Reduced model for superstrings on $AdS_3 \times S^3$

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Based on:

M.G. and A.A. Tseytlin, arXiv:0711.0155, Nucl. Phys. B, (2008)
proceedings, Osaka 2008
"Reduced model for superstrings on $AdS_n \times S^n$", to appear

Motivations

- IIB superstrings on $AdS_5 \times S^5$; difficulties at the quantum level
- The analog *(Metsaev, Thorn, Tseytlin, 01)* of the flat space light-cone gauge breaks the 2d Lorentz invariance. This makes it hard to apply the standard methods of 2d intergable fields theories.
- Another approach: FR reduction *(Fadeev-Reshetikhin 86)* (known in the case of S^3-sigma model) also breaks 2d Lorentz. Not clear how to generalize to other cosets.
- Remarkably, for rather general coset models there is an alternative – Pohlmeyer reduction. Although intermediate steps are not Lorentz invariant the resulting action is.
- Generalization to $AdS_5 \times S^5$ and $AdS_2 \times S^2$ *(M.G., A. Tseytlin, 07)*

- Known cases: superstrings on $AdS_n \times S^n$ for $n = 2, 3, 5$. $n = 5$ too complicated, $n = 2$ too simple but promissing (gives $N = 2$ susy sine-Gordon model)

- Can be intersting to study $AdS_3 \times S^3$ as a kind of full-featured but still explicitly manageable example. However, the structure is slightly different and the procedure should be modified.

Pohlmeyer reduction

Pohlmeyer, 1976

S^2-sigma model:

$$S = \frac{1}{4\pi\alpha}\int d^2\sigma\ \partial_+ X^m \partial_- X^m - \Lambda(X^m X^m - 1)\ , \quad m = 1,2,3$$

Equations of motion:

$$\partial_+\partial_- X^m + \Lambda X^m = 0\,, \quad \Lambda = \partial_+ X^m \partial_- X^m\,, \quad X^m X^m = 1$$

Stress tensor:

$$\mathrm{T}_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m, \qquad \mathrm{T}_{+-} = 0$$

Stress tensor conservation

$$\partial_+ \mathrm{T}_{--} = 0\,, \qquad \partial_- \mathrm{T}_{++} = 0$$

implies: $\mathrm{T}_{++} = f(\sigma_+), \mathrm{T}_{--} = h(\sigma_-)$ so that using the appropriate conformal transformation one can achieve:

$$\partial_+ X^m \partial_+ X^m = \mu^2\,, \qquad \partial_- X^m \partial_- X^m = \mu^2\,, \qquad \mu = \text{const}\ .$$

We then have 3 unit vectors in 3-dimensional Euclidean space:

$$X^m\,, \qquad X^m_+ = \mu^{-1}\partial_+ X^m\,, \qquad X^m_- = \mu^{-1}\partial_- X^m\,,$$

In fact X^m is orthogonal ($X^m\partial_\pm X^m = 0$) to both X^m_+ and X^m_- and therefore is not independent. The only $SO(3)$ invariant quantity is then

$$\mu^2\cos 2\varphi = \partial_+ X^m \partial_- X^m\,.$$

The equations of motion imply $\partial_+\partial_-\varphi + \frac{\mu^2}{2}\sin 2\varphi = 0$ following from

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \frac{\mu^2}{2}\cos 2\varphi$$

– sine-Gordon model(SG).

Analogous consideration for $S^3 = SO(4)/SO(3)$ leads to

$$\widetilde{L}_{CSG} = \partial_+\varphi\partial_-\varphi + \tan^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

– Complex sine-Gordon model (CSG).

Here φ, θ are the following $SO(4)$-invariants:

$$\mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m$$

$$\mu^3 \sin^2\varphi\,\partial_\pm\theta = \mp\frac{1}{2}\epsilon_{mnkl} X^m \partial_+ X^n \partial_- X^k \partial_\pm^2 X^l$$

Instead of sigma model on S^n one can consider bosonic strings on $S^n \times \mathbb{R}^1$ and use the gauge $t = \mu\tau$ along with the conformal gauge. The conditions

$$\partial_\pm X^m \partial_\pm X^m = \mu^2$$

are then the *Virasoro* constraints.

Comments:

- Virasoro constraints are solved by a special choice of variables related nonlocally to the original coordinates
- Although the reduction is not explicitly Lorentz invariant the resulting Lagrangian turns out to be 2d Lorentz invariant
- The reduced theory is formulated in terms of manifestly $SO(n)$ invariant variables “blind” to the original global symmetry
- The reduced theory is equivalent to the original theory as an integrable system: the respective Lax connections are gauge-equivalent
- PR may be thought of as a formulation in terms of physical d.o.f. – coset space analog of flat-space l.c. gauge (in the known l.c. gauges for $AdS_5 \times S^5$ the 2d Lorentz is broken)
- In general the reduced theory is not quantum-equivalent to the original one (e.g., conformal symmetry was assumed in the reduction procedure)

Pohlmeyer reduction of the F/G-coset models

For $SO(3)/SO(2)$ or $SO(4)/SO(3)$ it is not needed. But we need higher dimensions and better understanding.

F/G-coset sigma model:

Let $\mathfrak{f}$, $\mathfrak{g}$ – respective Lie algebras. The symmetric space condition (Z_2-grading)

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}\,, \qquad [\mathfrak{g},\mathfrak{p}] \subset \mathfrak{p}\,, \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{g}$$

along with $\langle \mathfrak{g}, \mathfrak{p} \rangle = 0$ (in our setting $\langle a, b \rangle = \mathrm{Tr}(ab)$).

The Lagrangian:

$$L = -\mathrm{Tr}(P_+ P_-)\,, \qquad P_\pm = (f^{-1}\partial_\pm f)_{\mathfrak{p}}\;,$$

where

$$J = f^{-1}df = \mathcal{A} + P\;, \qquad \mathcal{A} = J_{\mathfrak{g}} \in \mathfrak{g}\,, \qquad P = J_{\mathfrak{p}} \in \mathfrak{p}\;.$$

G gauge transformation $f \to fg$; global F-symmetry: $f \to f_0 f$ for any constant $f_0 \in F$; conformal invariance.

First step: equations of motion in terms of currents

Pohlmeyer, Lund, Rehren, Regge, D'Auria, Sciutto, Eichenherr, Forger...

The fundamental variables are now $J = \mathcal{A} + P$. The full set of equations of motion involves now:

$D_+ P_- = 0\,, \quad D_- P_+ = 0$ – Equations of motion

$-\frac{1}{2}\mathrm{Tr}(P_+ P_+) = \mu^2\,, \quad -\frac{1}{2}\mathrm{Tr}(P_- P_-) = \mu^2$ – Virasoro const.

$D_- P_+ - D_+ P_- + [P_+, P_-] + [\mathcal{A}_-, \mathcal{A}_+] = 0$ – Maurer-Cartan

Here e.g. $D_+ P_- = \partial_+ P_- + [\mathcal{A}_+, P_-]$.

Main idea: – first solve EOMs and Virasoro using special choice of G gauge condition and special parametrization of currents

Then find reduced action giving eqs. resulting from MC

Special gauge where the first Virasoro constraint is solved by

$$P_+ = \mu\, T\,, \quad \mu = const \in \mathbb{R}$$

$$T = const \in \mathfrak{p} = \mathfrak{f} \ominus \mathfrak{g}, \qquad \mathrm{Tr}(TT) = -1$$

"polar decomposition" theorem.

Lie algebra decomposition

The choice of an element T determines the following decomposition ($\mathfrak{a} = \{T\}$)

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad \mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}\,, \qquad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}\,, \qquad [\mathfrak{a}, \mathfrak{a}] = 0\,,$$

such that

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,, \qquad [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}\,, \qquad [\mathfrak{m}, \mathfrak{a}] \subset \mathfrak{n}\,, \qquad [\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{m}\,.$$

i.e. $\mathfrak{h}$ is a centraliser of T in $\mathfrak{g}$.

Well-known “triple” of Lie groups for F/G coset sigma model:

$$H \subset G \subset F$$

(Bakas et all, Miramontes et all)

New parametrization

Using the decomposition above the first EOM $D_- P_+ = 0$ is solved by $\mathcal{A}_- = A_- \in \mathfrak{h}$

The second Virasoro constraint is solved by

$$P_- = \mu g^{-1} T g$$

with g being a new G-valued field.

Finally, the EOM $D_+ P_- = 0$ is solved by

$$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g\,, \quad \mathfrak{h}\text{-valued } A_+$$

.

To summarise:

we have solved all EOMs and Virasoro constraints. The new parametrisation is in terms of

$$G\text{-valued field } g, \qquad \mathfrak{h}\text{-valued fields} \quad A_+, A_-.$$

The only remaining equation is the Maurer-Cartan equation.

Relation to gauged WZW model

Maurer-Cartan equation in terms of new parametrization:

$$\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- \\ + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g] + \mu^2[g^{-1}Tg, T] = 0$$

Recall: $P_+ = \mu\, T\,, \qquad P_- = \mu\, g^{-1}Tg\,,$

$$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1}A_+ g\,, \qquad \mathcal{A}_- = A_-$$

MC eq. has "on-shell" $H \times H$ gauge symmetry:

$$g \to h^{-1}g\bar{h}\,,$$

$$A_+ \to h^{-1}A_+ h + h^{-1}\partial_+ h\,, \quad A_- \to \bar{h}^{-1}A_-\bar{h} + \bar{h}^{-1}\partial_-\bar{h}\,,$$

can choose a gauge:

$$A_+ = (g^{-1}\partial_+ g + g^{-1}A_+ g)_{\mathfrak{h}}, \quad A_- = (g\partial_- g^{-1} + gA_- g^{-1})_{\mathfrak{h}}$$

G/H gWZW action with potential: *(Bakas, Park, Shin 95)*

$$
\begin{aligned}
L = \quad & - \quad \frac{1}{2}\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \text{WZ term} \\
& - \quad \mathrm{Tr}\big(A_+\,\partial_- g g^{-1} - A_-\, g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\big) \\
& - \quad \mu^2 \mathrm{Tr}(T g^{-1} T g)
\end{aligned}
$$

Remains left-right H gauge symmetry: $h = \bar{h}$.

– Action and gauge symmetries of the Pohlmeyer-reduced theory for F/G coset sigma model . Also for strings on $R_t \times F/G$ or $F/G \times S^1_\psi$

integrable potential: relation at the level of Lax pairs

special case of non-abelian Toda theory:

“symmetric space Sine-Gordon model”

(Fernandez-Pousa, Gallas, Hollowood, Miramontes 96)

Structure of the action

The action of the reduced theory can be written as:

$$L = L_{gWZW} + L_{add}\,, \qquad L_{add} = -\mathrm{Tr}(P_+ P_-)$$

where

$$P_+ = P_+(g) = \mu T\,, \qquad P_- = \mu g^{-1} T g$$

Note:

L_{add} – original Lagrangian of the F/G coset model written in terms of new parametrization.

L_{gWZW} – the Lagrangian of the gauged WZW model encoding the MC equation

Elimination of $A_\pm$

$A_\pm$–auxiliary fields. What to do about A_+, A_-: integrate out or gauge-fix?

Gauge $A_\pm = 0$: reduced EOM's in the "on-shell" gauge:

On-shell $\partial_- A_+ - \partial_+ A_- + [A_-, A_+] = 0$ so can set $A_\pm = 0$

$$\partial_-(g^{-1}\partial_+ g) - \mu^2 [T, g^{-1} T g] = 0 \ ,$$
$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = 0 \ , \qquad (\partial_- g g^{-1})_{\mathfrak{h}} = 0 \ .$$

$F/G = SO(n+1)/SO(n) = S^n : \ G/H = SO(n)/SO(n-1)$

Parametrising g as

$$g = \begin{pmatrix} k_1 & k_2 & \dots & k_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \ , \qquad \sum_{1=1}^{n} k_l k_l = 1$$

One gets (in general non-Lagrangian) EOM for k_m

$$\partial_- \frac{\partial_+ k_\ell}{\sqrt{1-\sum_{m=2}^n k_m k_m}} = -\mu^2 k_\ell\,, \qquad \ell = 2,\ldots,n\,.$$

Linearising around the vacuum $g=1$ (i.e. $k_1 = 1,\ k_\ell = 0$)

$$\partial_+\partial_- k_\ell + \mu^2 k_\ell + O(k_\ell^2) = 0$$

Massive spectrum, $H = SO(n-1)$ global symmetry

Integrating out $A_\pm$: gauge condition on g field

$F/G = SO(n+1)/SO(n) = S^n$: parametrization of g
in terms of Euler angles

$$g = e^{T_{n-1}\theta_{n-1}} ... e^{T_2\theta_2} e^{2T\varphi} e^{T_2\theta_2} ... e^{T_{n-1}\theta_{n-1}}$$

and integrating out $H = SO(n-1)$ gauge field $A_\pm$
leads to reduced theory that generalizes SG and CSG

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + G_{pq}(\varphi,\theta)\partial_+\theta^p\partial_-\theta^q + \frac{\mu^2}{2}\cos 2\varphi$$

no B_{mn} coupling
similar for $F/G = SO(2,n-1)/SO(1,n-1) = AdS_n$ case:
$G/H = SO(1,n-1)/SO(n-1)$

For $n = 2,3$: SG and CSG models
and their AdS counterparts.
For $n = 4,5$ explicit G_{pq} were given in
(Fradkin, Linetsky ; Bars, Sfetsos 91-92)

Bosonic strings on $AdS_n \times S^n$

The Lagrangian and the Virasoro constraints:

$$L^{AS} = \mathrm{Tr}(P_+^A P_-^A) - \mathrm{Tr}(P_+^S P_-^S)\,, \qquad \mathrm{Tr}(P_\pm^S P_\pm^S) - \mathrm{Tr}(P_\pm^A P_\pm^A) = 0$$

Using the conformal transformation one can assume

$$-\frac{1}{2}\mathrm{Tr}(P_\pm^S P_\pm^S) = -\frac{1}{2}\mathrm{Tr}(P_\pm^A P_\pm^A) = \mu^2$$

The rest of the Pohlmeyer reduction goes in each sector independently giving the direct product of the reduced systems for S^n and AdS_n respectively.

Except for μ the AdS_n and S^n sectors do not see each other.

Example: in the case of $AdS_2 \times S^2$ one gets:

$$L^{AdS_2\times S^2} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$

Note that for some n (e.g. $n = 2, 3, 5$) one can also use the representation by unitary matrices instead of orthogonal ones. For instance SG corresponds to $SU(2)/U(1)$

$AdS_5 \times S^5$ superstring sigma-model

$AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$

supercoset GS sigma model *(Metsaev, Tseytlin, 98)*

$$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)}$$

basic superalgebra $\widehat{\mathfrak{f}} = psu(2,2|4)$

bosonic part $\mathfrak{f} = su(2,2) \oplus su(4) \cong so(2,4) \oplus so(6)$

admits Z_4-grading: *(Berkovits, Bershadsky, et al 89)*

$$\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}_0 \oplus \widehat{\mathfrak{f}}_1 \oplus \widehat{\mathfrak{f}}_2 \oplus \widehat{\mathfrak{f}}_3 \,, \qquad [\widehat{\mathfrak{f}}_i, \widehat{\mathfrak{f}}_j] \subset \widehat{\mathfrak{f}}_{i+j \bmod 4}$$

$\widehat{\mathfrak{f}}_0 = \mathfrak{g} = sp(2,2) \oplus sp(4)$

current ($J = f^{-1}\partial_a f, \ f \in \widehat{F}$) decomposes as

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a}$$

$$\mathcal{A} \in \widehat{\mathfrak{f}}_0, \quad Q_1 \in \widehat{\mathfrak{f}}_1, \quad P \in \widehat{\mathfrak{f}}_2, \quad Q_2 \in \widehat{\mathfrak{f}}_3 \,.$$

GS Lagrangian:

$$L_{\rm GS} = \frac{1}{2}{\rm STr}(\sqrt{-g}g^{ab}P_aP_b + \varepsilon^{ab}Q_{1a}Q_{2b})\,,$$

Very simple structure – but not standard coset model:
fermionic currents in WZ term only
leads to κ-symmetry:

$$\delta_\kappa J_a = \partial_a\epsilon + [J_a, \epsilon], \qquad (\delta_\kappa\sqrt{-g}g^{ab})^{ab} = ...$$

$$\epsilon = \{P_{(+)a}, ik^a_{1(-)}\} + \{P_{(-)a}, ik^a_{2(+)}\}$$

Conformal gauge:

$$L_{\rm GS} = {\rm STr}[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})]$$

$${\rm STr}(P_+P_+) = 0\,, \qquad\qquad {\rm STr}(P_-P_-) = 0$$

Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring

In terms of current $J = \mathcal{A} + P + Q_1 + Q_2$

EOM :
$$\partial_+ P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0\,,$$
$$\partial_- P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0\,,$$
$$[P_+, Q_{1-}] = 0\,, \qquad [P_-, Q_{2+}] = 0\,.$$

Virasoro :
$$\mathrm{STr}(P_+ P_+) = 0\,, \qquad \mathrm{STr}(P_- P_-) = 0$$

MC :
$$\partial_- J_+ - \partial_+ J_- + [J_-, J_+] = 0\,.$$

PR procedure: solve first EOM and Virasoro

κ-gauge condition: $Q_{1-} = 0\,, \qquad Q_{2+} = 0$

solves the last (fermionic) pair of EOM

As in the bosonic case the remaining EOM:

$$\partial_+ P_- + [\mathcal{A}_+, P_-] = 0\,, \qquad \partial_- P_+ + [\mathcal{A}_-, P_+] = 0$$

are solved by fixing the "reduction gauge" and using the conformal symmetry. Namely one gets:

$$P_+ = \mu\, T\,, \qquad T = \frac{i}{2}\mathrm{diag}(1,1,-1,-1|1,1,-1,-1)$$

$$P_- = \mu\, g^{-1} T g\,, \quad \mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g\,, \quad \mathcal{A}_- = A_-$$

T defines $\mathfrak{h}$ by $[\mathfrak{h}, T] = 0$:

$$\mathfrak{h} = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$$

New parametrisation:

$G = Sp(2,2) \times Sp(4)$ – valued field g , $\quad \mathfrak{h}$ – valued field $A_\pm$

In the new parametrization MC eqs. become:

$$
\begin{aligned}
\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g]& \\
= -\mu^2[g^{-1}Tg, T] + [Q_{1+}, Q_{2-}]\,&, \\
\partial_- Q_{1+} + [A_-, Q_{1+}] =& \mu[T, Q_{2-}]\,, \\
\partial_+ Q_{2-} + [g^{-1}\partial_+ g + g^{-1}A_+ g, Q_{2-}] =& \mu[g^{-1}Tg, Q_{1+}]
\end{aligned}
$$

AdS_5 and S^5 sectors now coupled by fermions

remains residual κ-symmetry to be fixed

use T to generalise decomposition of bosonic part

$\mathfrak{f} = T \oplus \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{m}$ to superalgebra $psu(2,2|4)$

$$
\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}^{\parallel} \oplus \widehat{\mathfrak{f}}^{\perp}\,, \qquad [T,[T,\widehat{\mathfrak{f}}^{\perp}]] = 0
$$

define

$$
\Psi_1 = Q_{1+}\,, \qquad \Psi_2 = gQ_{2-}g^{-1}
$$

$\Psi_1^{\perp}, \Psi_2^{\perp}$ can be set =0 by residual κ-symmetry

The remaining fermionic components

$$\Psi_{_R} = \frac{1}{\sqrt{\mu}} \Psi_1^{\parallel} , \qquad\qquad \Psi_{_L} = \frac{1}{\sqrt{\mu}} \Psi_2^{\parallel} ,$$

transform under $H \times H$ as $\Psi_{_R} \to \bar{h}^{-1} \Psi_{_R} \bar{h}\, ,\ \Psi_{_L} \to h^{-1} \Psi_{_L} h\, .$

Equations of motion (MC equation) of reduced theory are thus:

$$\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g]$$
$$= \ -\mu^2 [g^{-1}Tg, T] - \mu [g^{-1}\Psi_{_L} g, \Psi_{_R}]\, ,$$

$$[T, D_-\Psi_{_R}] = -\mu (g^{-1}\Psi_{_L} g)^{\parallel}\, , \qquad [T, D_+\Psi_{_L}] = -\mu (g\Psi_{_R} g^{-1})^{\parallel}\, .$$

Pohlmeyer reduced system at the level of EOMs

Lagrangian of PR theory for $AdS_5 \times S^5$ superstring

(MG, Tseytlin 07; similar action: Mikhailov, Schafer-Nameki 07)

fermionic generalization of "gWZW+ potential" theory for
$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$

$$
\begin{aligned}
L &= L_{\rm gWZW}(g, A_+, A_-) + \mu^2\, {\rm STr}(g^{-1}TgT) \\
&+ {\rm STr}\,(\Psi_{_L}[T, D_+\Psi_{_L}] + \Psi_{_R}[T, D_-\Psi_{_R}]) \\
&+ \mu\, {\rm STr}\left(g^{-1}\Psi_{_L} g \Psi_{_R}\right)
\end{aligned}
$$

Direct sum of PR theories for AdS_5 and S^5 "glued together" by components of fermions

$$
\begin{aligned}
L &= \widetilde{L}_{S^5}(g, A_+, A_-) + \widetilde{L}_{AdS_5}(g, A_+, A_-) \\
&+ \psi_{_L} D_+ \psi_{_L} + \psi_{_R} D_+ \psi_{_R} + \mu\ (\text{interaction terms})
\end{aligned}
$$

standard kin. terms for bosons and fermions (cf. GS action)

Comments:

- gWZW model coupled to the fermions interacting minimally and through the "Yukawa term"
- 8 real bosonic and 16 real fermionic independent variables
- 2d Lorentz invariant with $\Psi_{_R}, \Psi_{_L}$ as 2d Majorana spinors
- 2d supersymmetry? yes, at the linearised level, and yes in $AdS_2 \times S^2$ case: $n = 2$ super sine-Gordon
- quadratic in fermions (like susy version of gWZW); integrating out $A_\pm$ gives quartic fermionic terms (reflecting curvature)
- linearisation of EOM in the gauge $A_\pm = 0$ around $g = \mathbf{1}$ describes 8+8 massive bosonic and fermionic d.o.f. with mass μ: same as in BMN limit. $H = [SU(2)]^4$ global symmetry

The structure of the potential:

Like in the bosonic case:

$$L = L_{WZW} + L_{add}\,, \quad L_{add} = \mathrm{STr}(P_+ P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+}))$$

where L_{WZW} – Lagrangian of gWZW with fermions and

$$P_+ = \mu T, \quad P_- = \mu g^{-1} T g, \quad Q_{1+} = \sqrt{\mu}\Psi_{_R}, \quad Q_{2-} = \sqrt{\mu} g^{-1} \Psi_{_L} g$$

and $Q_{1+} = Q_{2-} = 0$ due to the κ-gauge.

Path integral derivation via change from fields to currents?

Lorentz invariance:

Variables $\Psi_{_R}$ and $\Psi_{_L}$ originate from Lorentz vectors (fermionic components of currents).

Consistently assigning the Lorentz transformation properties.
If $\Psi_{_R}, \Psi_{_L}$ 2d Majorana spinors then L is Lorentz invariant.
(Contrary to the bosonic case where no change is needed)

Example: $AdS_2 \times S^2$

Explicit parametrisation:

$$T = \frac{1}{2}\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

$$g = exp\begin{pmatrix} 0 & \phi & 0 & 0 \\ \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & i\varphi \\ 0 & 0 & i\varphi & 0 \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & \cos\varphi & i\sin\varphi \\ 0 & 0 & i\sin\varphi & \cos\varphi \end{pmatrix}$$

Fermions:

$$\Psi_R = \begin{pmatrix} 0 & 0 & 0 & i\gamma \\ 0 & 0 & -\beta & 0 \\ 0 & i\beta & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix} \qquad \Psi_L = \begin{pmatrix} 0 & 0 & 0 & \rho \\ 0 & 0 & -i\nu & 0 \\ 0 & \nu & 0 & 0 \\ i\rho & 0 & 0 & 0 \end{pmatrix}$$

The Lagrangian ($N = 2$ supersymmetric sine-Gordon):

$$
\begin{aligned}
L_{tot} &= \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi) \\
&\quad + \beta\partial_-\beta + \gamma\partial_-\gamma + \nu\partial_+\nu + \rho\partial_+\rho \\
&\quad - 2\mu\left[\cosh\phi\ \cos\varphi\,(\beta\nu + \gamma\rho) + \sinh\phi\ \sin\varphi\,(\beta\rho - \gamma\nu)\right] .
\end{aligned}
$$

In more conventional ($N = 2$ susy) terms can be rewritten as:

$$
\begin{aligned}
L_{tot} &= \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2 + \psi_{_L}^*\partial_+\psi_{_L} + \psi_{_R}^*\partial_-\psi_{_R} \\
&\quad + \left[W''(\Phi)\psi_{_L}\psi_{_R} + W^{*\prime\prime}(\Phi^*)\psi_{_L}^*\psi_{_R}^*\right] .
\end{aligned}
$$

where

$$
W(\Phi) = \mu\cos\Phi\ , \qquad |W'(\Phi)|^2 = \frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)\ .
$$

and

$$
\psi_{_L} = \nu + i\rho\,, \qquad \psi_{_R} = -\beta + i\gamma\ ,
$$

Superstrings on $AdS_3 \times S^3$

Supercoset:

$$\frac{PSU(1,1|2) \times PSU(1,1|2)}{SU(1,1) \times SU(2)}$$

Rahmfeld, A. Rajaraman; Pesando; Park, Rey, Berkovits, Vafa, Witten; Metsaev, Tseytlin

Note: $AdS_3 \times S^3 \cong SU(1,1) \times SU(2)$

To apply the general scheme one needs to identify the Z_4 grading on

$$\widehat{\mathfrak{f}} = psu(1,1|2) \oplus psu(1,1|2)$$

such that $\widehat{\mathfrak{f}}_0 = su(1,1) \oplus su(2) \subset psu(1,1|2) \oplus psu(1,1|2)$

Z_4 grading was discussed in the literature but we need its matrix form along with the compatible decomposition $\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}^{\|} \oplus \widehat{\mathfrak{f}}^{\perp}$.

Simple example

G - Principal Chiral Model (PCM). It can be identified with

$$\frac{G \times G}{\bar{G}}$$

coset sigma model with $\bar{G}$ being diagonal subgroup.

More generally: $\bar{G}$ – a subgroup formed by elements of the form $(g, \widehat{\chi}(g))$; $\widehat{\chi}$ – automorphism compatible with the trace.

The respective orthogonal decomposition:

$$\mathfrak{f} = \bar{\mathfrak{g}} \oplus \mathfrak{p}\,, \quad \mathfrak{g} = \{(a, \chi(a))\}\,, \quad \mathfrak{p} = \{(a, -\chi(a))\}\,.$$

The symmetric space conditions $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] \subset \bar{\mathfrak{g}}$, $[\bar{\mathfrak{g}}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \bar{\mathfrak{g}}$ are satisfied.

Under some algebraic conditions (satisfied in the AdS_3 or S^3 case) one can apply Pohlmeyer reduction.

Pohlmeyer reduction of PCM

Using gauge symmetry and new parametrization:

$$P_+ = \mu T, \quad P_- = \mu \bar{g}^{-1} T \bar{g}, \qquad T = (t, -\chi(t)), \quad \bar{g} = (g, \widehat{\chi}(g))$$

with t in Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

The resulting gWZW model can be formulated entirely in terms of G-valued field g and $\mathfrak{h}$

$$L = L_{gWZW}[g, A_+, A_-] - \mu^2 \mathrm{Tr}(g^{-1} t g t)$$

(does not depen on χ).

That is what happens in the bosonic sector.

Our aim now is to identify Z_4 grading on $psu(2,2|4)$ such that in the bosonic part $\widehat{\mathfrak{f}}_0 = su(1,1) \oplus su(2)$ and $\widehat{\mathfrak{f}}_2$ its orthogonal complement in $su(1,1) \oplus su(2) \oplus su(1,1) \oplus su(2)$.

Z_4 grading

The automorphism inducing Z_4 grading:

$$\begin{pmatrix} a & \alpha & 0 & 0 \\ \beta & b & 0 & 0 \\ 0 & 0 & c & \gamma \\ 0 & 0 & \delta & d \end{pmatrix}^{\Omega} = - \begin{pmatrix} c^t & -\delta^t & 0 & 0 \\ \gamma^t & d^t & 0 & 0 \\ 0 & 0 & a^t & -\beta^t \\ 0 & 0 & \alpha^t & b^t \end{pmatrix} \qquad (1)$$

$M^{\Omega} = i^k M$ for $M \in \widehat{\mathfrak{f}}^{\mathbb{C}}_k$ so that

$$\widehat{\mathfrak{f}}^{\mathbb{C}} = \widehat{\mathfrak{f}}^{\mathbb{C}}_0 \oplus \widehat{\mathfrak{f}}^{\mathbb{C}}_1 \oplus \widehat{\mathfrak{f}}^{\mathbb{C}}_2 \oplus \widehat{\mathfrak{f}}^{\mathbb{C}}_3$$

The decomposition induces that of the real form $\widehat{\mathfrak{f}}$ of $\widehat{\mathfrak{f}}^{\mathbb{C}}$ (elements satisfying $a^* = -a$). Note that in the bosonic part $\chi(a) = -a^t$.

Once Z_4 grading is identified the general reduction procedure is aplicable. In the $AdS_3 \times S^3$ case we can explicitly eliminate $A_{\pm}$ fields and arrive at the Lagrangian for physical degrees of freedom only.

Reduced Lagrangian

Imposing the gauge on g and parametrizing g in terms of the Euler angles

$$g = \left(\begin{array}{cc} g_A & 0 \\ 0 & g_S \end{array} \right) ,$$

$$g_A = \left(\begin{array}{cc} e^{i\vartheta} \cosh\phi & \sinh\phi \\ \sinh\phi & e^{-i\vartheta} \cosh\phi \end{array} \right) , \; g_S = \left(\begin{array}{cc} e^{i\theta} \cos\varphi & \sin\varphi \\ -\sin\varphi & e^{-i\theta} \cos\varphi \end{array} \right)$$

One can solve for $A_\pm = A_\pm[\phi, \varphi, \theta, \vartheta, \text{fermions}]$.

The reduced Lagrangian has the following structure:

$$L = L_B + L_F$$

In more details:

$$L_B = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \partial_+\phi\partial_-\phi + \coth^2\phi\,\partial_+\vartheta\partial_-\vartheta + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)\ .$$

i.e. the sum of CSG Lagrangian and its hyperpolic counterpart.

$$\begin{aligned} L_F = {}& \alpha\partial_-\alpha+\beta\partial_-\beta+\gamma\partial_-\gamma+\delta\partial_-\delta+\lambda\partial_+\lambda+\nu\partial_+\nu+\rho\partial_+\rho+\sigma\partial_+\sigma \\ & + \cot^2(\phi)(\partial_-\theta(\alpha\beta-\gamma\delta)-\partial_+\theta(\lambda\nu-\rho\sigma)) \\ & - \coth^2(\varphi)(\partial_-\vartheta(\alpha\beta-\gamma\delta)-\partial_+\vartheta(\lambda\nu-\rho\sigma)) \\ & 2(\alpha\beta-\gamma\delta)(\lambda\nu-\rho\sigma)(\frac{1}{\sin^2\phi}+\frac{1}{\sinh^2\phi}) \\ & - 2\mu\Big(\sinh\phi\sin\varphi(\lambda\beta-\nu\alpha+\rho\delta-\sigma\gamma)+\cosh\phi\cos\varphi \\ & \times\big[\cos(\vartheta+\theta)(\sigma\alpha-\rho\beta+\lambda\delta-\nu\gamma)+\sin(\vartheta+\theta)(\rho\alpha+\sigma\beta-\lambda\gamma-\nu\delta)\big]\Big) \end{aligned}$$

– explicitly 2d Lorentz invariant, contains 4 real bosons and 4 real fermions.

– in the limit $\theta = \vartheta = 0$ and half of the fermons set to zero reproduces the $N = 2$ susy sine-Gordon model.

– L_B admits susy extension in both AdS_3 and S^3 sectors but this is different becuase the two sectors do not see each other in contrast to our case.

Open questions

- SUSY in $AdS_3 \times S^3$ case. If yes the same for $AdS_5 \times S^5$. There are indications but it seems nontrivial.
- Clarifying the relationship between the original and the reduced system. E.g. symmetries, vacua, values of conserved charges etc.
- Generalising the Pohlmeyer reduction to other models including flat space GS superstring, gWZW models, etc. Note that already $AdS_5 \times S^5$ superstring contains WZ term for fermions
- Integrate out the $A_\pm$ fields in the $AdS_5 \times S^5$ case. This involves identification of a right vacua at the Lagrangian level and possibly asymmetrically gauged form of the gWZW model with fermions.

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