MHV amplitudes in N=4 SUSY Yang-Mills theory and quantum geometry of the momentum space

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MHV amplitudes are the simplest objects to discuss within the gauge/string duality

Simplification at large N - MHV amplitudes are described by the single function of the kinematical variables

Properties of the tree amplitudes

- Holomorphy - it depends only on the "'half"' of the momentum variables $p_{\alpha,\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$
- Fermionic representation (Nair,88) - tree amplitudes are the correlators of the chiral fermions of the sphere

- Tree amplitudes admit the twistor representation(Witten,04). Tree MHV amplitudes are localized on the curves in the twistor space. Localization follows from the holomorphycity of the amplitude. Stringy interpretation - fermions are the degrees of freedom on the D1-D5 open strings ended on the Euclidean D1 instanton.
- The generating function for the tree MHV amplitudes - solution to the self-duality equation with the particular boundary conditions (Bardeen 96, Rosly-Selivanov 97).The same solution provides the canonical transformation from the light-cone YM lagrangian to the MHV Lagrangian (Rosly-A.G., 04)

Properties of the loop MHV amplitudes

- Exponentiation of the ratio $\frac{M_{all-loop}}{M_{tree}}$ which contains the IR divergent and finite parts.
- BDS conjecture for the all loop answer

$$\log \frac{M_{all-loop}}{M_{tree}} = (IR_{div} + \Gamma_{cusp}(\lambda) M_{one-loop})$$

  . It fails starting from six external legs. It involves two ingredients - one-loop amplitude and all-loop Γ_{cusp}
- $\Gamma_{cusp}(\lambda)$ obeys the integral equation (Beisert-Eden-Staudacher) and can be derived recursively

- There is conjecture that $\frac{M_{all-loop}}{M_{tree}}$ coincides with the Abelian Wilson polygon built from the external light-like momenta p_i. The conjecture was formulated at strong coupling (Alday-Maldacena, 06) and checked at weak coupling (one and two loops) as well (Drummond- Henn- Korchemsky- Sokachev, Bern-Dixon-Kosover, Brandhuber-Heslop-Travagnini 07).
- There is no satisfactory stringy twistor explanation of the loop MHV amplitudes. Expectation - closed string modes contribute (Cachazo-Swrchk-Witten)

Main Questions

- Is there fermionic representation of the loop MHV amplitudes similar to the tree case?
- Is there link with integrability at generic kinematics ? The integrability is known at low-loop Regge limit (Lipatov 93, Faddeev-Korchemsky 94) only
- What is the stringy origin of the BDS conjecture?
- What is the origin of MHV amplitude-Wilson polygon duality?

- Consider c=1 string (1d-target space + Liouville direction). The only degrees of freedom - massless tachyons with the discrete momemnta
- Exact answer for the tachyonic amplitudes (Dijkgraaf,Plesser,Moore 94)
- Generating function for the amplitude - τ function for the Toda integrable systems. "'Times"'- generating parameters for the tachyon operators with the different momenta
- Generating function admits representation via chiral fermions on the Riemann surface

$$x^2 - y^2 = 1$$

in the particular abelian gauge field A(z)

$$\tau(t_k) =< 0|exp(\sum t_k V_k)exp\int(\bar{\psi}A\psi)exp(\sum t_{-k}V_{-k})|0 >$$

- The amplitude can be represented in terms of the "'Wilson polygon"' for the auxiliary abelian gauge field! This gauge field has nothing to do with the initial tachyonic scalar degrees of freedom. The auxiliary abelian gauge field A(z) corresponds to the "point of Grassmanian" and yields the choice of the vacuum state in the theory.
- Riemann surface reflects the hidden moduli space of the theory (chiral ring) and it is quantized. Equation of the Riemann surface becomes the operator acting on the wave function.The following commutation relation is implied

$$[x, y] = i\hbar$$

- Consider the moduli space of the complex structures for genus zero surface with n marked points, $M_{0,n}$. Inequivalent triangulations of the surface can be mapped into set of geodesics on the upper half-plane
- This manifold has the Poisson structure and can be quantized in the different coordinates (Kashaev-Fock-Chekhov, 97-01). The generating function of the special canonical transformations (flip) on this symplectic manifold is provided by $Li_2(z)$ where z- is so-called shear coordinate related to the conformal cross-ratio of four points on the real axe

$$exp(z) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}$$

- Quantum mechanically there is flip-generating operator of the "'duality"' K acting on this phase space with the property $\hat{K}^5 = 1$. It is the analogue of the Q-operator in the theory of the integrable systems since it is build from the eigenfunction of the "'quantum spectral curve operator"'

$$e^u + e^v + 1 = 0$$

gets transformed into the Baxter equation

$$(e^{i\partial_v} + e^v + 1)Q(v) = 0$$

with the Poisson bracket

$$[v, u] = i\hbar$$

- In the integrable systems the Baxter equations yield the separated variables

- Let us use the representation for the one-loop amplitude as the some of the following dilogariphms

$$\sum_i \sum_r Li_2\left(1 - \frac{x_{i,i+r}^2 x_{i+1,i+r+1}^2}{x_{i,i+r+1}^2 x_{i-1,i+r}^2}\right)$$

$$x_{i,k} = p_i - p_k$$

where p_i are the external on-shell momenta

- Conjecture One-loop amplitude with n-gluons is described in terms of the fermions on the spectral curve which is embedded into the mirror of the topological vertex. Amplitude - fermion correlator on the spectral curve. Spectral curve parametrizes the moduli space $M_{0,n}$

- Fermions represents the coordinates of the D1 instantons. In the mirror dual geometry fermions represents Lagrangian branes. Fermions live on the moduli space of the complex structures. They transforms nontrivially under the change of patches on the surface because of its quantum nature
- The spectral curve is embedded as the holomorphic surface in the internal 3-dimensional complex space

$$xy = e^u + e^v + 1$$

- Quantization of the spectral curve involves the coupling constant

$$\frac{1}{g_{YM}^2} = \frac{B_{NS-NS}}{g_s}$$

Usually it is assumed the g_s yields the "Planck constant" for the quantization of the moduli space of the complex structures. However equally Yanh-Mills coupling can be considered as the quantization parameter.

- The expression for the Γ_{cusp} involves the wave function depending on the eigenvalue of the length operator on the moduli phase space.

$$\frac{d < W(\theta) >}{dlogm^2} = \Gamma_{cusp}(\alpha, \theta)$$

where $cosh(\theta) = Trg_1 g_2^{-1}$ with SL(2,R) group elements. Γ_{cusp} at one loop coincides with the wave function on the quantized moduli space

- The variable θ corresponds to the length of the geodesics which the wave function depends on. It is related to the variables x_i introduced above

- Quasiclassics for the solution to the Baxter equation

$$\Psi(z,\hbar) = \int \frac{e^{ipz}}{p.sinh(\pi p)sinh(\pi\hbar p)} dp$$

reduces to

$$\Psi(x) \to exp(\hbar^{-1} Li_2(x) + ...)$$

Arguments of the Li_2 in the expression for the amplitudes correspond to the shear coordinates on the moduli space.

- The one-loop MHV amplitude can be presented in the following form

$$M_{one-loop} \propto < 0|\Psi(z_1)...\Psi(z_n) exp(\psi_k A_{nk} \psi_k)|0 >$$

- The variables ψ_k are the modes of the fermion on the spectral curve. The matrix $A_{n,k}$ for the corresponding spectral curve is known (Aganagic-Vafa-Klemm-Marino 03)

- Solution to the Baxter equation is the operator for the Backlund transformation
- In the Regge limit the Baxter operator plays the similar role it provides the Lipatov,s duality transformation. In the thermodynamical limit of the one-loop spin chain one gets the worldsheet T-duality (Korchemsky- A.G. to appear)
- From the worldsheet viewpoint one considers the discretization of the Liuville mode and the Faddeev-Volkov model yields the good candidate for the correct S-matrix.

Conclusion

- The representation of the one-loop MHV amplitude as the fermion correlator on the spectral curve is found. Effect of closed string degrees of freedom(Kodaira-Spencer gravity)
- Link to the integrability behind generic MHV amplitudes - 3-KP integrable system
- BDS conjecture can be reformulated in terms of the quantum geometry of the momentum space
- Abelian Wilson loop - MHV amplitude duality is based on the fermionic representation of the amplitude