and $\mathcal{I}$ fixed.

Examples of rational maps

B	symmetry	$R(z)$
1	$O(3)$	z
2	$D_{\infty h}$	z^2
3	T_d	$\frac{\sqrt{3}iz^2-1}{z^3-\sqrt{3}iz}$
4	O_h	$\frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$

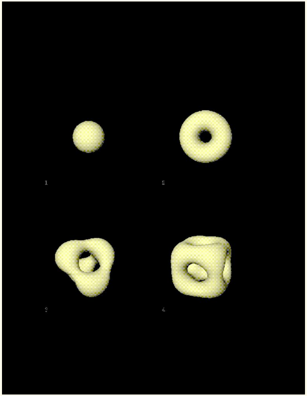


Figure: Energy density level sets for $B = 1, \ldots, 4$.

More examples

- For small B the symmetry is sufficient to determine the relevant rational map.
- For larger B, the parameters have to be found by minimizing $\mathcal{I}$ numerically.
- The rational map ansatz is very successful for predicting energy and symmetries of Skyrmions (in particular for $m = 0$).

B	symmetry	$R(z)$
5	D_{2d}	$\frac{z(z^4+ibz^2+a)}{az^4+ibz^2+1}$
6	D_{4d}	$\frac{z^4+ic}{z^2(icz^4+1)}$
7	Y_h	$\frac{7z^5+1}{z^2(z^5-7)}$
8	D_{6d}	$\frac{z^6-id}{z^2(idz^6-1)}$

The parameters a, b, c and d are numerically determined as -3.07, 3.94, 0.16 and 0.14 respectively.

Generalizations of the Rational Map Ansatz

- The restriction that that $R(z) = p(z)/q(z)$ is a quotient of two holomorphic polynomials can be relaxed. By allowing p and q to be functions of z and $\bar{z}$ the rational map ansatz can be 'improved'. This leads to best approximations to the numerical solutions for $B = 3$ and 4 know to date (Houghton, S.K.).
- The price to pay is that the energy now contains two functions which have to be minimized ($\mathcal{I}$ and $\mathcal{J}$).
- Another option is to "deform the sphere", so that the energy density in localized around a squashed sphere. The baryon density remains the same, but the energy density picks up even more terms (work in progress).

Finkelstein-Rubinstein constraints

- How do we quantize a scalar field theory and obtain fermions?
- The key observation is that the configuration space of the Skyrme model is topologically non-trivial:

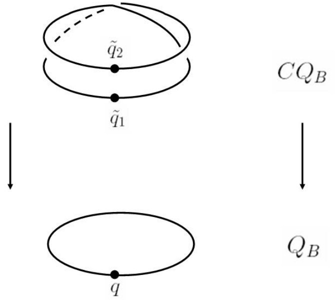
$$\pi_1(Q_B) = \mathbb{Z}_2,$$

where Q_B denotes the space of Skyrme configurations with topological charge B.

- Define wavefunctions ψ on the covering space of configuration space:

$$\psi : CQ_B \to \mathbb{C}.$$

- Impose $\psi(\tilde{q}_1) = -\psi(\tilde{q}_2)$.
- Symmetries of Skyrmions induce loops in configuration space.



Finkelstein-Rubinstein constraints

- Induced action of $SO(3) \times SO(3)$ symmetries on ψ :

$$\exp\left(-i\alpha\,\mathbf{n}\cdot\mathbf{L}\right)\exp\left(-i\beta\,\mathbf{N}\cdot\mathbf{K}\right)\psi(\tilde{q}) = \chi_{FR}\psi(\tilde{q}),$$

$$\text{where } \chi_{FR} = \begin{cases} 1 & \text{if the induced loop is contractible,} \\ -1 & \text{otherwise.} \end{cases}$$

- Here **L** and **K** are the body-fixed angular momentum operators in space and target space, respectively.

- Can we calculate $\chi_{FR} \in \pi_1(Q_B)$?

Theorem (S.K.): *The rational map ansatz induces a surjective homomorphism* $\pi_1(Rat_B) \to \pi_1(Q_B)$.

- This theorem implies that if we can calculate the Finkelstein-Rubinstein constraints for rational maps $R(z) \in Rat_B$ then we also know the constraints for the full configuration space Q_B of the Skyrme model.

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- Luckily, we can calculate the constraints for rational maps:

Finkelstein-Rubinstein constraints:

$$\chi_{FR} = (-1)^{\mathcal{N}} \quad \text{where} \quad \mathcal{N} = \frac{B}{2\pi}\left(B\alpha - \beta\right).$$

- Recall:

$$\exp\left(-i\alpha\,\mathbf{n}\cdot\mathbf{L}\right)\exp\left(-i\beta\,\mathbf{N}\cdot\mathbf{K}\right)\psi(\tilde{q}) = \chi_{FR}\psi(\tilde{q}),$$

Semiclassical collective coordinate quantization

- Recall
$$T = \frac{1}{2} a_i U_{ij} a_j - a_i W_{ij} b_j + \frac{1}{2} b_i V_{ij} b_j \,,$$
- The conjugate momenta corresponding to b_i and a_i are the body-fixed spin and isospin angular momenta L_i and K_i:
$$\begin{aligned} L_i &= -W^T_{ij} a_j + V_{ij} b_j \,, \\ K_i &= U_{ij} a_j - W_{ij} b_j \,. \end{aligned}$$
(Hamiltonian picture)
- We denote the space-fixed spin and isospin angular momenta by J_i and I_i respectively. Note $\mathbf{J}^2 = \mathbf{L}^2$ and $\mathbf{I}^2 = \mathbf{K}^2$.
- We now regard L_i, K_i, J_i and I_i as quantum operators, each individually satisfying the $\mathfrak{su}(2)$ commutation relations.

Skyrmion wavefunctions

- The idea is now to find the lowest values of spin and isospin which are compatible with all the Finkelstein-Rubinstein constraints. Then calculate the energy of these states.
- The Finkelstein-Rubinstein constraints arise from the symmetries of the classical minimal energy configuration.
- A basis for the wavefunctions is given by $|J, J_3, L_3\rangle \otimes |I, I_3, K_3\rangle$.
- The kinetic energy T can be expressed in terms of angular momentum operators.
- The matrices U_{ij}, V_{ij} and W_{ij} can be evaluated using

$$\Sigma_{ij} = 2\int \sin^2 f \, \frac{C_{\Sigma_{ij}}}{(1+|R|^2)^2}\left(1 + f'^2 + \frac{\sin^2 f}{r^2}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2\right) d^3x \,,$$

where $\Sigma = (U, V, W)$ and $C_{\Sigma_{ij}}$ only depends on angular variables.

Good results and Challenges

- The groundstates and possible excited states have been calculated using Finkelstein-Rubinstein constraints for $B \leq 22$. This approach works very well for even B, but not so well for odd B.
- Manko, Manton and Wood calculated the energy levels for $1 \leq B \leq 8$.
- The results correspond well to experiment, and energies and spins of a few as yet unobserved states have been predicted.

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- However, the quantitative results are not very accurate. There is also an ongoing discussion of how best to fix the three physical parameters (F_π, e and m_π) in the Skyrme model.
- For odd $B = 5$ and 7, the correct ground states can only be calculated by deforming the minimal energy solutions significantly.
- For $B = 10$, 18 and 22, the approach gives the wrong groundstates.

Possible ways out

- The zero-mode quantization is the simplest approximation to the quantum states (also known as rigid rotator approximation).
- The pion mass has a larger influence than expected. For $B > 8$, the minimal energy solutions (and their symmetries) change dramatically if the pion mass is increased.
- We have to take into account that Skyrmions deform when they are spinning. This can lead to changes in the symmetries of minimal energy solutions.

Note, different symmetries can lead to different allowed quantum states!

Relative Equilibria and Spinning Skyrmions

- The zero mode quantization has been quite successful. However, it is clear that Skyrmions are deforming when they are spinning, and this effect needs to be taken into account.
- The theory of relative equilibria was developed for rotating molecules. It gives strong theorems as to what kind of symmetry bifurcations can occur.
- Numerical simulation of Spinning Skyrmions is very time-consuming. Therefore, a good understanding of what kind of behaviour is to be expected is very valuable.

Relative equilibria

- Let $\mathcal{C}$ be a smooth manifold with a smooth action of a finite dimensional Lie group G. Let $< .,. >_q$ denote the G-invariant inner product on $T_q\mathcal{C}$.
- Consider the G-invariant Lagrangian Lagrangian

$$L = \frac{1}{2} < \dot{q}, \dot{q} >_q - V(q),$$

where $V(q)$ is also G-invariant.

- Let $q(t) = \exp(t\xi)r(t)$. This induces the Lagrangian L_ξ

$$L_\xi = \frac{1}{2} < \dot{r}, \dot{r} > - < \dot{r}, \xi \cdot r > + \frac{1}{2} < \xi \cdot r, \xi \cdot r > - V(r),$$

where

$$\xi \cdot r = \frac{d}{dt} \left(\exp(t\xi) r \right)_{t=0}.$$

Slice Coordinates

- An equilibrium point of L_ξ is called a relative equilibrium of L.
- A good choice of coordinates are the slice coordinates, splitting the dynamics into components along the group orbit $G \cdot q_0$ and transverse to it where q_0 is any point in $\mathcal{C}$. In a neighbourhood of q_0, we can write $q = g \cdot s$ where $g \in G$ and s is in the "slice" S transverse to the group action.
- For the tangent vectors, we obtain

$$g^{-1}\dot{q} = \xi \cdot s + \dot{s},$$

  where $\xi = g^{-1}\dot{g}$.
- These coordinates have the advantage that rotational and vibrational modes decouple at the point q_0.

Theorem:

1. In a neighbourhood of an equilibrium point q_0 the corresponding Hamiltonian H can be expressed as

$$H = h(\mu) + Q(s, \sigma),$$

where μ and σ are the momenta conjugate to ξ and s, respectively.

2. The function $h(\mu)$ is even and its Taylor series at $0 \in \mathfrak{g}^*$ is

$$h(\mu) = V(q_O) + \tfrac{1}{2}\mu^T \mathbf{I}(q_0)^{-1}\mu - \tfrac{1}{16} V_2^{-1}\left(\mu^T \mathbf{I}_s^{-1}(q_0)\mu, \mu^T \mathbf{I}_s^{-1}(q_0)\mu\right) + O(\mu^6),$$

where $\mathbf{I}$ is the locked inertia tensor, and V_2 and $\mathbf{I}_s^{-1}$ are certain derivatives with respect to s.

Preliminary Results

- This theorem gives as a way to systematically proceed with the quantization of Skyrmions.
- Here, μ corresponds to the angular momenta $\mathbf{K}$ and $\mathbf{L}$, and s parametrizes the vibrations/deformations, which we have ignored so far. The zero-mode quantization corresponds to only considering the quadratic approximation to $h(\mu)$.
- Symmetries place further restrictions on $h(\mu)$ and enable us to make more general statements about possible bifurcation patterns.

Outlook

- Calculate symmetry bifurcations explicitly for various baryon numbers (with Mark Roberts)
- Evaluate higher order terms in $h(\mu)$ using generalized rational map approximations.
- Find interesting configurations and calculate relative equilibria numerically (with Richard Battye and Paul Sutcliffe.)
- For given B analyze the bifurcation pattern as the angular velocity increases. (Better understanding of fermions)