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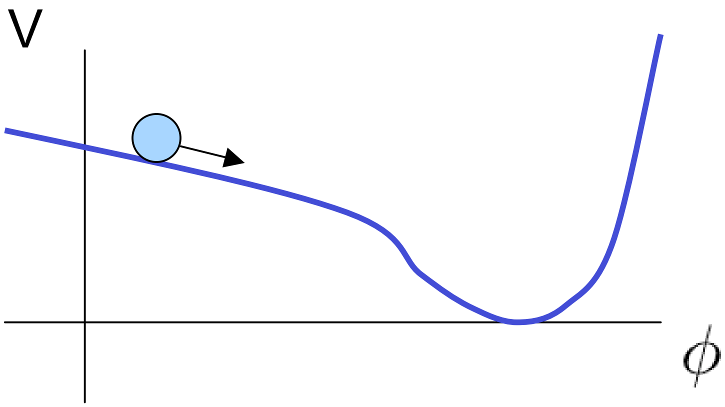
The phase transition to eternal inflation

with Sergei Dubovsky, Alberto Nicolis,
Leonardo Senatore and Matias Zaldarriaga,

arXiv:0802.1067 [hep-th].

The setup: slow-roll inflation



$$\ddot{\phi} + \underline{\underline{3H\dot{\phi}}} + V'(\phi) = 0 \qquad H \equiv \frac{\dot{a}}{a}$$

Friction is dominant

To have ~ dS space the potential must be VERY flat:

$$\epsilon, \eta, \ldots \ll 1$$

$$\epsilon = \frac{1}{2} M_P^2 \left(\frac{V'}{V}\right)^2 \qquad \eta = M_P^2 \frac{V''}{V}$$

- This gives a period of inflation: $\ddot{a} > 0$ Curvature, inhomogeneities and relics are diluted away. For $\hbar = 0$ we have a completely smooth Universe.

- For $\hbar \neq 0$ we have quantum fluctuations of all the light degrees of freedom $(m^2 < H^2)$. Inflaton itself (scalar perturbations) and graviton (tensor modes).

In the observable Universe these initial quantum fluctuations are small ~ 10^{-5}

What happens when they are ~1?

Outline

- Motivations:
  1. Experimental support of inflation
  2. Global geometry of spacetime
  3. Landscape + probability
  4. Does dS make sense quantum mechanically?

- The qualitative picture of slow-roll eternal inflation

- Is it possible to make quantitative progress? Is the system under **perturbative control**?
  Yes, quantitative results in an expansion in slow-roll parameters.

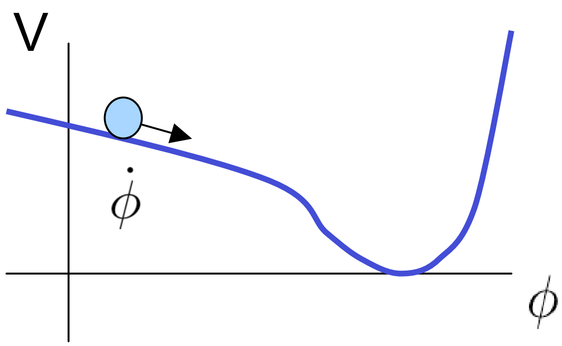
- Transition to eternal inflation: a kind of phase transition at a given value of the parameters

- **Volume of the Universe at reheating as "order parameter":**
  All moments $\langle V^n \rangle$ of the volume diverge at the same critical point

- Discretization of the system: branching process.
  At the same critical point a **non-zero probability for an infinite volume develops**

Slow-roll eternal inflation

Vilenkin 83, Linde 86



In the $\delta\phi=0$ gauge, scalar perturbations are described by ζ: $g_{ij} = a^2(t)(1 + 2\zeta)\,\delta_{ij}$

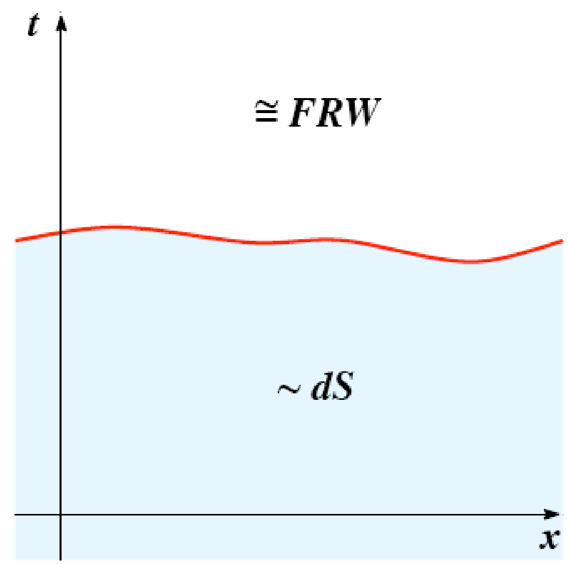
$$S_\zeta = \int dt d^3x \sqrt{-g}\,\frac{\dot{\phi}^2}{H^2}\,\tfrac{1}{2}(\partial\zeta)^2$$

Typical size of quantum fluctuations:

$$\langle\zeta^2\rangle_H \sim \frac{H^4}{\dot{\phi}^2} \sim \frac{1}{\epsilon}\,\frac{H^2}{M_{Pl}^2}$$

Ratio between quantum and classical motion

$$\epsilon = \frac{1}{2}M_P^2\left(\frac{V'}{V}\right)^2$$



What happens if we make ε smaller and smaller, until $\zeta \sim 1$?

Quantum motion as important as classical one. Reheating surface more and more curved.

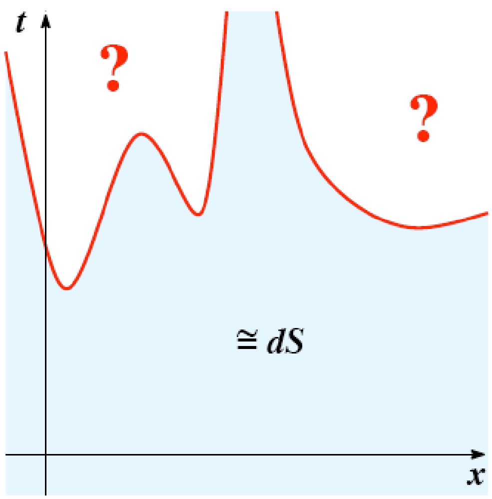
Always some points which are still inflating: eternal inflation

Is the system perturbative for $\frac{H}{M_P\sqrt{\epsilon}} \sim 1$?

1) ζ~1: large curvature? The large curvature comes from the embedding, the background geometry is still close to dS

In a spatially flat gauge, δφ: $g_{ij} = a^2(t)\delta_{ij}$ $\quad \delta g_{00} = \frac{\dot\phi}{H M_{\rm Pl}^2}\,\delta\phi \sim \sqrt{\epsilon}\,\frac{H}{M_{Pl}}$,

$$\delta g_{0i} = -\frac{\dot\phi^2}{2H^2 M_{\rm Pl}^2}\frac{a^2(t)}{\nabla^2}\nabla_i \frac{d}{dt}\left(\frac{H}{\dot\phi}\,\delta\phi\right) \sim a(t)\cdot\sqrt{\epsilon}\,\frac{H}{M_{Pl}}$$



As we make ε small the geometry gets closer to dS:
indeed the potential is flatter and flatter. Only δφ gets large.

Goncharov, Linde, Mukhanov 87

No control of the geometry after inflation:
it differs of order 1 from unperturbed FRW!

2) Large quantum fluctuations are not a problem, unless they signal strong coupling.
Dominant interactions come through gravity (the potential is very flat)

Maldacena 02

$$S_3 \sim \int \frac{\dot\phi}{H M_{\rm Pl}^2}\,\delta\phi\,\partial\delta\phi\,\partial\delta\phi \qquad \frac{S_3}{S_{\delta\phi}} \sim \sqrt{\epsilon}\,\frac{H}{M_{\rm Pl}}$$

Small!

Reheating volume: smoothing

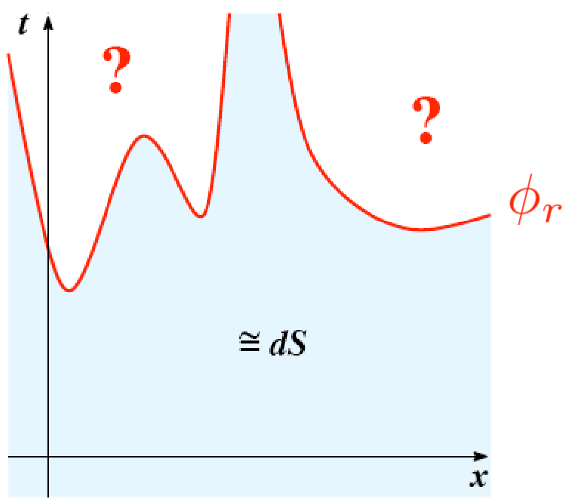
For small ε free scalar, living in exact dS

$$S_{\delta\phi} = \int dt d^3x \sqrt{-g}\, \tfrac{1}{2}(\partial\delta\phi)^2 \qquad ds^2 = -dt^2 + e^{2Ht} dx_i^2$$

At leading order in slow-roll, $\dot\phi$ and H are constant

Only dimless parameter is the ratio: $\dot\phi/H^2$

Characterize the onset of eternal inflation studying the reheating volume



Reheating at: $\phi = \phi_r$

Look at the inflaton configuration smoothed over physical distances Λ^{-1}

$$\phi_\Lambda(\vec{x}, t) = \int d^3r\; f_\Lambda(r)\, \phi(\vec{x} + \vec{r}/a(t), t)$$

As Universe expands more and more modes enter in the filter function: "quantum motion"

If Λ<< H, the surface is space-like $(\partial\phi)^2 > 0 \qquad \langle(\partial\phi)^2\rangle = \dot\phi^2 + \langle(\partial\delta\phi)^2\rangle$

$$\langle(\partial\delta\phi)^2\rangle = -\int^{\Lambda\cdot a(t)} \frac{d^3k}{(2\pi)^3}\frac{1}{a^2(t)}\frac{H^2}{2k} = -\frac{H^2}{8\pi^2}\Lambda^2$$

+ similar for the variance

Diffusion equation

Smoothing keeps only modes which are large wrt H^{-1}

Quantum effects are exp suppressed: classical stochastic variables

$$\langle \delta\phi(\vec{x},t)\,\delta\phi(\vec{y},t)\rangle = \frac{1}{(2\pi)^2}\frac{1}{|\vec{x}-\vec{y}|^2\,a^2(t)} - \frac{H^2}{(2\pi)^2}\log\frac{|\vec{x}-\vec{y}|}{L}$$

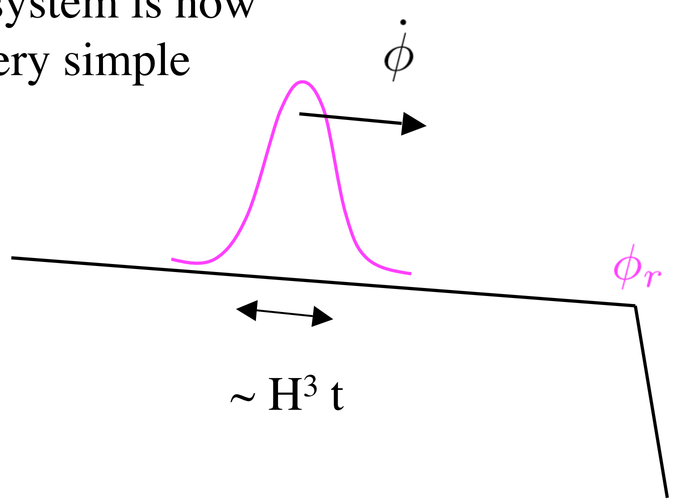
Starobinsky, Linde, Ford, Vilenkin 82

For points which are not resolved by the smoothing:

$$\langle(\delta\phi^2(\vec{x},t))\rangle = \alpha\Lambda^2 + \frac{H^2}{(2\pi)^2}\big(\log\Lambda L + \beta + Ht\big) = \boxed{\frac{H^3}{(2\pi)^2}\,t + \text{const}}$$

Diffusion: like brownian motion

The system is now very simple



+ exponential expansion

$a \sim e^{Ht}$

The reheating volume is a very non-linear function of my field.
Let us study its statistical properties

The average reheating volume

Unable to study directly ρ(V). Let us study its momenta.

$$\langle V \rangle = \int dV\, V \rho(V)$$

$$\langle V \rangle = \left\langle \int d^3x\, e^{3Ht_r(\vec{x})} \right\rangle = \int d^3x\, \left\langle e^{3Ht_r(\vec{x})} \right\rangle$$

Reheating time at x

$$\langle V \rangle = L^3 \int dt\, e^{3Ht} p_r(t)$$

$$p_r(t) = -\frac{d}{dt} \int_{-\infty}^{\phi_r} d\phi\, P(\phi, t)$$

Pdf of being at ϕ at time t

$$\frac{\partial P}{\partial \sigma^2} = \frac{1}{2} \frac{\partial^2 P}{\partial \psi^2}$$

$$\sigma^2 = \frac{H^3}{4\pi^2} t$$

$$\psi = \phi - \dot{\phi} t$$

Diffusion equation with boundary condition at ϕ_r

$$P_1(\psi, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \left(e^{-\psi^2/2\sigma^2} - e^{8\pi^2 \dot{\phi}\, \phi_r / H^3} e^{-(\psi - 2\phi_r)^2/2\sigma^2} \right)$$

Image term

$$p_r(t) = -\frac{H^3}{8\pi^2} \frac{\partial P}{\partial \phi} \bigg|_{\psi = \phi_r - \dot{\phi} t}$$

$$p_r(t) \sim e^{-2\pi^2 \frac{(\phi_r - \dot{\phi} t)^2}{H^3 t}}$$

At large t: $\langle V \rangle \simeq L^3 \int_0^{\infty} dt\, \alpha(t)\, e^{3Ht - \frac{2\pi^2 \dot{\phi}^2}{H^3} t + O(1)}$ It converges iff

$$\boxed{\Omega \equiv \frac{2\pi^2}{3} \frac{\dot{\phi}^2}{H^4} > 1}$$

Independent of initial condition!

see also Winitzki 02

What happens at Ω=1?

We have identified a critical point. What happens there?

Anticipating the conclusion: **at Ω=1 a non-zero probability of infinite volume develops (and this probability is 1 is the limit of large initial volume)**

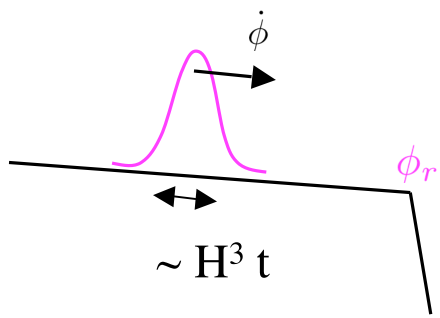
Do all moments diverge at Ω=1 ? NO!

$\langle V^n \rangle$ for sufficiently large n, diverges for any value of Ω!

Indeed consider the probability of being at $\phi = \phi_r - \dot{\phi} t$ (backwards wrt classical motion!)

$$e^{-A \cdot Ht} \qquad A \sim \dot{\phi}^2/H^4 \sim \Omega^2$$

The classical volume from that point is ~ $e^{3Ht} H^{-3}$



For $n > A/3 \sim \Omega^2$ the momentum diverges

It requires the possibility of fluctuating infinitely far away from the barrier
The large time limit does not commute with the infinite interval limit

All moments diverge at Ω=1 on a finite interval

The divergence of the variance

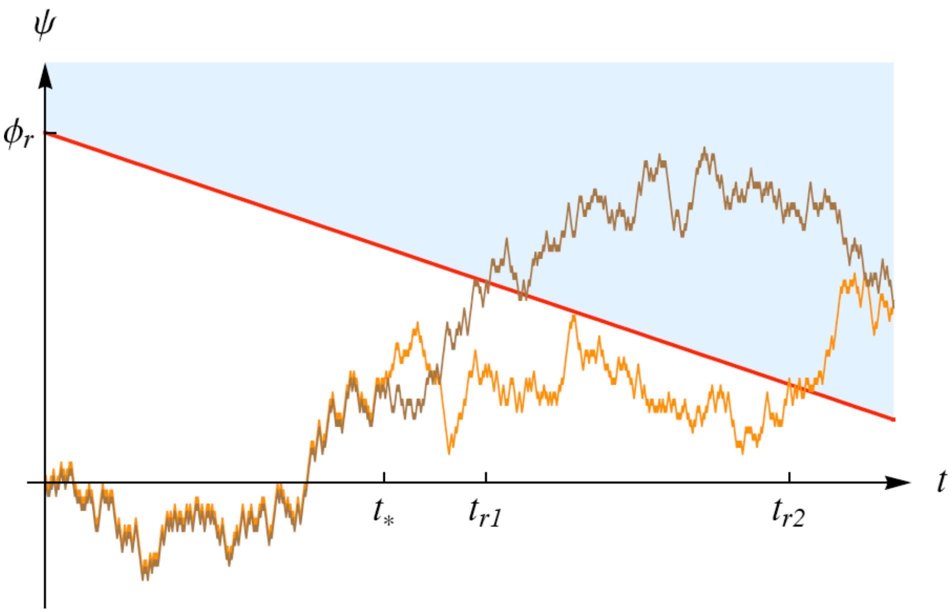
The average was simple! $\langle V^2 \rangle = \left\langle \int d^3x\, d^3y\, e^{3Ht_r(\vec{x})}\, e^{3Ht_r(\vec{y})} \right\rangle = L^3 \int dt_1 dt_2 d^3x\; e^{3Ht_1} e^{3Ht_2}\; p_{2r}(t_1, t_2, |\vec{x}|)$

I need the joint reheating pdf: $p_{2r}(t_1, t_2, |\vec{x}_1 - \vec{x}_2|) = -\partial_{t_1}\partial_{t_2} \int_{-\infty}^{\phi_r} d\phi_1 \int_{-\infty}^{\phi_r} d\phi_2\; P_2(\phi_1, \phi_2, |\vec{x}_1 - \vec{x}_2|, t_1, t_2)$

2 point pdf

Solution is a convolution of random walks $P_2(\psi_1, \psi_2, |\vec{x}|, t_1, t_2) = \int_{-\infty}^{\phi_r - \dot{\phi} t_*} d\psi\, G(\psi_1, \psi, t_1 - t_*) G(\psi_2, \psi, t_2 - t_*) P_1(\psi, t_*)$

For $t_1, t_2 \ll t_* \equiv -H^{-1} \log H|\vec{x}|$ the 2 points are not resolved and evolve together.
Then they separate and evolve independently.



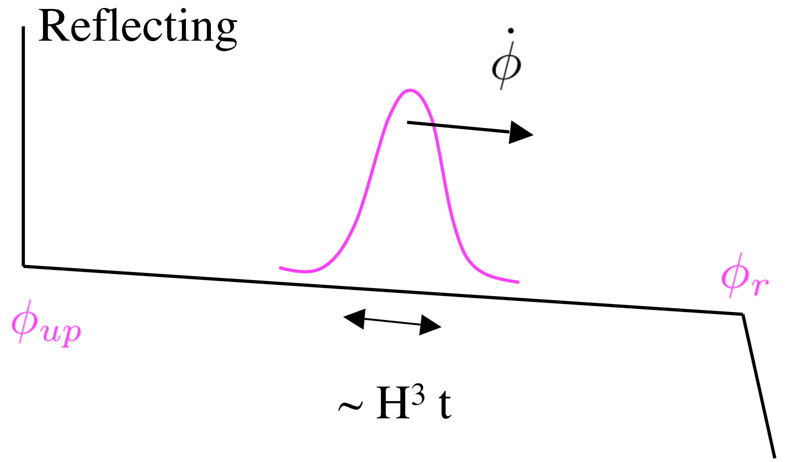
The integrals can be done in saddle point.

The variance converges for Ω > 9/8 > 1

Check. The most likely splitting point is

$\phi = -\dot{\phi} t_r / 6$ dominated by infinitely backwards fluctuations

Moments on a finite segment



Diffusion eq: $$\frac{\partial P}{\partial t} = -\dot\phi\frac{\partial P}{\partial \phi} + \frac{H^3}{8\pi^2}\frac{\partial^2 P}{\partial \phi^2}$$

Boundary cond:
$$\dot\phi P - \frac{H^3}{8\pi^2}\frac{\partial P}{\partial \phi} = 0 \quad \text{at} \quad \phi = \phi_{up}$$
$$\phi = 0 \quad \text{at} \quad \phi = \phi_r$$

Separation of variables: $$\exp\left[\frac{4\pi^2\dot\phi\,\phi - 2\pi^2\dot\phi^2 t}{H^3}\right]\sum_n a_n \sin\left[\frac{\alpha_n\pi(\phi_r-\phi)}{\phi_r-\phi_{up}}\right]\exp\left[-\frac{H^3 t}{8\pi^2}\cdot\frac{\alpha_n^2\pi^2}{(\phi_r-\phi_{up})^2}\right]$$

$$\mathrm{tg}(\alpha_n\pi) = -\frac{H^3}{4\pi^2\dot\phi}\cdot\frac{1}{\phi_r-\phi_{up}}\cdot\alpha_n\pi$$

Removing ϕ_{up} we recover previous results

But if we keep the segment finite, long t dynamics dominated by lowest eigenvalue

$$\exp\left[-\frac{2\pi^2\dot\phi^2 t}{H^3} - \frac{H^3 t}{8(\phi_r-\phi_{up})^2}\right] \sim \exp\left[-\frac{2\pi^2\dot\phi^2 t}{H^3}\right] = \exp(-3H\Omega t)$$

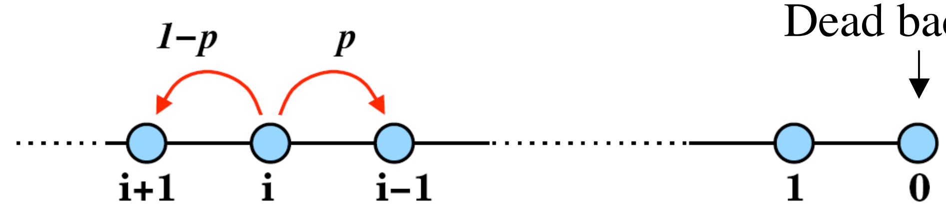
Straightforward proof all $\langle V^n\rangle$ diverge at Ω=1

Limits do not commute: $t \to \infty$ diffusion knows if we live on a finite segment

A discrete "toy" model

(see also Winitzki 05)

Discrete version of the model:
multi-type branching process



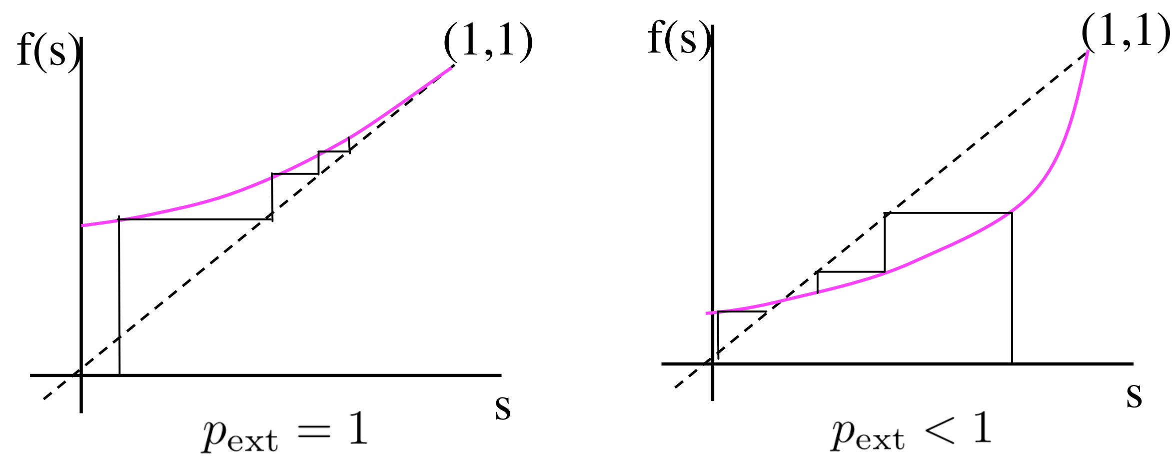
Extinction probability

N children at each step

Galton-Watson: Single type model with offspring probability p_k (e.g. family names)

Generating function: $f(s) \equiv \sum_k p_k \, s^k$ Gen function at n-th step is functional iterate of f: $f^n(s) = f(\ldots f(f(s)))$.

Extinction probability < 1
iff $\langle k \rangle > 1$



For finite number of sites: maximum λ of the matrix of averages

- λ<1 : subcritical, extinction probability = 1
- λ>1 : supercritical, extinction probability < 1

The chain and the real case

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ Np & 0 & N(1-p) & 0 & \cdots \\ 0 & Np & 0 & N(1-p) & \cdots \\ 0 & 0 & Np & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Studying the behavior of M^n at large n

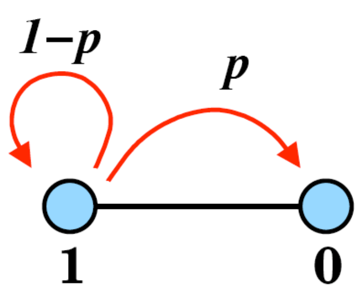
$$p_c \equiv \frac{1}{2}\left(1+\sqrt{1-\frac{1}{N^2}}\right)$$

Below this value all moments diverge
+ non-zero probability of never terminating

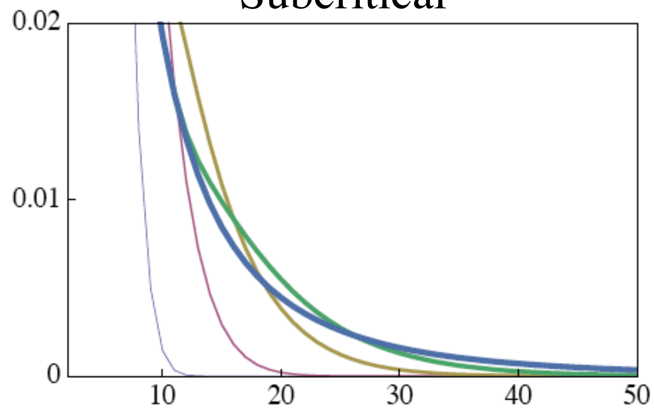
Continuum limit to reproduce diffusion equation

Reproduce the critical value of Ω, the factor 9/8 for the variance in the infinite case

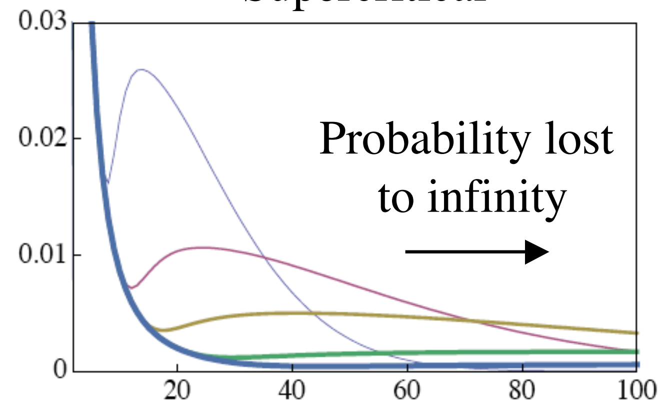
How does a probability for infinite volume develops?



Subcritical



Supercritical



Conclusions

1. The regime of eternal inflation $\zeta \sim 1$ can be perturbatively studied (during inflation). Slow-roll expansion

2. Reheating volume (smoothed) as "order parameter"

3. Diffusion equation, <V> converges for $\boxed{\Omega \equiv \frac{2\pi^2}{3} \frac{\dot{\phi}^2}{H^4} > 1}$

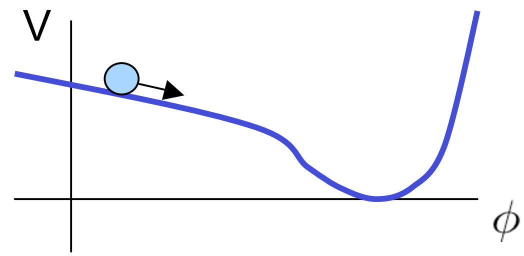
4. Spurious divergences of higher moments for infinite line

5. From the discrete model, **at Ω=1 a non-zero probability of infinite V develops,**
   this probability is 1 for large initial volume

6. Subleading terms in slow-roll?

Generalization

1. In slow-roll $\dot{\phi}$ and H are ~ constant in H^{-1}, but vary over a **long interval**



Eternal inflation (non-zero prob of infinite volume)
iff $\Omega<1$ for $\Delta\phi > H$

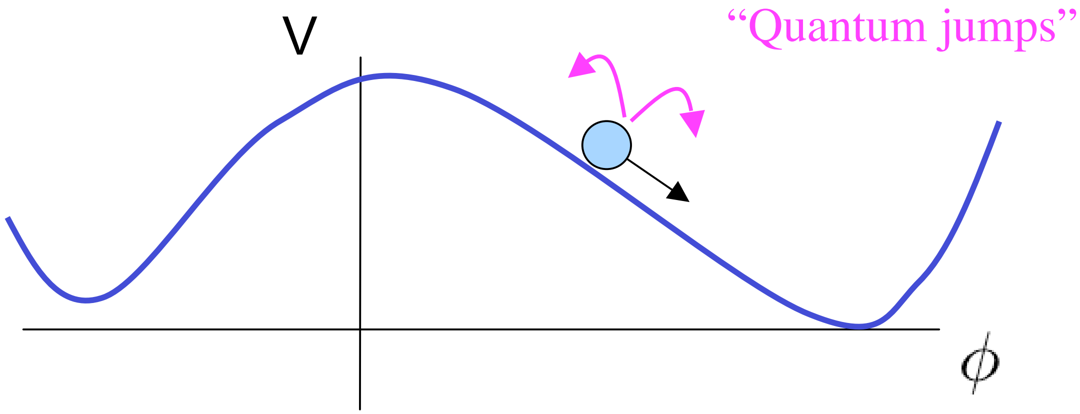
2. What is the effect of **slow-roll corrections**?

- They do not completely change the picture: no $t^{1+\varepsilon}$. It makes sense to perturb in ε.
- Random walk picture not valid anymore. Coupling among different modes.
- Is the transition still sharp? The possibility of infinite V is still a good criterion?

3. Non-minimal models of inflation.

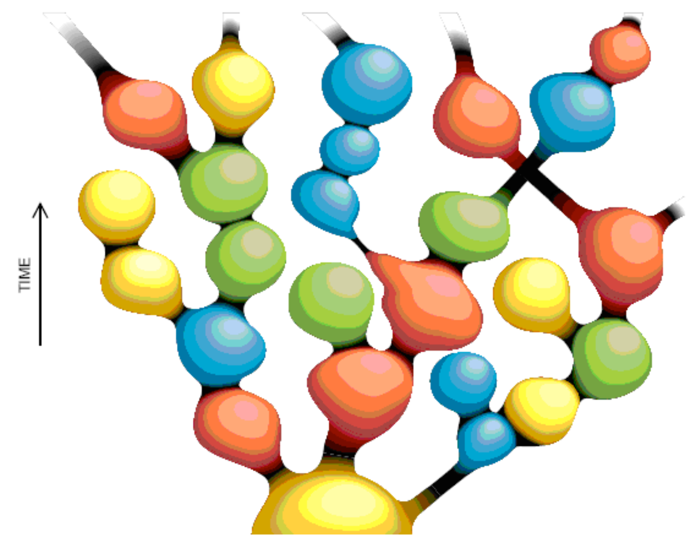
- **Eternal inflation out of the regime of validity of EFT if $c_S<1$** and for ghost inflation
- Multi-field models

Eternal inflation



Steinhardt, Vilenkin, Linde...

- Jumps backwards give a longer inflation and thus more volume
  For suff. slow classical motion we expect an eternal process



- Global structure of the Universe

- Population of the landscape, probabilities...

+ tunnelling among vacua