Hypergeometric functions, Feynman diagrams and special functions.

How to evaluate Feynman diagrams withous integration by parts.

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- Feynman Diagrams via Hypergeometric functions
- reduction to the basis
- construction of ε-expansion
- particular values of argument

Introduction

In 1965 Regge make proposal, that
any Feynman diagram is a special class of hypergeometric functions. [Kershaw, 1973; Wu, 1974; Golubeva, 1976]

According to Horn, a formal (Laurent) power series in r-variables,

$$\Phi(\vec{x}) = \sum_{\vec{m}} c(\vec{m}) \vec{x}^m = \sum_{m_1, \cdots, m_r} c(m_1, \cdots m_r) x_1^{m_1} \cdots x_r^{m_r} \,,$$

is called hypergeometric if for all $i = 1, \cdots, r$ the ration

$$\frac{c(\vec{m} + e_i)}{c(\vec{m})}$$

is a rational function of $m_1, \cdots, m_r$ and e_i is vector

$$e_i = 0, \cdots, 0, 1, 0, \cdots, 0.$$

(unit in the i-th place).

Ore[1930], Sato[1990]

$$c(\vec{m}) = \Pi_{i=1}^r \lambda_i^{m_i} R(\vec{m}) \left(\Pi_{j=1}^N \Gamma(\mu_j(\vec{m}) + \gamma_j + 1) \right)^{-1}$$

R is an arbitrary rational function.

Formulation of problem

The elaboration of the algorithm for reduction and analytical evaluation of the higher order terms of the ε-expansion of any hypergeometric functions of several variables with arbitrary set of parameters.

There is not universal agreement on what it means to express a solution in terms of known special functions. One reasonable answer has been presented by Kitaev, when he quotes R. Askye's Forward to the book *Symmetries and Separation of Variables* by W. Miller, Jr., which says "One term which has not been defined so far is 'special function'. My definition is simple, but not time invariant. A function is a special function if it occurs often enough so that it gets a name".
Kitaev adds, "... most of the people who apply them . . . understand, under the notion of special functions, a set of functions which can be found in one of the well-known reference books. . . ." To this, we may add "functions which can be found in one of the well-known computer algebra systems."

"Quantum" problem

One of the *classical tasks* in mathematics is to find the full set of parameters and arguments for which hypergeometric functions are expressible in terms of algebraic functions. Quantum field theory makes a *quantum* generalisation of this classical task: to find the full set of parameters and arguments so that the all-order ε-expansion is expressible in terms of known special functions or identify the full set of functions which must be invented in order to construct the all-order ε-expansion of generalized hypergeometric functions.

The Gauss hypergeometric function $\omega(z) \equiv {}_2F_1(a,b;c;z)$ could be defined as

- Solution of second-order differential equation of Fuchsian type with three regular singular points at $z = 0, 1, \infty$:

$$\frac{d}{dz}\left(z\frac{d}{dz} + c - 1\right)\omega(z) = \left(z\frac{d}{dz} + a\right)\left(z\frac{d}{dz} + b\right)\omega(z) \; ,$$

- Series

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \; ,$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol.

- Integral:
  1. Euler-Pochammer type:

$$\omega(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt$$

  2. Mellin-Barnes:

$$\omega(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s ds$$

M.K., JHEP, 2006

As is known, for any three contiguous Gauss hypergeometric functions there is a contiguous relation, which is a linear relation with coefficients being rational functions in the parameters A, B, C and argument z.

$$
\begin{aligned}
& P_1(A,B,B,z)\ {}_2F_1\left(\begin{array}{c} A \pm 1, B \\ C \end{array} \middle| z \right) \\
& + P_2(A,B,B,z)\ {}_2F_1\left(\begin{array}{c} A, B \pm 1 \\ C \end{array} \middle| z \right) \\
& + P_3(A,B,B,z)\ {}_2F_1\left(\begin{array}{c} A, B \\ C \pm 1 \end{array} \middle| z \right) = 0
\end{aligned}
$$

Any Gauss hypergeometric function with arbitrary parameters is reduced to the combination of we are able to reduce an original Gauss hypergeometric function to the linear combination of two (our basis):

$$
\begin{aligned}
& P(a,b,c,z) {}_2F_1(a+I_1, b+I_2; c+I_3; z) \\
& = \left\{ Q_1(a,b,c,z) \frac{d}{dz} + Q_2(a,b,c,z) \right\} {}_2F_1(a,b;c;z) \ ,
\end{aligned}
$$

where $a, b, c,$ are any fixed numbers, P, Q_1, Q_2 are polynomial in parameters a, b, c and argument z, and I_1, I_2, I_3 any integer numbers. These basis functions are related by a differential identity:

$$
\frac{d}{dz}\ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| z \right) = \frac{ab}{c}\ {}_2F_1\left(\begin{array}{c} a+1, b+1 \\ c+1 \end{array} \middle| z \right) \ .
$$

In the case when some of the parameters are positive integers (let us put $B = m$), we get one function with the value of one of the parameter equal to unity and some polynomial with respect to z (parameter $B = 0$). In this case, the following two relations should be used for further reduction:

$$a(1-z)\; {}_2F_1\left(\begin{array}{c} 1, a+1 \\ c \end{array} \middle| z \right)$$
$$= (c-1) + (1+a-c)\; {}_2F_1\left(\begin{array}{c} 1, a \\ c \end{array} \middle| z \right) ,$$
$$(a-c)z\; {}_2F_1\left(\begin{array}{c} 1, a \\ c+1 \end{array} \middle| z \right) = -c\left[1-(1-z)\; {}_2F_1\left(\begin{array}{c} 1, a \\ c \end{array} \middle| z \right) \right] .$$

In this way, if one of the upper parameters is an integer, then the result of reduction is expressible in terms of one Gauss hypergeometric function and a polynomials (the function ${}_1F_0$). For case $c = b$, we should apply the Kummer relation:

$$\begin{aligned} {}_2F_1\left(\begin{array}{c} A, b \\ 1+b \end{array} \middle| z \right) &= \frac{1}{(1-z)^A}\; {}_2F_1\left(\begin{array}{c} 1, A \\ 1+b \end{array} \middle| -\frac{z}{1-z} \right) \\ &= (1-z)^{1-A}\; {}_2F_1\left(\begin{array}{c} 1, 1+b-A \\ 1+b \end{array} \middle| z \right) . \end{aligned}$$

M.K., JHEP, 2006

Let us consider as basis the Gauss hypergeometric function with the following set of parameters:

$$\omega(z) = {}_2F_1\left(\frac{p_1}{q_1}+a_1\varepsilon, \frac{p_2}{q_2}+a_2\varepsilon; 1-\frac{p_3}{q_3}+c\varepsilon; z\right);.$$

It is the solution of the differential equation

$$\left[z\frac{d}{dz}+\frac{p_1}{q_1}+a_1\varepsilon\right]\left[z\frac{d}{dz}+\frac{p_2}{q_2}+a_2\varepsilon\right]\omega(z) = \frac{d}{dz}\left[z\frac{d}{dz}-\frac{p_3}{q_3}+c\varepsilon\right]\omega(z) \; .$$

with boundary conditions $w(0)=1$ and $z\frac{d}{dz}w(z)\big|_{z=0}=0$. Due to analyticity of Gauss hypergeometric function with respect to parameters, this equation is valid in each order of ε, so that in terms of coefficients functions $w_k(z)$ defined as

$$w(z) = \sum_{k=0}^{\infty} w_k(z)\varepsilon^k.$$

In terms of coefficients functions $\omega_k(z)$, we have

$$\left[(1-z)\frac{d}{dz}-\left(\frac{p_1}{q_1}+\frac{p_2}{q_2}\right)-\frac{1}{z}\frac{p_3}{q_3}\right]\left(z\frac{d}{dz}\right)\omega_k - \frac{p_1p_2}{q_1q_2}\omega_k = \left(a_1+a_2-\frac{c}{z}\right)\left(z\frac{d}{dz}\right)\omega_{k-1} + \left(a_1\frac{p_2}{q_2}+a_2\frac{p_1}{q_1}\right)\omega_{k-1}+a_1a_2\omega_{k-2} \; .$$

Differential Equation approach for construction of ε-expansion (II)

M.K., JHEP, 2006

The main idea is to find a new parametrization (change of variable) $z \to \xi(z)$, and to define a new functions $\rho_k(\xi)$, related with a first derivative of original functions $\omega_k(\xi)$,

$$\rho_k(\xi) = \sum_j \Gamma_{kj}(\xi) \frac{d}{d\xi} \omega_j(\xi) \ ,$$

so that original equation can be rewritten as system of linear differential equations of the first order with an rational coefficients:

$$\begin{aligned}
\frac{d}{d\xi} w_k(\xi) &= \rho_k(\xi) \sum_j \frac{A_j}{\xi - \alpha_j} \ , \\
\frac{d}{d\xi} \rho_k(\xi) &= \rho_{k-1}(\xi) \sum_j \frac{B_j}{\xi - \beta_j} \\
&\quad + \omega_{k-1}(\xi) \sum_j \frac{C_j}{\xi - \gamma_j} + \omega_{k-2}(\xi) \sum_j \frac{D_j}{\xi - \sigma_j} \ ,
\end{aligned}$$

where $A_j, B_j, C_j, D_j, \alpha_j, \beta_j, \gamma_j, \sigma_j$ Then the iterative solution of this system can be constructed. Under condition, $\omega_0(z) = 1 (\rho_0 = 0)$, this solution are expressible in terms of hyperlogarithms depending on parameters $\alpha_j, \beta_j, \gamma_j, \sigma_j$, (possible) times on powers of logarithms.

The main problem is to find general algorithm for getting of this parametrisation and enumerate all possible values of parameters. We are not able to proof that we got solution of this problem for all possible values of parameters. But for some special set of parameters the solution is done.

Iterative solution of differential equation: integer values of parameters

M.K.,Ward,Yost, JHEP, 07.

$${}_2F_1(a_1\varepsilon, a_2\varepsilon; 1+c\varepsilon; z)$$

Starting equation is

$$
\begin{aligned}
&(1-z)\frac{d}{dz}\left(z\frac{d}{dz}\right) w_k(z) \\
&= \left(a_1+a_2-\frac{c}{z}\right)\left(z\frac{d}{dz}\right) w_{k-1}(z)+a_1a_2w_{k-2}(z) \; .
\end{aligned}
$$

Let us introduce a new function: and rewrite original equation as

$$
\begin{aligned}
(1-z)\frac{d}{dz}\rho_i(z) &= \left(a_1+a_2-\frac{c}{z}\right)\rho_{i-1}(z)+a_1a_2w_{i-2}(z) \; , \\
z\frac{d}{dz}w_i(z) &= \rho_i(z) \; .
\end{aligned}
$$

The solution of this system can be presented in an iterated form:

$$
\begin{aligned}
\rho_i(z) &= (a_1+a_2-c)\int_0^z \frac{dt}{1-t}\rho_{i-1}(t) \\
&\quad +a_1a_2\int_0^z \frac{dt}{1-t}w_{i-2}(t)-cw_{i-1}(z) \; , \quad i \geq 1 \; , \\
w_i(z) &= \int_0^z \frac{dt}{t}\rho_i(t) \; , \quad i \geq 1 \; .
\end{aligned}
$$

and written in terms of generalized polylogarithms:

$$
\begin{aligned}
&\mathrm{Li}_{k_1,\cdots,k_n}(z) \\
&= \int_0^z \underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{k_1-1 \text{ times}}\circ\frac{dt}{1-t}\circ\cdots\circ\underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{k_n-1 \text{ times}}\circ\frac{dt}{1-t} \\
&= \sum_{m_1>m_2>\cdots m_n>0} \frac{z^{m_1}}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}} \; .
\end{aligned}
$$

Iterative solution of differential equation: half-integer values of parameters

M.K.,Ward,Yost, JHEP, 07.

$$ {}_2F_1\left(\begin{array}{c} a_1\varepsilon, a_2\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| z \right) . $$

Original equation is

$$ \left[(1-z)\frac{d}{dz} - \frac{1}{2z}\right]\left(z\frac{d}{dz}\right) w_i(z) $$
$$ = \left[(a_1+a_2) - \frac{f}{z}\right]\left(z\frac{d}{dz}\right) w_{i-1}(z) + a_1 a_2 w_{i-2}(z) . $$

Let us introduce the new variable y

$$ y = \frac{1-\sqrt{\frac{z}{z-1}}}{1+\sqrt{\frac{z}{z-1}}} , \quad z = -\frac{(1-y)^2}{4y} , $$

and define a set of a new functions $\rho_i(y)$

$$ \begin{aligned} y\frac{d}{dy}\rho_i(y) &= (a_1+a_2)\frac{1-y}{1+y}\rho_{i-1}(y) \\ &\quad +2f\left(\frac{1}{1-y} - \frac{1}{1+y}\right)\rho_{i-1}(y) + a_1 a_2 w_{i-2}(y) , \\ y\frac{d}{dy}w_k(y) &= -\rho_k(y) . \end{aligned} $$

The solution of these differential equations has the form

$$ \begin{aligned} \rho_i(y) &= \int_1^y dt \left[2f\frac{1}{1-t} - 2(a_1+a_2-f)\frac{1}{1+t}\right]\rho_{i-1}(t) \\ &\quad -(a_1+a_2)\left[w_{i-1}(y) - w_{i-1}(1)\right] \\ &\quad +a_1 a_2 \int_1^y \frac{dt}{t} w_{i-2}(t) , \quad i \geq 1 , \\ w_i(y) &= -\int_1^y \frac{dt}{t}\rho_i(t) , \quad i \geq 1 . \end{aligned} $$

Iterative solution of differential equation: half-integer values of parameters

There is a new type of function, coming from the integral $\int f(t)dt/(1+t)$.

$$
\begin{aligned}
w_k(y) &= \sum_{j=0}^{k} c(\vec{s},\vec{\sigma},k) \ln^{k-j}(y) \left[\mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y) - \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(1) \right] , \\
\rho_k(y) &= \sum_{j=0}^{k-1} \tilde{c}(\vec{s},\vec{\sigma},k) \ln^{k-j}(y) \left[\mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y) - \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(1) \right] ,
\end{aligned}
$$

$c(\vec{s},\vec{\sigma},k)$ and $\tilde{c}(\vec{s},\vec{\sigma},k)$ are numerical coefficients, $\vec{s}$ and $\vec{\sigma}$ are multi-index, $\vec{s} = (s_1, \cdots s_n)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_n)$, $\sigma_k = \pm 1$, $\mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y)$ is a coloured multiple polylogarithm of one variable defined as

$$
\mathrm{Li}_{\binom{\sigma_1,\sigma_2,\cdots,\sigma_k}{s_1,s_2,\cdots,s_n}}(z) = \sum_{m_1 > m_2 > \cdots m_n > 0} z^{m_1} \frac{\sigma_1^{m_1} \cdots \sigma_n^{m_n}}{m_1^{s_1} m_2^{s_2} \cdots m_n^{s_n}} .
$$

It has an iterated integral representation w.r.t. three differential forms,

$$
\begin{aligned}
\omega_0 &= \frac{dy}{y}, \quad \sigma = 0, \\
\omega_\sigma &= \frac{dy}{y-\sigma}, \quad \sigma = \pm 1,
\end{aligned}
$$

$$
\mathrm{Li}_{\binom{\sigma_1,\sigma_2,\cdots,\sigma_k}{s_1,s_2,\cdots,s_k}}(y) = \int_0^1 \omega_0^{s_1-1} \omega_{\sigma_1} \omega_0^{s_2-1} \omega_{\sigma_1 \sigma_2} \cdots \omega_0^{s_k-1} \omega_{\sigma_1 \sigma_2 \cdots \sigma_k} ,
$$

Hyperlogarithms (Multiple polylogarithms)

The starting point of our consideration is integral

$$
\begin{aligned}
& I(a_k, a_{k-1} \cdots, a_1; z) \\
& = \int_0^z \frac{dt_k}{t_k - a_k} \int_0^{t_k} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \cdots \int_0^{t_2} \frac{dt_1}{t_1 - a_1} \\
& = \int_0^z \frac{dt}{t - a_k} I(a_{k-1} \cdots, a_1; t) \;,
\end{aligned}
$$

where we put that all $a_k \neq 0$. In early consideration by Kummer, Poincare, Lappo-Danilevky this integral was called as hyperlogarithms It was treated as analytical functions of one variable z, the upper limit of integration. Goncharov has analysed it as multivalued analytical functions on $a_1, \cdots, a_k, z$. By definition, the multiple polylogarithm

$$
\mathrm{Li}_{k_1, k_2, \cdots, k_n}(x_1, x_2, \cdots, x_n) = \sum_{m_n > \cdots > m_1 > 0}^{\infty} \frac{x_1^{m_1}}{m_1^{k_1}} \frac{x_2^{m_2}}{m_2^{k_2}} \cdots \frac{x_n^{m_n}}{m_n^{k_n}} \;,
$$

where **weight** $k = k_1 + k_2 + \cdots + k_n$ and **depth** is equal to n.
The multiple polylogarithm is a special case of iterated integral

$$
\begin{aligned}
& G_{m_n, m_{n-1}, \cdots, m_1}(x_n, \cdots, x_1; z) \\
& \equiv I(\underbrace{0, \cdots, 0}_{m_n - 1 \text{ times}}, x_n, \underbrace{0, \cdots, 0}_{m_{n-1} - 1 \text{ times}}, x_{n-1}, \cdots, \underbrace{0, \cdots, 0}_{m_1 - 1 \text{ times}}, x_1; z) \\
& = (-1)^n \mathrm{Li}_{m_1, m_2, \cdots, m_n}\left(\frac{x_2}{x_1}, \frac{x_3}{x_2}, \cdots, \frac{z}{x_n} \right) \;.
\end{aligned}
$$

The inverse relation is

$$
\begin{aligned}
& \mathrm{Li}_{k_1, k_2, \cdots, k_n}(y_1, y_2, \cdots, y_n) \\
& = (-1)^n G_{k_n, k_{n-1}, \cdots, k_2, k_1}\left(\frac{1}{y_n}, \frac{1}{y_n y_{n-1}}, \cdots, \frac{1}{y_1 \cdots y_n}; 1 \right) \;.
\end{aligned}
$$

M.K., JHEP 06.

Let us consider a Gauss hypergeometric functions with integer or half-integer values of ε-independent parameters. We will call these basis functions as functions of type **A, B, C, D, E, F**. For each type the values of a, b, c, parameters of our basis, are presented in Table I:

$$ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| z \right) $$

Table I						
	A	**B**	**C**	**D**	**E**	**F**
a	$a_1\varepsilon$	$a\varepsilon$	$a\varepsilon$	$\frac{1}{2}+b_1\varepsilon$	$a_1\varepsilon$	$\frac{1}{2}+b_1\varepsilon$
b	$a_2\varepsilon$	$\frac{1}{2}+b\varepsilon$	$\frac{1}{2}+b\varepsilon$	$\frac{1}{2}+b_2\varepsilon$	$a_2\varepsilon$	$\frac{1}{2}+b_2\varepsilon$
c	$\frac{1}{2}+f\varepsilon$	$1+c\varepsilon$	$\frac{1}{2}+f\varepsilon$	$\frac{1}{2}+f\varepsilon$	$1+c\varepsilon$	$1+c\varepsilon$

The number of independent basis hypergeometric functions, enumerated in Table I, can be reduced by help of the Kummer transformations of variable z.

$$ z \to \frac{1}{z}, 1-z, \frac{1}{1-z}, \frac{-z}{1-z}, 1-\frac{1}{z} $$

With respect to this transformations the functions of type **A, B, C, D** are transformed into each other. This allows us to reduce the number of independent hypergeometric functions. The functions of type **E, F** transform into functions of the same type.

M.K., JHEP 06.

Let us illustrate how functions of type **B, C, D** can be expressed in terms of functions of type **A**:

D-type:

$$
{}_2F_1\left(\begin{array}{c} \frac{1}{2}+b_1\varepsilon, \frac{1}{2}+b_2\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| z\right)
= \frac{(1-z)^{(f-b_1-b_2)\varepsilon}}{(1-z)^{1/2}} \, {}_2F_1\left(\begin{array}{c} (f-b_1)\varepsilon, (f-b_2)\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| z\right) ,
$$

C-type:

$$
{}_2F_1\left(\begin{array}{c} \frac{1}{2}+b, a\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| z\right)
= \frac{1}{(1-z)^{a\varepsilon}} \, {}_2F_1\left(\begin{array}{c} a\varepsilon, (f-b)\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| -\frac{z}{1-z}\right)
$$

B-type:

$$
{}_2F_1\left(\begin{array}{c} \frac{1}{2}+b\varepsilon, a\varepsilon \\ 1+c\varepsilon \end{array} \middle| z\right) =
\frac{\Gamma(1+c\varepsilon)\Gamma\left(-\frac{1}{2}-(c-a-b)\varepsilon\right)}{\Gamma(a\varepsilon)\Gamma\left(\frac{1}{2}+b\varepsilon\right)} \frac{(1-z)^{1/2+(c-a-b)\varepsilon}}{z^{1-(a-c)\varepsilon}}
{}_2F_1\left(\begin{array}{c} 1+(c-a)\varepsilon, 1-a\varepsilon \\ \frac{3}{2}+(c-a-b)\varepsilon \end{array} \middle| 1-\frac{1}{z}\right)
+\frac{1}{z^{a\varepsilon}} \frac{\Gamma(1+c\varepsilon)\Gamma\left(\frac{1}{2}+(c-a-b)\varepsilon\right)}{\Gamma(1+(c-a)\varepsilon)\Gamma\left(\frac{1}{2}+(c-b)\varepsilon\right)}
{}_2F_1\left(\begin{array}{c} a\varepsilon, (a-c)\varepsilon \\ \frac{1}{2}+(a+b-c)\varepsilon \end{array} \middle| 1-\frac{1}{z}\right)
$$

As a result, we get the following statement:

Any functions of type A, B, C, D can be expressed in an algebraic way in terms of just one of these types.

There are several important master-integrals expressible in terms of ${}_2F_1$ hypergeometric functions: This set of integrals includes one-loop propagator type diagram with arbitrary values of mass and momentum; two-loop bubble integral with an arbitrary values of masses, and one-loop massless vertex diagram with three non-zero external momenta. For these diagrams, all order ε-expansions can be written in terms of Nielsen polylogarithms only.

A.I.Davydychev , Phys.Rev.(1999);
A.I.Davydychev & M.K., Nucl.Phys.Proc.Suppl.89 (2000);
A.I.Davydychev & M.K., Nucl.Phys.B605 (2001)

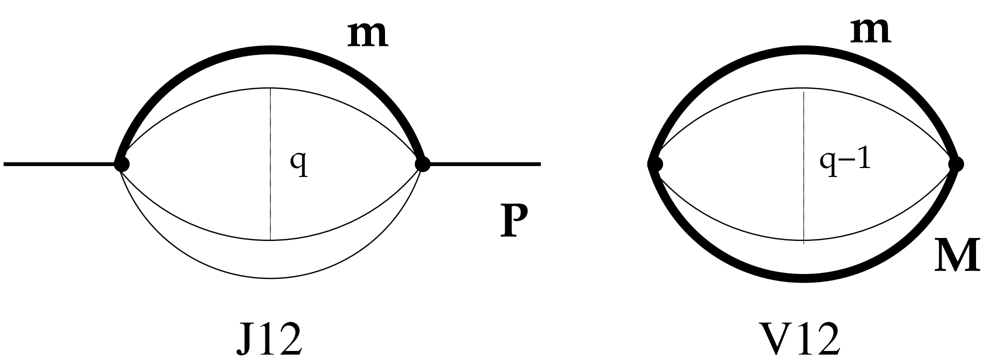

Figure 1: Bold and thin lines correspond to massive and massless propagators, respectively.

Diagrams of this type suffer, in general, from irreducible numerator, so that the solution of recurrence relations is nontrivial problem (besides one-loop propagator and two-loop bubble cases). Using the algorithm of Davydychev-Tarasov any tensor integral can be presented in terms of scalar integrals with the shifted space-time dimension and arbitrary (positive) powers of propagators.

M.K., JHEP, 2006

$$
\begin{aligned}
&J_{12}(\alpha_1, \alpha_2, \cdots, \alpha_q, \beta, m^2, p^2) = \\
&\int \frac{d^n k_1 d^n k_2 \cdots d^n k_q}{[k_1^2 + m^2]^\beta [(k_1 - k_2 - \cdots - k_q - p)^2]^{\alpha_1} [k_2^2]^{\alpha_2} \cdots [k_q^2]^{\alpha_q}} \\
&= \left[\Pi_{l=1}^q \frac{\Gamma\left(\frac{n}{2} - \alpha_l\right)}{\Gamma(\alpha_l)} \right] \pi^{qn/2} (m^2)^{qn/2 - \beta - \alpha} \\
&\times \frac{\Gamma\left(\alpha + \beta - \frac{n}{2} q\right) \Gamma\left(\alpha - \frac{n}{2}(q-1)\right)}{\Gamma(\beta) \Gamma\left(\frac{n}{2}\right) \Gamma^q\left(3 - \frac{n}{2}\right)} \\
&\quad \times {}_2F_1\left(\begin{array}{c} \alpha - \frac{n}{2}(q-1), \alpha + \beta - \frac{n}{2} q \\ \frac{n}{2} \end{array} \middle| - \frac{p^2}{m^2} \right) , \qquad (1)
\end{aligned}
$$

where

$$
\alpha = \sum_{r=1}^q \alpha_r \; .
$$

For given type of diagram there are only two nontrivial master-integrals. In the parametrization $n = 2m - 2\varepsilon$ with integer m, the basis is

$$
\begin{aligned}
&{}_2F_1\left(\begin{array}{c} 1 + \varepsilon(q-1), 1 + \varepsilon q \\ 2 - \varepsilon \end{array} \middle| - \frac{p^2}{m^2} \right) , \\
&{}_2F_1\left(\begin{array}{c} \varepsilon(q-1), \varepsilon q \\ 1 - \varepsilon \end{array} \middle| - \frac{p^2}{m^2} \right) .
\end{aligned}
$$

q–loop bubble with $q-1$ massless lines

M.K., JHEP, 2006

$$
\begin{aligned}
& V_{12}(\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, m^2, M^2) = \\
& \int \frac{d^n k_1 d^n k_2 \cdots d^n k_q}{[k_p^2 + m^2]^{\beta_1} [(k_{p-1}^2 + M^2]^{\beta_2} [k_1^2]^{\alpha_1} \cdots [(k_p - k_1 - k_2 - \cdots - k_{p-1})^2]^{\alpha_q}} \\
& = \left[\Pi_{l=1}^{q-1} \frac{\Gamma\left(\frac{n}{2} - \alpha_l\right)}{\Gamma(\alpha_l)} \right] \frac{\pi^{qn/2} (m^2)^{qn/2 - \beta_1 - \beta_2 - \alpha}}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma\left(\frac{n}{2}\right)\Gamma^q\left(3 - \frac{n}{2}\right)} \\
& \quad \times \Bigg\{ \Gamma\left(\frac{n}{2} - \beta_2\right) \Gamma\left(\alpha + \beta_2 - \frac{n}{2}(q-1)\right) \Gamma\left(\alpha + \beta_1 + \beta_2 - \frac{n}{2} q\right) \\
& \quad \times {}_2F_1\left(\begin{array}{c} \beta_1 + \beta_2 + \alpha - \frac{n}{2} q, \alpha + \beta_2 - \frac{n}{2}(q-1) \\ 1 + \beta_2 - \frac{n}{2} \end{array} \Bigg| \frac{M^2}{m^2} \right) \\
& \quad + \Gamma\left(\beta_2 - \frac{n}{2}\right) \Gamma\left(\alpha + \beta_1 - \frac{n}{2}(q-1)\right) \Gamma\left(\alpha - \frac{n}{2}(q-2)\right) \\
& \quad \left(\frac{M^2}{m^2}\right)^{n/2 - \beta_2} {}_2F_1\left(\begin{array}{c} \alpha - \frac{n}{2}(q-2), \alpha + \beta_1 - \frac{n}{2}(q-1) \\ 1 - \beta_2 + \frac{n}{2} \end{array} \Bigg| \frac{M^2}{m^2} \right) \Bigg\}
\end{aligned}
$$

where

$$
\alpha = \sum_{r=1}^{q-1} \alpha_r \ .
$$

q–loop bubble with $q - 1$ massless lines (II)

The results of the reduction are expressible in terms of four Gauss hypergeometric functions. In the parametrization $n = 2m - 2\varepsilon$, where is m is an integer number we get four basis functions:

$$
\begin{aligned}
&{}_2F_1\left(\begin{array}{c} 1+\varepsilon(q-1), 1+\varepsilon q \\ 2+\varepsilon \end{array} \middle| \frac{M^2}{m^2}\right) \;, \\
&{}_2F_1\left(\begin{array}{c} \varepsilon(q-1), \varepsilon q \\ 1+\varepsilon \end{array} \middle| \frac{M^2}{m^2}\right) \;, \\
&{}_2F_1\left(\begin{array}{c} 1+\varepsilon(q-2), 1+\varepsilon(q-1) \\ 2-\varepsilon \end{array} \middle| \frac{M^2}{m^2}\right) \;, \\
&{}_2F_1\left(\begin{array}{c} \varepsilon(q-2), \varepsilon(q-1) \\ 1-\varepsilon \end{array} \middle| \frac{M^2}{m^2}\right) \;.
\end{aligned}
$$

Only for $q = 2$ (two-loop case) these four hypergeometric functions are expressible in terms of one Gauss hypergeometric function and the function ${}_1F_0$, so that only one nontrivial master-integral exists. For $q > 2$ (3-loop or more) there are four independent Gauss hypergeometric functions. As a consequence, there are four nontrivial master-integrals for diagrams of this type at 3-loop or more.

Generalized hypergeometric functions

The generalized hypergeometric function can be written as series

$$
{}_PF_Q\left(\begin{array}{c} \{A_1+a_1\varepsilon\}, \{A_2+a_2\varepsilon\}, \cdots \{A_P+a_P\varepsilon\} \\ \{B_1+b_1\varepsilon\}, \{B_2+b_2\varepsilon\}, \cdots \{B_Q+b_q\varepsilon\} \end{array} \middle| z \right)
= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\Pi_{s=1}^{P}(A_s+a_s\varepsilon)_j}{\Pi_{r=1}^{Q}(B_r+b_r\varepsilon)_j} \,,
$$

where $(\alpha)_j \equiv \Gamma(\alpha+j)/\Gamma(\alpha)$ is the Pochhammer symbol.

We want to construct the ε-expansion of this series.

$$
{}_PF_Q = \begin{cases} P \leq Q & \text{converges for all finite z} \\ P = Q+1 & \text{converges for all } |z|<1 \\ P > Q+1 & \text{diverges for all } z \neq 0 \end{cases}
$$

Reduction of hypergeometric function

It is well known that any function

$$
{}_pF_{p-1}(\vec{a}+\vec{m}; \vec{b}+\vec{k}; z)
$$

is expressible in terms of p other functions of the same type:

$$
R_{p+1}(\vec{a},\vec{b},z)\,{}_pF_{p-1}(\vec{a}+\vec{m}; \vec{b}+\vec{k}; z) = \sum_{k=1}^{p} R_k(\vec{a},\vec{b},z)\,{}_pF_{p-1}(\vec{a}+\vec{e}_k; \vec{b}+\vec{E}_k; z) \,,
$$

where $\vec{m}$, $\vec{k}$, $\vec{e}_k$, and $\vec{E}_k$ are lists of integers and R_k are polynomials in parameters $\vec{a}$, $\vec{b}$, and z.

Construction of all-order ε-expansion via Differential equation

M.K.,Ward,Yost, JHEP, 07.

$$\omega(z) = {}_pF_{p-1}(\vec{a}\varepsilon; \vec{1} + \vec{b}\varepsilon; z)$$

Defining the coefficients functions $w_k(z)$ at each order by

$$\omega(z) = \sum_{k=0}^{\infty} w_k(z)\varepsilon^k,$$

The differential equation is

$$\left[(1-z)\frac{d}{dz}\right]\left(z\frac{d}{dz}\right)^{p-1} w_k(z)$$
$$= \sum_{i=1}^{p-1}\left[P_i(\vec{a}) - \frac{1}{z}Q_i(\vec{b})\right]\left(z\frac{d}{dz}\right)^{p-i} w_{k-i}(z) + P_p(\vec{a})w_{k-p}(z) \; ,$$

where $P_j(\vec{a})$ and $Q_j(\vec{b})$ are polynomials of order j depending on vectors $\vec{a}$ and $\vec{b}$, respectively.

$$z\frac{d}{dz}\rho_k^{(j)}(z) = \rho_k^{(j+1)}(z) \; , \quad j = 0, 1, \cdots, p-1$$
$$(1-z)\frac{d}{dz}\rho_k^{(p-1)}(z) = \sum_{i=1}^{p}\left[P_i(\vec{a}) - \frac{1}{z}Q_i(\vec{b})\right]\rho_{k-i}^{(p-i)}(z) \; ,$$

The solution is iterated integral:

$$\rho_k^{(p-1)}(z) = \sum_{i=1}^{p}\left[P_i(\vec{a}) - Q_i(\vec{b})\right]\int_0^z \frac{dt}{1-t}\rho_{k-i}^{(p-i)}(t)$$
$$- \sum_{i=1}^{p-2} Q_i(\vec{b})\rho_{k-i}^{(p-i-1)}(z)$$
$$- Q_{p-1}(\vec{b})[w_{k-p+1}(z) - \delta_{0,k-p+1}] \; ,$$
$$\rho_k^{(j-1)}(z) = \int_0^z \frac{dt}{t}\rho_k^{(j)}(t) \; , \quad k \geq 1 \; , \quad j = 1, 2, \cdots, p-1 \; ,$$

The pioneering systematic activity in studying the Laurent series expansion of hypergeometric functions at particular values of the argument $(z = 1)$ was started by David Broadhurst. in the context of Euler-Zagier sums (or multidimensional zeta values). This activity has received further consideration for another, physically interesting point, $z = 1/4$ and also for the "primitive sixth roots of unity". Over time, other types of sums have been analysed in a several publications:

- *harmonic* sums
- *generalized harmonic sums*
- *binomial* sums
- *inverse binomial* sums

How to calculate this sums analytically?

Generating function approach

H.S. Wilf, *Generatingfunctionology*, Academic Press, London, 1994.
Let us rewrite an arbitrary serie as

$$\Sigma_{\vec{A}}(\vec{z}) = \sum_{j=1}^{\infty} \vec{z}^j \eta_{\vec{A}}(j) \ ,$$

where $\vec{A}$ denote the collective sets of indices, whereas $\eta_{\vec{A}}(j)$ is the coefficient of $\vec{z}^j$.

The idea is to find a recurrence relation with respect to j, for the coefficients $\eta_{vecA}(j)$, and then transform it into a differential equation for the *generating* function $\Sigma_{\vec{A}}(z)$. In this way, the problem of summing the series would be reduced to solving a differential equation.

M.K.,Ward,Yost, JHEP, 07.

$$\left.\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \right|_{u=-\frac{(1-y)^2}{y}}$$

$$= \frac{1-y}{1+y} \sum_{p,\vec{s}} c_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y)$$

$$\left.\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \right|_{u=-\frac{(1-y)^2}{y}}$$

$$= \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y) \ , \quad c \geq 2$$

$$\left.\sum_{j=1}^{\infty} \binom{2j}{j} u^j S_{a_1}(j-1) \cdots S_{a_k}(j-1) \right|_{u=\frac{\chi}{(1+\chi)^2}}$$

$$= \sum_{p,\vec{s}} \left[\frac{c_{p,\vec{s}}}{1-\chi} + d_{p,\vec{s}} \right] \ln^p \chi \ \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(\chi) \ ,$$

$$\left.\sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \right|_{u=\frac{\chi}{(1+\chi)^2}}$$

$$= \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p \chi \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(\chi) \ , \quad c \geq 1$$

where c is a positive integer, $c_{p,\vec{s}}$, $\tilde{c}_{p,\vec{s}}$ and $d_{p,\vec{s}}$ are rational coefficients, $\mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(z)$ is the multiple polylogarithm of a square root of unity and

$$S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a} \ ,$$

Special values of agrument

M.K. & Davydychev 99/05.

It is evident that some (or all, if the basis is complete) of the alternating or non-alternating multiple Euler-Zagier sums (or multiple zeta values) can be written in terms of multiple (inverse) binomial sums of special values of arguments. Two arguments where such a representation is possible are trivially obtained by setting the arguments of the harmonic polylogarithms y, χ to ± 1:

$$\begin{aligned} u &= 4 \, , \quad y = -1 \, , \\ u &= \frac{1}{4} \, , \quad \chi = 1 \, . \end{aligned}$$

Another such point is "golden ratio",

$$u = -1 \, , \quad y = \frac{3 - \sqrt{5}}{2}$$

has been discussed intensively in the context of Apéry-like expressions for Riemann zeta functions. For two other points

$$\begin{aligned} u &= 1 \, , \quad y = \exp\left(i\frac{\pi}{3}\right) \, , \\ u &= 2 \, , \quad y = i \, , \end{aligned}$$

the relation between multiple inverse binomial sums and multiple zeta values was analysed mainly by the method of experimental mathematics.

Let us make a few comments about harmonic polylogarithms of a complex argument. For the case $0 \leq u \leq 4$, the variable y belongs to a complex unit circle, $y = \exp(i\theta)$. In this case, the coloured polylogarithms of a square root of unity can be split into real and imaginary parts and generalized log-sine functions are generated.

M.K., B.Ward,S.Yost, JHEP07.

The all order ε-expansion of the generalized hypergeometric functions

$$
\begin{aligned}
{}_pF_{p-1} & \left(\vec{A}+\vec{a}\varepsilon; \vec{B}+\vec{b}\varepsilon, \tfrac{1}{2}+I_1; z\right), \\
{}_pF_{p-1} & \left(\vec{A}+\vec{a}\varepsilon, \tfrac{1}{2}+I_2; \vec{B}+\vec{b}\varepsilon; z\right),
\end{aligned}
$$

where $\vec{A}$, $\vec{B}$ are lists of integers and I_1, I_2 are integers, are expressible in terms of the harmonic polylogarithms with coefficients that are ratios of polynomials.

At the present moment it is unclear is there some limitation on the type of functions generated by Feynman diagrams or a zoo of a new functions is artifact of using technique?