Quantum Deformations of the One-Dimensional Hubbard Model

Quarks 2008 Sergiev Posad

May 25th 2008

Peter Koroteev

in collaboration with N. Beisert and F. Spill

ArXiv:0802.0777, work in progress

INR & ITEP, Moscow and AEI, Potsdam

Outine

- Introduction
- AdS/CFT correspondence. SYM spin chain

  Spectrum of Planar $\mathcal{N} = 4$ SYM
- Quantum deformations

  $\mathrm{U}_q(\mathfrak{su}(2|2) \ltimes \mathbb{R}^2)$ chain

  Hubbard model and its deformations
- Open Questions

  Dynamical transformations

  Universal R-matrix

Why Integrabilty?

- Easy models: Classical Mechanics (ossilator, free point particle)
- Integrable models: not necessarily easy/complicated.
- "Complicated", chaotic models. Almost untractable.

Integrable models can be completely solved

What is Integrabilty?

Exists an infinite set of independent commuting charges

$$\{\mathfrak{Q}_\alpha\}, \quad [\mathfrak{Q}_\alpha, \mathfrak{Q}_\beta] = 0$$

Finite dimensional: mechanical model of order n (n d.o.fs) – exists n-1 local first integrals in involution

Field Theory: Infinite dimensional symmetry, S–matrix satisfies integrability constraints.

We are focused on QFTs and spin chains

Signatures of Integrabilty

Explicitely find all commuting charges

Investigate S-matrix (field theory, spin chains) S-matrix satisfies

Yang-Baxter equation, Unitarity conditions

Some methods

- Analytic Bethe Ansatz. [Leningrad school, 70-80s]
- Coordinate Bethe Ansatz. Mostly for spin chains
- Algebraic integrability.

AdS/CFT correspondence. String side

Conjectured duality between IIB string theory (closed strings and open strings bound to D-branes; no tachyon; massless chiral fermions) on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM on its boundary $S^3 \times \mathbb{R}$

Coset space

$$AdS_5 \times S^5 \times fermions = \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$$

String theory as coset sigma model on a cylinder (GS superstring)

[Metsaev, Tseytlin hep-th/9805028]

N=4 4D Super Yang Mills

$\mathfrak{su}(N)$ gauge connection A, 4 Majorana gluinos as 16 component 10d Majorana-Weyl spinor χ, 6 scalars Φ_i. All adjoint!

$$S = \frac{2}{g_{YM}^2} \int d^4x \,\mathrm{Tr} \left\{ \frac{1}{4} F^2 + \frac{1}{2} (\nabla \Phi_i)^2 - \frac{1}{4} [\Phi_i, \Phi_j]^2 + \frac{1}{2} \bar{\chi} \not{\nabla} \chi - \frac{i}{2} \bar{\chi} \Gamma_i [\phi_i, \chi] \right\}$$

- Completely fixed by supersymmetry – two parameters g and N
  $g \sim \sqrt{\lambda} \sim g_{YM} \sqrt{N}$
- All fileds massless
- Finiteness: beta function is exactly zero, no running
- Unbroken conformal symmetry.
  Superconformal symmetry $\mathfrak{psu}(2,2|4)$

Establishing the Correspondence

String side

Solution of classical EOM + Quantum corrections

Gauge side

States: Local operators $\mathcal{O} = Tr\Phi\nabla\nabla\Phi F\ldots$

Energies: Scaling dimensions

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \alpha |x-y|^{-2D(\lambda)}$$

Strong-weak duality. Strong $g \to \infty$ for strings

$$E(g) = gE_0 + E_1 + E_2/g + \ldots$$

Weak for SYM

$$D(g) = D_0 + g^2 D_1 + g^4 D_2 + \ldots$$

Problems in comparison. Integrabilty may help

Anomalous Dimensions

Consider local operators of many fields.

$$\mathcal{O}(x) = Tr\Phi_i(x)\Phi_j(x)F(x)\nabla_k(x)\Psi_m(x)\ldots$$

Scaling dimensions $[\Phi]=1$, $[\Psi]=1$, $[\nabla]=1$ affect D_0

Correlators

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \alpha|x-y|^{-2D(g)}$$

Renormalization: claim correlators to be finite.

Calculate anomalous dimension: double series g^2–loops, N^{-2g}–genera

$$D(g) = D_0 + \sum_{l=1} g^{2l} \sum_{g=0} N^{-2g} D_{l,g}$$

Complicated beyond scalar sector (nonplanar contributions) Huge combinatorial problem, classifying Feynman dyagrams (even tree level), mixing

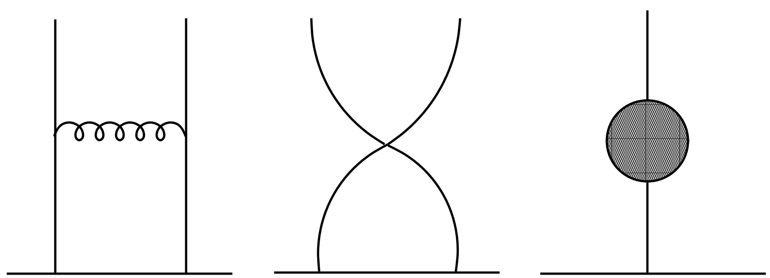
But simpler in planar limit $\mathcal{O}(N^0)$ graphs at arbitrary coupling

Dilatation Operator

Composite gauge invariant operators need renormalization $\mathscr{O} \to Z\mathscr{O}$

$$\langle \mathscr{O}(x)\mathscr{O}(y)\rangle = \alpha |x-y|^{-2(D_0+D(\lambda))}$$

No mixing in scalar sector at 1 loop [Minahan, Zarembo hep-th/0212208]



Anomalous dimension matrix (dilatation generator) $D = \frac{d\log Z}{d\log \Lambda}$ interpreted as a spin chain Hamiltonian

$$\mathfrak{D}(g)\mathscr{O} = g^2\mathscr{H}\mathscr{O} = D(g)\mathscr{O}$$

Spin Chains. Scalar Sector

$[\Phi]$=1. Length of spin chain = bare scaling dimension. Spectrum of Hamiltonian – anomalous dimension

Hibert space

$$\mathrm{H} = \mathbb{R}^6 \otimes \cdots \otimes \mathbb{R}^6$$

States – single trace operators

$$|\Phi_i \Phi_j \Phi_k \Phi_l \ldots\rangle = \mathrm{Tr} \Phi_i \Phi_j \Phi_k \Phi_l \ldots$$

Hamiltonian $\mathscr{H} = \sum \mathscr{H}_{k,k+1}$. Boiled down to $\mathfrak{su}(2)$ sector $XXX_{\frac{1}{2}}$ spin chain

$$\mathscr{H}_{k,k+1} = \mathscr{I}_{k,k+1} - \mathscr{P}_{k,k+1} = \frac{1}{2}(1 - \vec{\sigma}_k \vec{\sigma}_{k+1})$$

Integrable !

Higher loops

Dilatation operator

$$\mathfrak{D}(g) = \mathfrak{D}_0 + g^2\mathfrak{D}_1 + g^3\mathfrak{D}_2 + \ldots$$

- Action is homogeneous

Interaction with I in and O out legs is of order $\mathcal{O}(g^{I+O-2})$

- local
- long-ranged (range grows with g)
- dynamic (creation and annihilation of sites)

SYM Spin Chain Vacuum

Three complex scalars $\mathscr{X} = \Phi_1 + i\Phi_2, \mathscr{Y} = \Phi_3 + i\Phi_4, \mathscr{Z} = \Phi_5 + i\Phi_6$

tranform under $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$

Vacuum: $|\mathscr{X}\mathscr{X}\mathscr{X}\ldots\rangle$ breaks superconformal symmetry

Residual symmetry $\mathfrak{u}(1) \ltimes \mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2) \ltimes \mathbb{R}$

Excitations transform in reps of $\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)$

Can work with one copy of $\mathfrak{psu}(2|2)$. Leeds us to $\mathfrak{psu}(2|2)$ spin chain

On string side corresponds to

Light cone gauge using time form AdS_5 and great circle of S^5

Vacuum: Point particle moving along great circle

Excitations: 4 coordinates on S^5, 4 coordinates on AdS_5

$\mathfrak{su}(2|2)$ Spin Chain [Beisert hep-th/0511082]

- d.o.f: 2 bosons ϕ^1, ϕ^2, 2 fermions ψ^1, ψ^2
- S-matrix: $V_1 \otimes V_2 \to V_2' \otimes V_1'$ is fully constrained by symmetry up to overall scalar factor (peculiaraty of representation theory)
- Central extension to $\mathfrak{su}(2|2) \ltimes \mathbb{R}$ makes spectrum continuous
- Magnon dispersion relation merely from symmetry

$$E(p) = \sqrt{1 + g^2 \sin^2(\tfrac{1}{2}p)}$$

Lie Superalgebra

$\mathfrak{su}(2)\times\mathfrak{su}(2)$ generators $\mathfrak{R}^a{}_b$, $\mathfrak{L}^\alpha{}_\beta$ supercharges $\mathfrak{Q}^\alpha{}_b$, $\mathfrak{S}^a{}_\beta$ and central charge $\mathfrak{C}$

$$[\mathfrak{R}^a{}_b,\mathfrak{R}^c{}_d]=\delta^c_b\mathfrak{R}^a{}_d-\delta^a_d\mathfrak{R}^c{}_b,\qquad [\mathfrak{L}^\alpha{}_\beta,\mathfrak{L}^\gamma{}_\delta]=\delta^\gamma_\beta\mathfrak{L}^\alpha{}_\delta-\delta^\alpha_\delta\mathfrak{L}^\gamma{}_\beta,$$

$$[\mathfrak{R}^a{}_b,\mathfrak{Q}^\gamma{}_d]=-\delta^a_d\mathfrak{Q}^\gamma{}_b+\tfrac{1}{2}\delta^a_b\mathfrak{Q}^\gamma{}_d,\qquad [\mathfrak{L}^\alpha{}_\beta,\mathfrak{Q}^\gamma{}_d]=\delta^\gamma_\beta\mathfrak{Q}^\alpha{}_d-\tfrac{1}{2}\delta^\alpha_\beta\mathfrak{Q}^\gamma{}_d,$$

$$[\mathfrak{R}^a{}_b,\mathfrak{S}^c{}_\delta]=\delta^c_b\mathfrak{S}^a{}_\delta-\tfrac{1}{2}\delta^a_b\mathfrak{S}^c{}_\delta,\qquad [\mathfrak{L}^\alpha{}_\beta,\mathfrak{S}^c{}_\delta]=-\delta^\alpha_\delta\mathfrak{S}^c{}_\beta+\tfrac{1}{2}\delta^\alpha_\beta\mathfrak{S}^c{}_\delta.$$

$$\{\mathfrak{Q}^\alpha{}_b,\mathfrak{S}^c{}_\delta\}=\delta^c_b\mathfrak{L}^\alpha{}_\delta+\delta^\alpha_\delta\mathfrak{R}^c{}_b+\delta^c_b\delta^\alpha_\delta\mathfrak{C}.$$

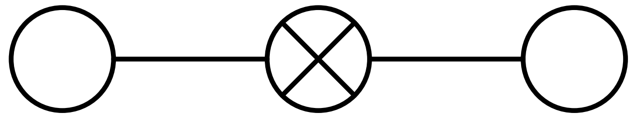
Central extention

$$\{\mathfrak{Q}^\alpha{}_b,\mathfrak{Q}^\gamma{}_d\}=\varepsilon^{\alpha\gamma}\varepsilon_{bd}\mathfrak{P},\qquad \{\mathfrak{S}^a{}_\beta,\mathfrak{S}^c{}_\delta\}=\varepsilon^{ac}\varepsilon_{\beta\delta}\mathfrak{K}.$$

denote

$$\mathfrak{h}:=\mathfrak{su}(2|2)\ltimes\mathbb{R}^2=\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3.$$

Distinguished OXO (Chevalley) basis



$$\mathfrak{H}_1 = \mathfrak{R}^2{}_2 - \mathfrak{R}^1{}_1 = 2\mathfrak{R}^2{}_2, \qquad \mathfrak{E}_1 = \mathfrak{R}^2{}_1, \qquad \mathfrak{F}_1 = \mathfrak{R}^1{}_2,$$
$$\mathfrak{H}_2 = -\mathfrak{C} - \tfrac{1}{2}\mathfrak{H}_1 - \tfrac{1}{2}\mathfrak{H}_3, \qquad \mathfrak{E}_2 = \mathfrak{Q}^2{}_2, \qquad \mathfrak{F}_2 = \mathfrak{S}^2{}_2,$$
$$\mathfrak{H}_3 = \mathfrak{L}^2{}_2 - \mathfrak{L}^1{}_1 = 2\mathfrak{L}^2{}_2, \qquad \mathfrak{E}_3 = \mathfrak{L}^1{}_2, \qquad \mathfrak{F}_3 = \mathfrak{L}^2{}_1.$$

Commutators

$$[\mathfrak{H}_j, \mathfrak{H}_k] = 0, \qquad [\mathfrak{H}_j, \mathfrak{E}_k] = +A_{jk}\mathfrak{E}_k, \qquad [\mathfrak{H}_j, \mathfrak{F}_k] = -A_{jk}\mathfrak{F}_k.$$

$$[\mathfrak{E}_1, \mathfrak{F}_1] = +\mathfrak{H}_1, \qquad \{\mathfrak{E}_2, \mathfrak{F}_2\} = -\mathfrak{H}_2, \qquad [\mathfrak{E}_3, \mathfrak{F}_3] = -\mathfrak{H}_3.$$

Symmetrized Cartan matrix – degenerate

$$A_{jk} = \begin{pmatrix} +2 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -2 \end{pmatrix}$$

Null vector $v_j = (1, 2, 1)$

$$\mathfrak{C} = -\frac{1}{2}\sum_{j=1}^{3} v_j \mathfrak{H}_j = -\tfrac{1}{2}\mathfrak{H}_1 - \mathfrak{H}_2 - \tfrac{1}{2}\mathfrak{H}_3$$

Central Charges and Serre Relations

Central charges

$$\mathfrak{P} = \left\{[\mathfrak{E}_1, \mathfrak{E}_2], [\mathfrak{E}_3, \mathfrak{E}_2]\right\}, \qquad \mathfrak{K} = \left\{[\mathfrak{F}_1, \mathfrak{F}_2], [\mathfrak{F}_3, \mathfrak{F}_2]\right\}$$

Serre relations

$$\begin{aligned} 0 &= [\mathfrak{E}_1, \mathfrak{E}_3] = \mathfrak{E}_2\mathfrak{E}_2 = \left[\mathfrak{E}_1, [\mathfrak{E}_1, \mathfrak{E}_2]\right] = \left[\mathfrak{E}_3, [\mathfrak{E}_3, \mathfrak{E}_2]\right] \\ &= [\mathfrak{F}_1, \mathfrak{F}_3] = \mathfrak{F}_2\mathfrak{F}_2 = \left[\mathfrak{F}_1, [\mathfrak{F}_1, \mathfrak{F}_2]\right] = \left[\mathfrak{F}_3, [\mathfrak{F}_3, \mathfrak{F}_2]\right] \end{aligned}$$

For $\mathrm{U}(\mathfrak{su}(2|2))$

$$\mathfrak{P} = \mathfrak{K} = 0$$

Quantum Deformation

Q-number ($q \in \mathbb{C}$)

$$[A]_q = \frac{q^A - q^{-A}}{q - q^{-1}}$$

Commutators

$$[\mathfrak{E}_1, \mathfrak{F}_1] = [\mathfrak{H}_1]_q\,, \qquad \{\mathfrak{E}_2, \mathfrak{F}_2\} = -[\mathfrak{H}_2]_q\,, \qquad [\mathfrak{E}_3, \mathfrak{F}_3] = -[\mathfrak{H}_3]_q\,,$$

Serre relations

$$\begin{aligned} 0 &= [\mathfrak{E}_1, \mathfrak{E}_3] = [\mathfrak{F}_1, \mathfrak{F}_3] = \mathfrak{E}_2\mathfrak{E}_2 = \mathfrak{F}_2\mathfrak{F}_2 \\ &= \mathfrak{E}_1\mathfrak{E}_1\mathfrak{E}_2 - (q+q^{-1})\mathfrak{E}_1\mathfrak{E}_2\mathfrak{E}_1 + \mathfrak{E}_2\mathfrak{E}_1\mathfrak{E}_1 = \mathfrak{E}_3\mathfrak{E}_3\mathfrak{E}_2 - (q+q^{-1})\mathfrak{E}_3\mathfrak{E}_2\mathfrak{E}_3 + \mathfrak{E}_2\mathfrak{E}_3\mathfrak{E}_3 \\ &= \mathfrak{F}_1\mathfrak{F}_1\mathfrak{F}_2 - (q+q^{-1})\mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_1 + \mathfrak{F}_2\mathfrak{F}_1\mathfrak{F}_1 = \mathfrak{F}_3\mathfrak{F}_3\mathfrak{F}_2 - (q+q^{-1})\mathfrak{F}_3\mathfrak{F}_2\mathfrak{F}_3 + \mathfrak{F}_2\mathfrak{F}_3\mathfrak{F}_3 \end{aligned}$$

Central charges

$$\begin{aligned} \mathfrak{P} &= \mathfrak{E}_1\mathfrak{E}_2\mathfrak{E}_3\mathfrak{E}_2 + \mathfrak{E}_2\mathfrak{E}_3\mathfrak{E}_2\mathfrak{E}_1 + \mathfrak{E}_3\mathfrak{E}_2\mathfrak{E}_1\mathfrak{E}_2 + \mathfrak{E}_2\mathfrak{E}_1\mathfrak{E}_2\mathfrak{E}_3 - (q+q^{-1})\mathfrak{E}_2\mathfrak{E}_1\mathfrak{E}_3\mathfrak{E}_2, \\ \mathfrak{K} &= \mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_3\mathfrak{F}_2 + \mathfrak{F}_2\mathfrak{F}_3\mathfrak{F}_2\mathfrak{F}_1 + \mathfrak{F}_3\mathfrak{F}_2\mathfrak{F}_1\mathfrak{F}_2 + \mathfrak{F}_2\mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_3 - (q+q^{-1})\mathfrak{F}_2\mathfrak{F}_1\mathfrak{F}_3\mathfrak{F}_2. \end{aligned}$$

Symmetry

Let $\mathfrak{g}$-(Lie) algebra. We want S-matrix to be invariant under $\mathfrak{g}$. How to elaborate?

S matrix intertwines two multiplets

$$\mathscr{S} : V_1 \otimes V_2 \longrightarrow V_2' \otimes V_1'$$

Representation of $\mathfrak{g}$

$$\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$$

Need to tensor multiply two reps. Coproduct in Hopf algebra of $\mathfrak{g}$

$$\Delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$$

consistent way to do it.

Braided coproduct

$$\Delta(\mathfrak{E}_2) = \mathfrak{E}_2 \otimes 1 + q^{-\mathfrak{H}_2}\mathfrak{U} \otimes \mathfrak{E}_2,$$
$$\Delta(\mathfrak{F}_2) = \mathfrak{F}_2 \otimes q^{\mathfrak{H}_2} + \mathfrak{U}^{-1} \otimes \mathfrak{F}_2,$$
$$\Delta(\mathfrak{P}) = \mathfrak{P} \otimes 1 + q^{2\mathfrak{C}}\mathfrak{U}^2 \otimes \mathfrak{P},$$
$$\Delta(\mathfrak{K}) = \mathfrak{K} \otimes q^{-2\mathfrak{C}} + \mathfrak{U}^{-2} \otimes \mathfrak{K},$$
$$\Delta(\mathfrak{U}) = \mathfrak{U} \otimes \mathfrak{U}.$$

Undraided for the rest generatiors (i=1,2,3, j=1,3)

$$\Delta(1) = 1 \otimes 1,$$
$$\Delta(\mathfrak{H}_i) = \mathfrak{H}_i \otimes 1 + 1 \otimes \mathfrak{H}_i,$$
$$\Delta(\mathfrak{E}_j) = \mathfrak{E}_j \otimes 1 + q^{-\mathfrak{H}_j} \otimes \mathfrak{E}_j,$$
$$\Delta(\mathfrak{F}_j) = \mathfrak{F}_j \otimes q^{\mathfrak{H}_j} + 1 \otimes \mathfrak{F}_j,$$

Unit Couniut Antipode

Unit element $\eta(1) = 1$. Counit $\varepsilon : \mathrm{U}_q(\mathfrak{h}) \to \mathbb{C}$ takes the form

$$\varepsilon(1) = 1, \qquad \varepsilon(\mathfrak{H}_j) = \varepsilon(\mathfrak{E}_j) = \varepsilon(\mathfrak{F}_j) = 0.$$

Antipode $S : \mathrm{U}_q(\mathfrak{h}) \to \mathrm{U}_q(\mathfrak{h})$ is uniquely fixed by the compatibility condition

$$\mu \circ (S \otimes 1) \circ \Delta(\mathfrak{J}) = \mu \circ (1 \otimes S) \circ \Delta(\mathfrak{J}) = \eta \circ \varepsilon(\mathfrak{J}),$$

for all $\mathfrak{J} \in \mathrm{U}_q(\mathfrak{h})$, $\mu : \mathrm{U}_q(\mathfrak{h}) \otimes \mathrm{U}_q(\mathfrak{h}) \to \mathrm{U}_q(\mathfrak{h})$ – product.

$$S(1) = 1, \qquad S(\mathfrak{H}_j) = -\mathfrak{H}_j, \qquad S(\mathfrak{E}_j) = -q^{\mathfrak{H}_j}\mathfrak{E}_j, \qquad S(\mathfrak{F}_j) = -\mathfrak{F}_j q^{-\mathfrak{H}_j}$$

for central charges

$$S(\mathfrak{C}) = -\mathfrak{C}, \qquad S(\mathfrak{P}) = -q^{-2\mathfrak{C}}\mathfrak{P}, \qquad S(\mathfrak{K}) = -q^{2\mathfrak{C}}\mathfrak{K}$$

Co-commutativity and R-matrix

Hopf algebra quasi-cocommutative if ($\mathscr{R} \in \mathfrak{g} \otimes \mathfrak{g}$)

$$\Delta_{\mathrm{op}}(\mathfrak{J})\mathscr{R} = \mathscr{R}\Delta(\mathfrak{J}), \quad \Delta_{\mathrm{op}} = \mathscr{P}\Delta\mathscr{P}$$

Comutative for center of $\mathrm{U}_q(\mathfrak{h})$ For central charges $\mathfrak{P}$ and $\mathfrak{K}$ more involved

$$\begin{aligned} \Delta(\mathfrak{P}) &= \mathfrak{P} \otimes 1 + q^{2\mathfrak{C}}\mathfrak{U}^2 \otimes \mathfrak{P}, \\ \Delta(\mathfrak{K}) &= \mathfrak{K} \otimes q^{-2\mathfrak{C}} + \mathfrak{U}^{-2} \otimes \mathfrak{K}, \end{aligned} \quad \text{but} \quad \begin{aligned} \Delta_{\mathrm{op}}(\mathfrak{P}) &= \mathfrak{P} \otimes \mathfrak{U}^2 q^{2\mathfrak{C}} + 1 \otimes \mathfrak{P}, \\ \Delta_{\mathrm{op}}(\mathfrak{K}) &= \mathfrak{K} \otimes \mathfrak{U}^{-2} + q^{-2\mathfrak{C}} \otimes \mathfrak{K}. \end{aligned}$$

have to be identified with $\mathfrak{C}$ and $\mathfrak{U}$

$$\mathfrak{P} = g\alpha(1 - q^{2\mathfrak{C}}\mathfrak{U}^2), \qquad \mathfrak{K} = g\alpha^{-1}(q^{-2\mathfrak{C}} - \mathfrak{U}^{-2}).$$

Here g and α are two global constants of the reduced algebra.

Braiding and R-matrix

Deform the coproduct according to $\mathbb{Z}$-grading of the algebra. Grading associates the charges $+2, +1, -1, -2$ to the generators $\mathfrak{P}, \mathfrak{E}_2, \mathfrak{F}_2, \mathfrak{K}$, the others uncharged.

Braiding leads to a non-trivial R-matrix.

Introduce abelian generator $\mathfrak{U}$ and deform the coproduct

$$\begin{aligned}
\Delta(\mathfrak{E}_2) &= \mathfrak{E}_2 \otimes 1 + q^{-\mathfrak{H}_2}\mathfrak{U} \otimes \mathfrak{E}_2, \\
\Delta(\mathfrak{F}_2) &= \mathfrak{F}_2 \otimes q^{\mathfrak{H}_2} + \mathfrak{U}^{-1} \otimes \mathfrak{F}_2, \\
\Delta(\mathfrak{P}) &= \mathfrak{P} \otimes 1 + q^{2\mathfrak{C}}\mathfrak{U}^2 \otimes \mathfrak{P}, \\
\Delta(\mathfrak{K}) &= \mathfrak{K} \otimes q^{-2\mathfrak{C}} + \mathfrak{U}^{-2} \otimes \mathfrak{K}, \\
\Delta(\mathfrak{U}) &= \mathfrak{U} \otimes \mathfrak{U}.
\end{aligned}$$

The exponents of $\mathfrak{U}$ follow the $\mathbb{Z}$-grading of the algebra.

Counit $\varepsilon(\mathfrak{U}) = 1$.

Antipode

$$S(\mathfrak{E}_2) = -q^{\mathfrak{H}_2}\mathfrak{U}^{-1}\mathfrak{E}_2, \qquad S(\mathfrak{F}_2) = -q^{-\mathfrak{H}_2}\mathfrak{U}\mathfrak{F}_2, \qquad S(\mathfrak{U}) = \mathfrak{U}^{-1}$$

for central charges

$$S(\mathfrak{C}) = -\mathfrak{C}, \qquad S(\mathfrak{P}) = -q^{-2\mathfrak{C}}\mathfrak{U}^{-2}\mathfrak{P}, \qquad S(\mathfrak{K}) = -q^{2\mathfrak{C}}\mathfrak{U}^2\mathfrak{K}.$$

Fundamental Reresentation

$$\mathfrak{H}_1|\phi^1\rangle = -|\phi^1\rangle, \quad \mathfrak{H}_2|\phi^1\rangle = -(C-\tfrac{1}{2})|\phi^1\rangle, \quad \mathfrak{E}_1|\phi^1\rangle = q^{+1/2}|\phi^2\rangle, \quad \mathfrak{E}_2|\phi^1\rangle = c|\psi^1\rangle,$$

$$\mathfrak{H}_1|\phi^2\rangle = +|\phi^2\rangle, \quad \mathfrak{H}_2|\phi^2\rangle = -(C+\tfrac{1}{2})|\phi^2\rangle, \quad \mathfrak{E}_2|\phi^2\rangle = a|\psi^2\rangle, \quad \mathfrak{F}_1|\phi^2\rangle = q^{-1/2}|\phi^1\rangle,$$

$$\mathfrak{H}_3|\psi^2\rangle = +|\psi^2\rangle, \quad \mathfrak{H}_2|\psi^2\rangle = -(C+\tfrac{1}{2})|\psi^2\rangle, \quad \mathfrak{E}_3|\psi^2\rangle = q^{-1/2}|\psi^1\rangle, \quad \mathfrak{F}_2|\psi^2\rangle = d|\phi^2\rangle,$$

$$\mathfrak{H}_3|\psi^1\rangle = -|\psi^1\rangle, \quad \mathfrak{H}_2|\psi^1\rangle = -(C-\tfrac{1}{2})|\psi^1\rangle, \quad \mathfrak{F}_2|\psi^1\rangle = b|\phi^1\rangle, \quad \mathfrak{F}_3|\psi^1\rangle = q^{+1/2}|\psi^2\rangle.$$

Constraints

$$ad = [C+\tfrac{1}{2}]_q, \qquad bc = [C-\tfrac{1}{2}]_q, \qquad ab = P, \qquad cd = K.$$

$$(ad - qbc)(ad - q^{-1}bc) = 1.$$

Parametrization

$$a = \sqrt{g}\,\gamma, \quad b = \frac{\sqrt{g}\,\alpha}{\gamma}\,\frac{1}{x^-}\left(x^- - q^{2C-1}x^+\right),$$

$$c = \frac{i\sqrt{g}\gamma}{\alpha}\,\frac{q^{-C+1/2}}{x^+}, \quad d = \frac{i\sqrt{g}}{\gamma}\,q^{C+1/2}\left(x^- - q^{-2C-1}x^+\right).$$

Fundamental R-matrix

$$\mathscr{R}|\phi^1\phi^1\rangle = A_{12}|\phi^1\phi^1\rangle$$

$$\mathscr{R}|\phi^1\phi^2\rangle = \frac{qA_{12}+q^{-1}B_{12}}{q+q^{-1}}|\phi^2\phi^1\rangle + \frac{A_{12}-B_{12}}{q+q^{-1}}|\phi^1\phi^2\rangle - \frac{q^{-1}C_{12}}{q+q^{-1}}|\psi^2\psi^1\rangle + \frac{C_{12}}{q+q^{-1}}|\psi^1\psi^2\rangle$$

$$\mathscr{R}|\phi^2\phi^1\rangle = \frac{A_{12}-B_{12}}{q+q^{-1}}|\phi^2\phi^1\rangle + \frac{q^{-1}A_{12}+qB_{12}}{q+q^{-1}}|\phi^1\phi^2\rangle + \frac{C_{12}}{q+q^{-1}}|\psi^2\psi^1\rangle - \frac{qC_{12}}{q+q^{-1}}|\psi^1\psi^2\rangle$$

$$\mathscr{R}|\phi^2\phi^2\rangle = A_{12}|\phi^2\phi^2\rangle$$

$$\mathscr{R}|\psi^1\psi^1\rangle = -D_{12}|\psi^1\psi^1\rangle$$

$$\mathscr{R}|\psi^1\psi^2\rangle = -\frac{qD_{12}+q^{-1}E_{12}}{q+q^{-1}}|\psi^2\psi^1\rangle - \frac{D_{12}-E_{12}}{q+q^{-1}}|\psi^1\psi^2\rangle + \frac{q^{-1}F_{12}}{q+q^{-1}}|\phi^2\phi^1\rangle - \frac{F_{12}}{q+q^{-1}}|\phi^1\phi^2\rangle$$

$$\mathscr{R}|\psi^2\psi^1\rangle = -\frac{D_{12}-E_{12}}{q+q^{-1}}|\psi^2\psi^1\rangle - \frac{q^{-1}D_{12}+qE_{12}}{q+q^{-1}}|\psi^1\psi^2\rangle - \frac{F_{12}}{q+q^{-1}}|\phi^2\phi^1\rangle + \frac{qF_{12}}{q+q^{-1}}|\phi^1\phi^2\rangle$$

$$\mathscr{R}|\psi^2\psi^2\rangle = -D_{12}|\psi^2\psi^2\rangle$$

$$\mathscr{R}|\phi^a\psi^\beta\rangle = G_{12}|\phi^a\psi^\beta\rangle + H_{12}|\psi^\beta\phi^a\rangle$$

$$\mathscr{R}|\psi^\alpha\phi^b\rangle = K_{12}|\phi^b\psi^\alpha\rangle + L_{12}|\psi^\alpha\phi^b\rangle$$

Fundamental R-matrix. Coefficients

$$A_{12} = R^0_{12}\,\frac{q^{C_1}U_1}{q^{C_2}U_2}\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}$$

$$B_{12} = R^0_{12}\,\frac{q^{C_1}U_1}{q^{C_2}U_2}\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\left(1-(q+q^{-1})q^{-1}\,\frac{x_2^+ - x_1^+}{x_2^+ - x_1^-}\,\frac{x_2^- - s(x_1^+)}{x_2^- - s(x_1^-)}\right)$$

$$C_{12} = R^0_{12}\,(q+q^{-1})\,\frac{ig\alpha^{-1}\gamma_2\gamma_1 q^{C_1}U_1}{q^{2C_2+3/2}U_2^2}\,\frac{ig^{-1}x_2^+ - (q-q^{-1})}{x_2^- - s(x_1^-)}\,\frac{s(x_2^+) - s(x_1^+)}{x_2^- - x_1^+}$$

$$D_{12} = -R^0_{12}$$

$$E_{12} = -R^0_{12}\left(1-(q+q^{-1})q^{-2C_2-1}U_2^{-2}\,\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\,\frac{x_2^+ - s(x_1^-)}{x_2^- - s(x_1^-)}\right)$$

$$F_{12} = -R^0_{12}\,(q+q^{-1})\,\frac{ig\alpha^{-1}\gamma_2\gamma_1 q^{C_1}U_1}{q^{2C_2+3/2}U_2^2}\,\frac{ig^{-1}x_2^+ - (q-q^{-1})}{x_2^- - s(x_1^-)}\,\frac{s(x_2^+) - s(x_1^+)}{x_2^- - x_1^+}$$
$$\cdot\,\frac{\alpha^2}{1-g^2(q-q^{-1})^2}\,\frac{U_2 q^{C_2+1/2}(x_2^+ - x_2^-)}{\gamma_2^2}\,\frac{U_1 q^{C_1+1/2}(x_1^+ - x_1^-)}{\gamma_1^2}$$

$$G_{12} = R^0_{12}\,\frac{1}{q^{C_2+1/2}U_2}\,\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}, \qquad H_{12} = R^0_{12}\,\frac{\gamma_1}{\gamma_2}\,\frac{x_2^+ - x_2^-}{x_2^- - x_1^+}$$

$$K_{12} = R^0_{12}\,\frac{q^{C_1}U_1}{q^{C_2}U_2}\,\frac{\gamma_2}{\gamma_1}\,\frac{x_1^+ - x_1^-}{x_2^- - x_1^+}, \qquad L_{12} = R^0_{12}\,q^{C_1+1/2}U_1\,\frac{x_2^- - x_1^-}{x_2^- - x_1^+}$$

Some q-notations

$$q^{2C} = q\,\frac{(q-q^{-1})/x^+ - ig^{-1}}{(q-q^{-1})/x^- - ig^{-1}} = q^{-1}\,\frac{(q-q^{-1})x^+ + ig^{-1}}{(q-q^{-1})x^- + ig^{-1}}\,.$$

$$u(y) = y + s(y), \qquad s(y) = \frac{ig^{-1} + (q-q^{-1})y}{ig^{-1}y - (q-q^{-1})}\,.$$

Descrete Symmetries of R-Matrix

Braiding unitarity $\mathscr{R}_{12}\mathscr{R}_{21} = 1 \otimes 1$ entails

$$A_{12}A_{21} = B_{12}B_{21} + C_{12}F_{21} = G_{12}L_{21} + H_{12}H_{21} = 1,$$
$$A_{12}D_{12} = B_{12}E_{12} - C_{12}F_{12} = H_{12}K_{12} - G_{12}L_{12}.$$

Yang–Baxter equation

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}.$$

Matrix Unitarity

$$(\mathscr{R}_{12})^{\dagger}\mathscr{R}_{12} = 1 \otimes 1.$$

Crossing Symmetry

$$(\mathscr{C}^{-1} \otimes 1)\mathscr{R}_{\underline{12}}^{\mathrm{ST}\otimes 1}(\mathscr{C} \otimes 1)\mathscr{R}_{12} = 1 \otimes 1.$$

imposes relations on scalar factor R^0_{12}

Near Neighbour Hamiltonian

Homogeneous Hamiltonian

$$\mathscr{H} = \sum_{k=1}^{L} \mathscr{H}_{k,k+1}.$$

The pairwise interaction $\mathscr{H}_{12}$ is the following logarithmic derivative of the R-matrix

$$\mathscr{H}_{12} = -i\frac{\left(x^+ - s(x^+)\right)\left(x^- - s(x^-)\right)}{q^{-1}x^+ s(x^+)}\left(\frac{du^*}{du}\right)^{-1/2} \mathscr{R}_{12}^{-1}\frac{d}{du_1}\mathscr{R}_{12}\bigg|_{x_{12}^{\pm}=x^{\pm}}.$$

The spectral parameters u_k are defined via $x_k^{\pm}$

$$u_k = q^{-1}u(x_k^+) - \frac{i}{2g} = qu(x_k^-) + \frac{i}{2g}.$$

The Hamiltonian

$$\mathscr{H}_{12}|\phi^1\phi^1\rangle = A|\phi^1\phi^1\rangle$$

$$\mathscr{H}_{12}|\phi^1\phi^2\rangle = \frac{qA+q^{-1}B}{q+q^{-1}}|\phi^1\phi^2\rangle + \frac{A-B}{q+q^{-1}}|\phi^2\phi^1\rangle + \frac{q^{-1}C}{q+q^{-1}}|\psi^1\psi^2\rangle - \frac{C}{q+q^{-1}}|\psi^2\psi^1\rangle$$

$$\mathscr{H}_{12}|\phi^2\phi^1\rangle = \frac{A-B}{q+q^{-1}}|\phi^1\phi^2\rangle + \frac{q^{-1}A+qB}{q+q^{-1}}|\phi^2\phi^1\rangle - \frac{C}{q+q^{-1}}|\psi^1\psi^2\rangle + \frac{qC}{q+q^{-1}}|\psi^2\psi^1\rangle$$

$$\mathscr{H}_{12}|\phi^2\phi^2\rangle = A|\phi^2\phi^2\rangle$$

$$\mathscr{H}_{12}|\psi^1\psi^1\rangle = D|\psi^1\psi^1\rangle$$

$$\mathscr{H}_{12}|\psi^1\psi^2\rangle = \frac{qD+q^{-1}E}{q+q^{-1}}|\psi^1\psi^2\rangle + \frac{D-E}{q+q^{-1}}|\psi^2\psi^1\rangle + \frac{q^{-1}F}{q+q^{-1}}|\phi^1\phi^2\rangle - \frac{F}{q+q^{-1}}|\phi^2\phi^1\rangle$$

$$\mathscr{H}_{12}|\psi^2\psi^1\rangle = \frac{D-E}{q+q^{-1}}|\psi^1\psi^2\rangle + \frac{q^{-1}D+qE}{q+q^{-1}}|\psi^2\psi^1\rangle - \frac{F}{q+q^{-1}}|\phi^1\phi^2\rangle + \frac{qF}{q+q^{-1}}|\phi^2\phi^1\rangle$$

$$\mathscr{H}_{12}|\psi^2\psi^2\rangle = D|\psi^2\psi^2\rangle$$

$$\mathscr{H}_{12}|\phi^a\psi^\beta\rangle = G|\psi^\beta\phi^a\rangle + H|\phi^a\psi^\beta\rangle$$

$$\mathscr{H}_{12}|\psi^\alpha\phi^b\rangle = K|\psi^\alpha\phi^b\rangle + L|\phi^b\psi^\alpha\rangle$$

The Hamiltonian

$$A = -D = \frac{1}{4g}\,\frac{(q^C U + q^{-C}U^{-1})(q^C U^{-1} + q^{-C}U)}{(q^C U - q^{-C}U^{-1})(q^C U^{-1} - q^{-C}U)}$$

$$A - B = E - D = \frac{q+q^{-1}}{g}\,\frac{1}{(q^C U - q^{-C}U^{-1})(q^C U^{-1} - q^{-C}U)}$$

$$C = F = (q+q^{-1})\sqrt{1-(q-q^{-1})^2 g^2}$$

$$G = \frac{q^{C+1/2}U^{-1} - q^{-C-1/2}U^{-1} - q^{C-1/2}U + q^{-C+1/2}U}{g(q-q^{-1})(q^C U - q^{-C}U^{-1})(q^C U^{-1} - q^{-C}U)}$$

$$L = \frac{q^{C+1/2}U - q^{-C-1/2}U - q^{C-1/2}U^{-1} + q^{-C+1/2}U^{-1}}{g(q-q^{-1})(q^C U - q^{-C}U^{-1})(q^C U^{-1} - q^{-C}U)}$$

$$H = K = 0$$

Bethe Equations and Spectrum

Generic for rank 3 algebra

$$1=\prod_{j=1}^{K} R^{\mathrm{I,II}}(x_j,y_k)\prod_{\substack{j=1\\ j\neq k}}^{N} R^{\mathrm{II,II}}(y_j,y_k)\prod_{j=1}^{M} R^{\mathrm{III,II}}(w_j,y_k),$$

$$1=\prod_{j=1}^{N} R^{\mathrm{I,III}}(x_j,w_k)\prod_{j=1}^{N} R^{\mathrm{II,III}}(y_j,w_k)\prod_{\substack{j=1\\ j\neq k}}^{M} R^{\mathrm{III,III}}(w_j,y_k),$$

For our case

$$1=\left(q^{-C-1/2}U^{-1}\frac{y_k-x^+}{y_k-x^-}\right)^K\prod_{j=1}^{M} q^{-1}\frac{qu(y_k)-w_j+\frac{i}{2}g^{-1}}{q^{-1}u(y_k)-w_j-\frac{i}{2}g^{-1}},$$

$$1=\prod_{j=1}^{N} q\,\frac{w_k-q^{-1}u(y_j)+\frac{i}{2}g^{-1}}{w_k-qu(y_j)-\frac{i}{2}g^{-1}}\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{q^{-1}w_k-qw_j-\frac{i}{2}(q+q^{-1})g^{-1}}{qw_k-q^{-1}w_j+\frac{i}{2}(q+q^{-1})g^{-1}}.$$

Energy

$$E=E_0K+\sum_{j=1}^{N}E(y_j).$$

$$E_0=A,\qquad E(y_k)=H+K-2A+Ge^{ip_k}+Le^{-ip_k},$$

1D Hubbard Model

Hamiltonian

$$\mathscr{H}_{j,k}^{\text{Hub}} = \sum_{\alpha=1,2} \left(c^\dagger_{\alpha,j} c_{\alpha,k} + c^\dagger_{\alpha,k} c_{\alpha,j} \right) + U n_{1,j} n_{2,j}.$$

exhibits $\mathfrak{su}(2) \times \mathfrak{su}(2) \in \mathfrak{su}(2|2)$ symmetry once

$$|\phi_k^1\rangle = |\circ\rangle, \qquad |\phi_k^2\rangle = \kappa c^\dagger_{1,k} c^\dagger_{2,k} |\circ\rangle, \qquad |\psi_k^1\rangle = c^\dagger_{1,k}|\circ\rangle, \qquad |\psi_k^2\rangle = c^\dagger_{2,k}|\circ\rangle.$$

Alcaraz and Bariev model [Alcaraz, Bariev '99]

$$
\begin{aligned}
\mathscr{H}^{\mathrm{AB}}_{j,k} = {} & (c^\dagger_{1,j}c_{1,k} + c^\dagger_{1,k}c_{1,j})(1 + t_{11}n_{2,j} + t_{12}n_{2,k} + t'_1 n_{2,j}n_{2,k}) \\
& + (c^\dagger_{2,j}c_{2,k} + c^\dagger_{2,k}c_{2,j})(1 + t_{21}n_{1,j} + t_{22}n_{1,k} + t'_2 n_{1,j}n_{1,k}) \\
& + J(c^\dagger_{1,j}c^\dagger_{2,k}c_{2,j}c_{1,k} + c^\dagger_{1,k}c^\dagger_{2,j}c_{2,k}c_{1,j}) \\
& + t_{\mathrm{p}}(c^\dagger_{1,j}c^\dagger_{2,j}c_{2,k}c_{1,k} + c^\dagger_{1,j}c^\dagger_{2,j}c_{2,k}c_{1,k}) \\
& + V_{11}n_{1,j}n_{1,k} + V_{12}n_{1,j}n_{2,k} + V_{21}n_{2,j}n_{1,k} + V_{22}n_{2,j}n_{2,k} + Un_{1,j}n_{2,j} \\
& + V^{(1)}_3 n_{2,j}n_{1,k}n_{2,k} + V^{(2)}_3 n_{1,j}n_{1,k}n_{2,k} \\
& + V^{(3)}_3 n_{1,j}n_{2,j}n_{2,k} + V^{(4)}_3 n_{1,j}n_{2,j}n_{1,k} \\
& + V_4 n_{1,j}n_{2,j}n_{1,k}n_{2,k},
\end{aligned}
$$

where

$$
\begin{aligned}
t_{11} &= t_4 - 1, & t_{12} &= t_3 - 1, & t'_1 &= t_5 - t_3 - t_4 + 1, \\
t_{21} &= t_1 - 1, & t_{22} &= t_2 - 1, & t'_2 &= t_5 - t_1 - t_2 + 1.
\end{aligned}
$$

and ...

Alcaraz and Bariev model [Alcaraz, Bariev '99]

... was found to be integrable in four cases A$^\pm$ and B$^\pm$

In the case $A^\pm$

$$t_1 = \varepsilon t_2 = t_3 = \varepsilon t_4 = \sin\vartheta, \quad t_5 = \varepsilon = \pm 1,$$

$$J = -\varepsilon t_{\mathrm{p}} = -\tfrac{1}{2}\varepsilon U = V_{12}e^{2\eta} = V_{21}e^{-2\eta} = \cos\vartheta,$$

$$V_{11} = V_{22} = V_3^{(1)} = V_3^{(2)} = V_3^{(3)} = V_3^{(4)} = V_4 = 0,$$

and in the case B$^\pm$

$$t_1 = \varepsilon t_2 = \varepsilon t_3 e^{2\eta} = t_4 e^{-2\eta} = \sin\vartheta, \quad t_5 = \varepsilon = \pm 1,$$

$$J = -\varepsilon t_{\mathrm{p}} = V_{12}e^{2\eta} = V_{21}e^{-2\eta} = \cos\vartheta, \quad U = 2t_p + \sin\vartheta\tan\vartheta(e^{\eta} - \varepsilon e^{-\eta})^2,$$

$$V_{11} = V_{22} = V_3^{(2)} = V_3^{(4)} = V_4 = 0, \quad V_3^{(1)} = -V_3^{(3)} = V_{12} - V_{21}.$$

with the free parameters ϑ, η.

Relation to CM notation

Four d.o.f. for each site

$$|\circ\rangle, \quad |\uparrow\rangle \sim c_1^\dagger|\circ\rangle, \quad |\downarrow\rangle \sim c_2^\dagger|\circ\rangle, \quad |\updownarrow\rangle \sim c_1^\dagger c_2^\dagger|\circ\rangle$$

or

$$|\phi_k^1\rangle = |\circ\rangle, \qquad |\phi_k^2\rangle = \kappa c_{1,k}^\dagger c_{2,k}^\dagger|\circ\rangle, \qquad |\psi_k^1\rangle = c_{1,k}^\dagger|\circ\rangle, \qquad |\psi_k^2\rangle = c_{2,k}^\dagger|\circ\rangle.$$

anticommutators

$$\{c_{\alpha,k}, c_{\beta,l}^\dagger\} = \delta_{\alpha\beta}\delta_{kl}, \qquad \{c_{\alpha,k}, c_{\beta,l}\} = \{c_{\alpha,k}^\dagger, c_{\beta,l}^\dagger\} = 0.$$

number operators

$$n_{\alpha,k} = c_{\alpha,k}^\dagger c_{\alpha,k}$$

Our Hamiltonian in electronic notation

$$
\begin{aligned}
\mathscr{H}_{j,k} = {} & \frac{A-B}{q+q^{-1}}\left(c^{\dagger}_{1,j}c^{\dagger}_{2,j}c_{2,k}c_{1,k}+c^{\dagger}_{1,k}c^{\dagger}_{2,k}c_{2,j}c_{1,j}\right)-\frac{D-E}{q+q^{-1}}\left(c^{\dagger}_{1,j}c^{\dagger}_{2,k}c_{2,j}c_{1,k}+c^{\dagger}_{1,k}c^{\dagger}_{2,j}c_{2,k}c_{1,j}\right) \\
& +\frac{1}{q+q^{-1}}\,c^{\dagger}_{1,j}c_{1,k}\big(q^{-1}\kappa^{-1}C(1-n_{2,j})n_{2,k}-q\kappa F n_{2,j}(1-n_{2,k})\big) \\
& +\frac{1}{q+q^{-1}}\,c^{\dagger}_{2,j}c_{2,k}\big(\kappa^{-1}C(1-n_{1,j})n_{1,k}-\kappa F n_{1,j}(1-n_{1,k})\big) \\
& +\frac{1}{q+q^{-1}}\,c^{\dagger}_{1,k}c_{1,j}\big(q^{-1}\kappa F(1-n_{2,j})n_{2,k}-q\kappa^{-1}C n_{2,j}(1-n_{2,k})\big) \\
& +\frac{1}{q+q^{-1}}\,c^{\dagger}_{2,k}c_{2,j}\big(\kappa F(1-n_{1,j})n_{1,k}-\kappa^{-1}C n_{1,j}(1-n_{1,k})\big) \\
& +c^{\dagger}_{1,j}c_{1,k}\big(G(1-n_{2,j})(1-n_{2,k})-L n_{2,j}n_{2,k}\big)+c^{\dagger}_{2,j}c_{2,k}\big(G(1-n_{1,j})(1-n_{1,k})-L n_{1,j}n_{1,k}\big) \\
& +c^{\dagger}_{1,k}c_{1,j}\big(L(1-n_{2,j})(1-n_{2,k})-G n_{2,j}n_{2,k}\big)+c^{\dagger}_{2,k}c_{2,j}\big(L(1-n_{1,j})(1-n_{1,k})-G n_{1,j}n_{1,k}\big) \\
& +A+(K-A)(n_{1,j}+n_{2,j})+(H-A)(n_{1,k}+n_{2,k})+(A+D-H-K)(n_{1,j}n_{1,k}+n_{2,j}n_{2,k}) \\
& +\left(A-2H+\frac{qA+q^{-1}B}{q+q^{-1}}\right)n_{1,k}n_{2,k}+\left(A-2K+\frac{q^{-1}A+qB}{q+q^{-1}}\right)n_{1,j}n_{2,j} \\
& +\left(A-H-K+\frac{qD+q^{-1}E}{q+q^{-1}}\right)n_{1,j}n_{2,k}+\left(A-H-K+\frac{q^{-1}D+qE}{q+q^{-1}}\right)n_{2,j}n_{1,k} \\
& +\left(-A-D+2H+2K-\frac{qA+q^{-1}B}{q+q^{-1}}-\frac{q^{-1}D+qE}{q+q^{-1}}\right)n_{2,j}n_{1,k}n_{2,k} \\
& +\left(-A-D+2H+2K-\frac{qA+q^{-1}B}{q+q^{-1}}-\frac{qD+q^{-1}E}{q+q^{-1}}\right)n_{1,j}n_{1,k}n_{2,k}
\end{aligned}
$$

Possible Transformations

One can: twist, add central elements... and change spectrum in controlable way

$$\mathscr{H}'_{12} = a_0 \mathscr{T} \mathscr{H}_{12} \mathscr{T}^{-1} + \tfrac{1}{2} a_1 \Delta(\mathfrak{H}_1) + a_2 \Delta(1) + \tfrac{1}{2} a_3 \Delta(\mathfrak{H}_3)$$
$$+ \tfrac{1}{2} b_1 (\mathfrak{H}_1 \otimes 1 - 1 \otimes \mathfrak{H}_1) + b_2 (\mathfrak{H}_1 \mathfrak{H}_1 \otimes 1 - 1 \otimes \mathfrak{H}_1 \mathfrak{H}_1)$$
$$+ \tfrac{1}{2} b_3 (\mathfrak{H}_3 \otimes 1 - 1 \otimes \mathfrak{H}_3)$$

Twist [Reshetikhin '90 Let.MathPhys]

$$\mathscr{T} = \exp\Bigg(i f_1 \sum_{j=1}^{K} (j-1) \mathfrak{H}_{1,j} + \tfrac{i}{2} f_2 \sum_{j<k=1}^{K} (\mathfrak{H}_{1,j} \mathfrak{H}_{3,k} - \mathfrak{H}_{3,j} \mathfrak{H}_{1,k})$$
$$+ i f_3 \sum_{j=1}^{K} (j-1) \mathfrak{H}_{3,k} \Bigg)$$

Transformations and Spectrum

...and change spectrum in controlable way

$$1 = \left(e^{i(f_3-f_1-f_2)} q^{-C-1/2} U^{-1} \frac{y_k - x^+}{y_k - x^-}\right)^K \prod_{j=1}^{M} e^{2if_2} q^{-1} \frac{qu(y_k) - w_j + \frac{i}{2} g^{-1}}{q^{-1}u(y_k) - w_j - \frac{i}{2} g^{-1}},$$

$$1 = e^{2i(f_2-f_3)K} \prod_{j=1}^{N} e^{-2if_2} q \frac{w_k - q^{-1}u(y_j) + \frac{i}{2} g^{-1}}{w_k - qu(y_j) - \frac{i}{2} g^{-1}} \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{q^{-1}w_k - qw_j - \frac{i}{2}(q+q^{-1})g^{-1}}{qw_k - q^{-1}w_j + \frac{i}{2}(q+q^{-1})g^{-1}}.$$

Energy

$$E' = (a_0 E_0 - a_1 + a_2)K + 2a_3 M + \sum_{j=1}^{N} \big(a_0 E(y_j) + a_1 - a_3\big).$$

Quantum Deformation of the 1D Hubbard

Q-deformation of the Hubbard model limit [PK, Beisert 0802.0777]

$$\begin{aligned}\mathscr{H}'_{j,k} = A' \sum_{\ell=j,k} \big((1-n_{1,\ell})(1-n_{2,\ell}) + n_{1,\ell}n_{2,\ell} - \tfrac{1}{2}\big) \\ + iq^{+1/2} c^\dagger_{1,j} c_{1,k} \big(1-(1-q^{+1/2})n_{2,j}\big)\big(1-(1-q^{-3/2})n_{2,k}\big) \\ + iq^{+1/2} c^\dagger_{2,j} c_{2,k} \big(1-(1-q^{-1/2})n_{1,j}\big)\big(1-(1-q^{-1/2})n_{1,k}\big) \\ - iq^{-1/2} c^\dagger_{1,k} c_{1,j} \big(1-(1-q^{+3/2})n_{2,j}\big)\big(1-(1-q^{-1/2})n_{2,k}\big) \\ - iq^{-1/2} c^\dagger_{2,k} c_{2,j} \big(1-(1-q^{+1/2})n_{1,j}\big)\big(1-(1-q^{+1/2})n_{1,k}\big).\end{aligned}$$

So What is Alcaraz–Bariev Case A? [PK, Beisert in progress]

- Same spectrua for A and B cases
- Relation between quntum deformations

$$\cosh 2\eta_B = \cosh 2\eta_a \cos\vartheta + \varepsilon \sin\vartheta$$

admits Möbius parametrisation (two fixed points ± 1)

$$e^{2\eta_A} = \varepsilon\xi \frac{\xi - \cos\vartheta}{1 - \xi\cos\vartheta}, \quad e^{2\eta_B} = \frac{\varepsilon}{\xi}\frac{\xi - \cos\vartheta}{1 - \xi\cos\vartheta}$$

- Similar structure of II level S-matrices
- Nonlocal dynamical (momenta dependent) transformation from A to B exists
- Solutions: Different coalgebra?