Hadronic Z- and τ-decays in order α_s^4

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Phys. Rev. Lett. 88 012001 (2002)
Phys. Rev. Lett. 95, 012003 (2005)
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Phys. Rev. Lett. 97, 061803 (2006)

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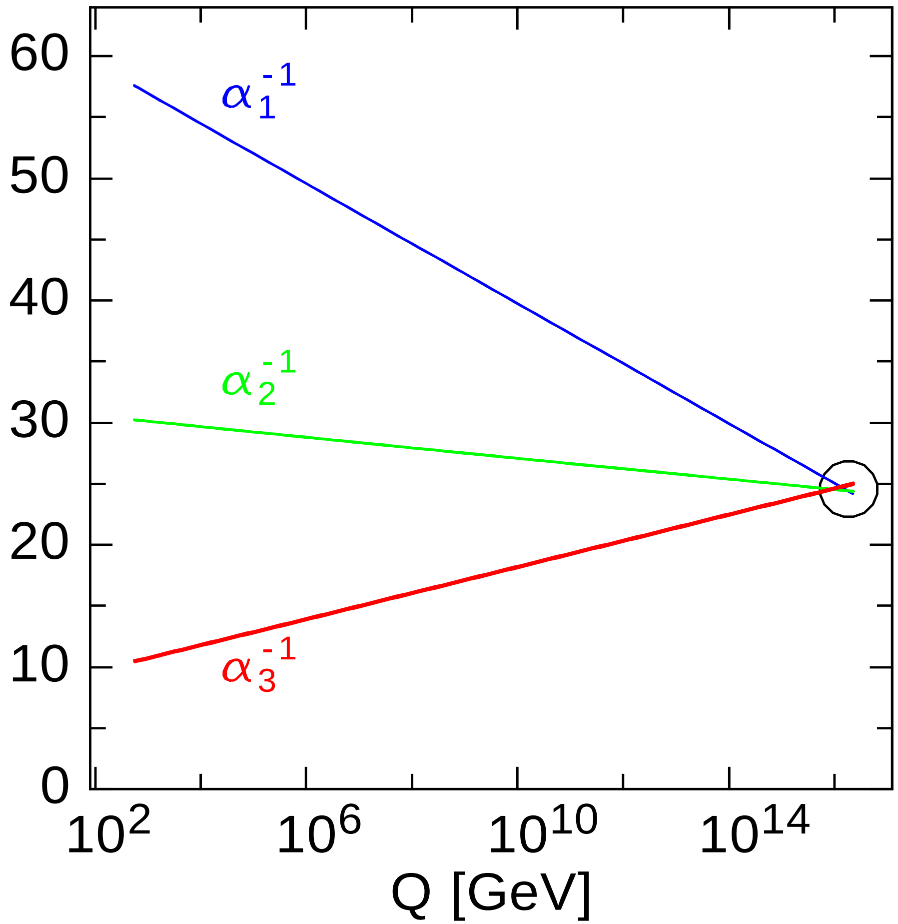
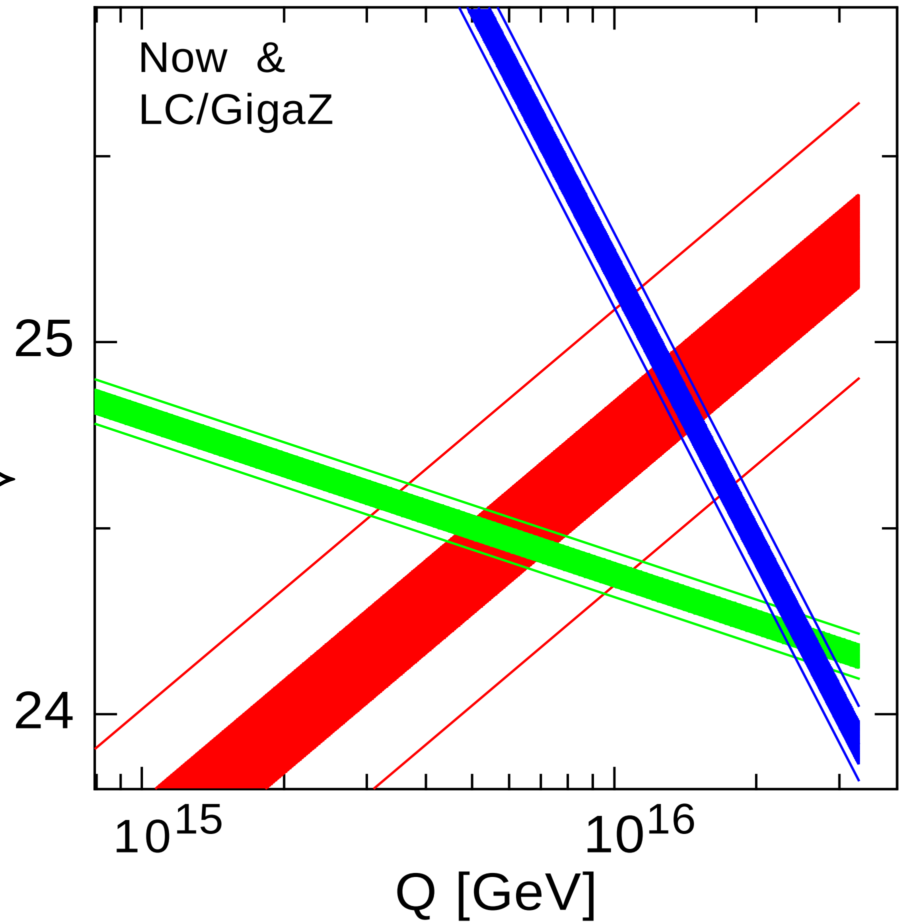
QUARKS 2008, Sergiev Posad

$\alpha_s \longleftrightarrow$ **least known fundamental constant of the Nature:**

(from Zerwas)



$\Rightarrow$



$$\alpha_s = 0.1185 \pm 0.0026 \quad \text{vs} \quad \pm 0.0009$$

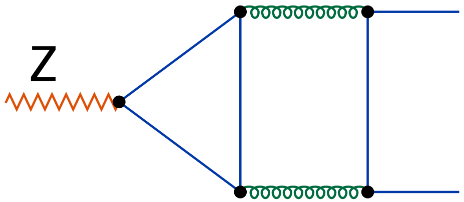
$\delta\alpha_s/\alpha_s = 2.3\,\%$ $\qquad$ $\delta\alpha_s/\alpha_s = 0.8\,\%$

α_s based on

$$\Gamma_{\text{had}} = \Gamma_0 \left(1 + a_s + 1.409\, a_s^2 - 12.767\, a_s^3\right) \qquad \left(a_s \equiv \frac{\alpha_s}{\pi}\right)$$

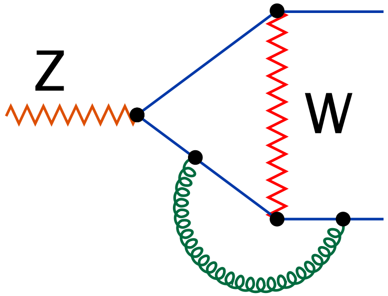
$+$ corrections

$\Big(\sim m_b^2/M_z^2$; singlet terms



$+ \ldots.$

mixed QCD $\otimes$ electroweak



$+ \ldots \Big)$

dominant theory error: $\delta\alpha_s/\alpha_s = 1.7\,\%$
from uncalculated higher orders! $\longrightarrow \alpha_s^4$

higher QCD corrections are even more important (theory error significantly larger than exp. uncertainity) for

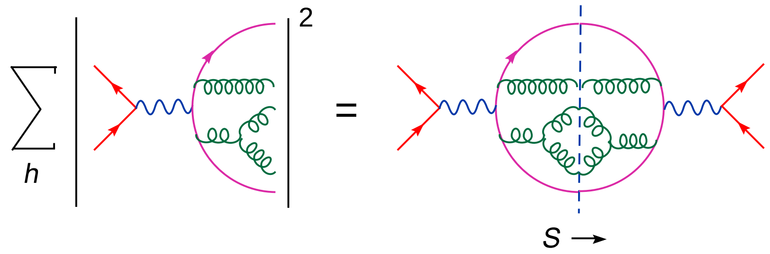
$$\mathbf{R}_\tau = \mathbf{\Gamma}(\tau \to \nu \text{ had})/\mathbf{\Gamma}(\tau \to \mathbf{e}\nu\nu)$$

due to much less energetic scale involved:

$$\mathbf{M_Z}/\mathbf{M}_\tau \approx \mathbf{50}!$$

Theoretical Framework

R(s) is related (via unitarity) to the correlator of the EM quark currents:



$$R(s) \approx \Im\,\Pi(s - i\delta)$$

$$3Q^2\Pi(q^2 = -Q^2) = \int e^{iqx}\langle 0|T[\ j^v_\mu(x) j^v_\mu(0)\ ]|0\rangle dx$$

To conveniently sum the RG-logs one uses the Adler function:

$$D(Q^2) = Q^2 \frac{d}{d\,Q^2}\Pi(q^2) = Q^2 \int \frac{R(s)}{(s+Q^2)^2}\,ds$$

or $(a_s \equiv \alpha_s/\pi)$

$$R(s) = \frac{1}{2\pi i}\int_{-s-i\delta}^{-s+i\delta} dQ^2 \frac{D(Q^2)}{Q^2} = D(s) - \pi^2 \frac{\beta_0^2 d_0}{3} a_s^3 + \ldots$$

Status of R(s) (before 01.08, $\overline{\text{MS}}$-scheme, $\mu^2 = s$):

$$\mathbf{R(s) = 1 + \frac{\alpha_s}{\pi} + (1.9857 - 0.1152\, n_f)\ \frac{\alpha_s^2}{\pi^2}}$$

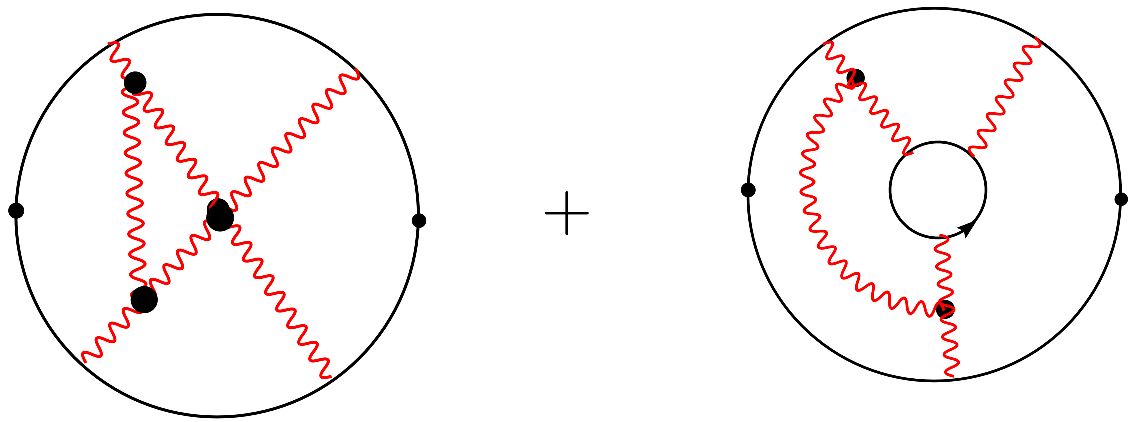
$$\mathbf{+(-6.6369 - 1.2001\, n_f - 0.00518\, n_f^2)\, \frac{\alpha_s^3}{\pi^3}}$$

/**Gorishnii, Kataev, Larin, (1991)**; in Feynman gauge

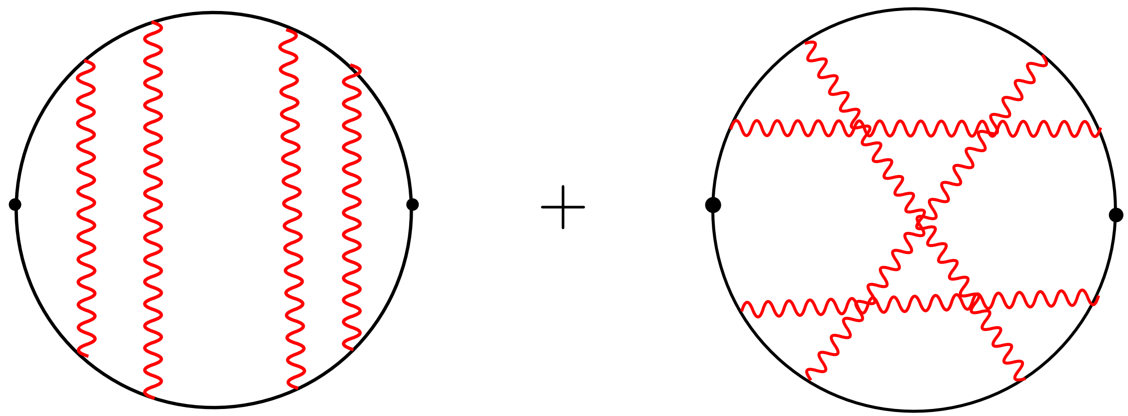
/**Surguladze, Samuel, (1991)**; in Feynman gauge

K.Ch, (1997); in general covariant gauge /

R(s) at five loops is contributed by $\approx 17 \cdot 10^3$ of nonabelian or/and non-quenched diagrams like



as well as 2671 purely abelian quenched diagrams like



massless props $\longleftrightarrow$ simplicity:

5-loop R(s) is reducible*

to 4-loop massless propagators ($\equiv$ p-integrals)

$\longleftarrow$

main object to compute

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* **in fact, any 5-loop anom. dim. or β-function in any theory**

reducible to 4-loop p-integrals with the R^*-operation

– a generalization of the IRR /A.A. Vladimirov, (1989); K. Ch., Smirnov (1984)/

COMMON STRATEGY

1. reduce (with the use of the traditional IBP method) to master integrals

2. evaluate masters (better analytically)

COMMON PROBLEMS

1. IBP identities are *extremely* complicated at higher loops/legs

2. master integrals are difficult to perform analytically (numerical integration is possible but not simple: an art by itself)

5 ways to reduce a Feynman integral to Masters

- Empiric /sit and think/ way, basically limited to 3 loops (/Mincer,Matad/);
- Arithmetic way: direct solution of /thousands or even millions!/ IBP eqs. /Laporta, Remiddi (96); Gehrmann, Schröder, Anastasiou, Czakon, F.Tkachov. . . , Sturm , Marquard. . . , A. Smirnov
- Gröbner Basis Technique /Tarasov (98-), . . . ,Smirnov & Smirnov (2006-)/
- New Representation for CF's /Baikov (96), Steinhauser, Smirnov . . . /

$$\Downarrow$$

- $1/D$ expansion of CF's /Baikov (98-04) /

Feynman parameters:

$$\frac{1}{m^2 - p^2} \approx \int d\ \alpha\ \mathrm{e}^{i\alpha(m^2-p^2)}$$

New parameters:

$$\frac{1}{m^2 - p^2} \approx \int\ \frac{d\ x}{x}\ \delta(x - (m^2 - p^2))$$

Now for a given topology one can make loop integrations once and forever with the result:

Baikov's Representation:

$$\boxed{F(\underline{n}) \ \sim \ \int \cdots \int \frac{\mathbf{d}x_1 \ldots \mathbf{d}x_N}{x_1^{n_1} \ldots x_N^{n_N}} \left[P(\underline{x})\right]^{(D-h-1)/2} ,}$$

where $P(\underline{x})$ is a polynomial on $x_1, \ldots, x_N$ (and masses and external momenta)

New representation obviously meets the same set IBP'id as the original integral but it has much more flexibility! (Due to choice of the integration contours)

MAIN IDEA: **to use (1) as a "template" for the very CF's!**

reduction to Masters: $1/D$ expansion[1]

- coefficient functions in front of *master integrals* depend on D in simple way:

$$C^\alpha(D) = \frac{P^n(D)}{Q^m(D)} \xlongequal[D\to\infty]{} \sum_k C_k^\alpha \ (1/D)^k$$

- The terms in the $\mathbf{1/D}$ expansion expressible (with the use of the Baikov's representation) through simple Gaussian integrals

- sufficiently many terms in $1/D$ and $C_k^\alpha \longrightarrow C^\alpha(D)$

- computing time and required resources: could be huge (the price for full automatization); to cope with it we use parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .)

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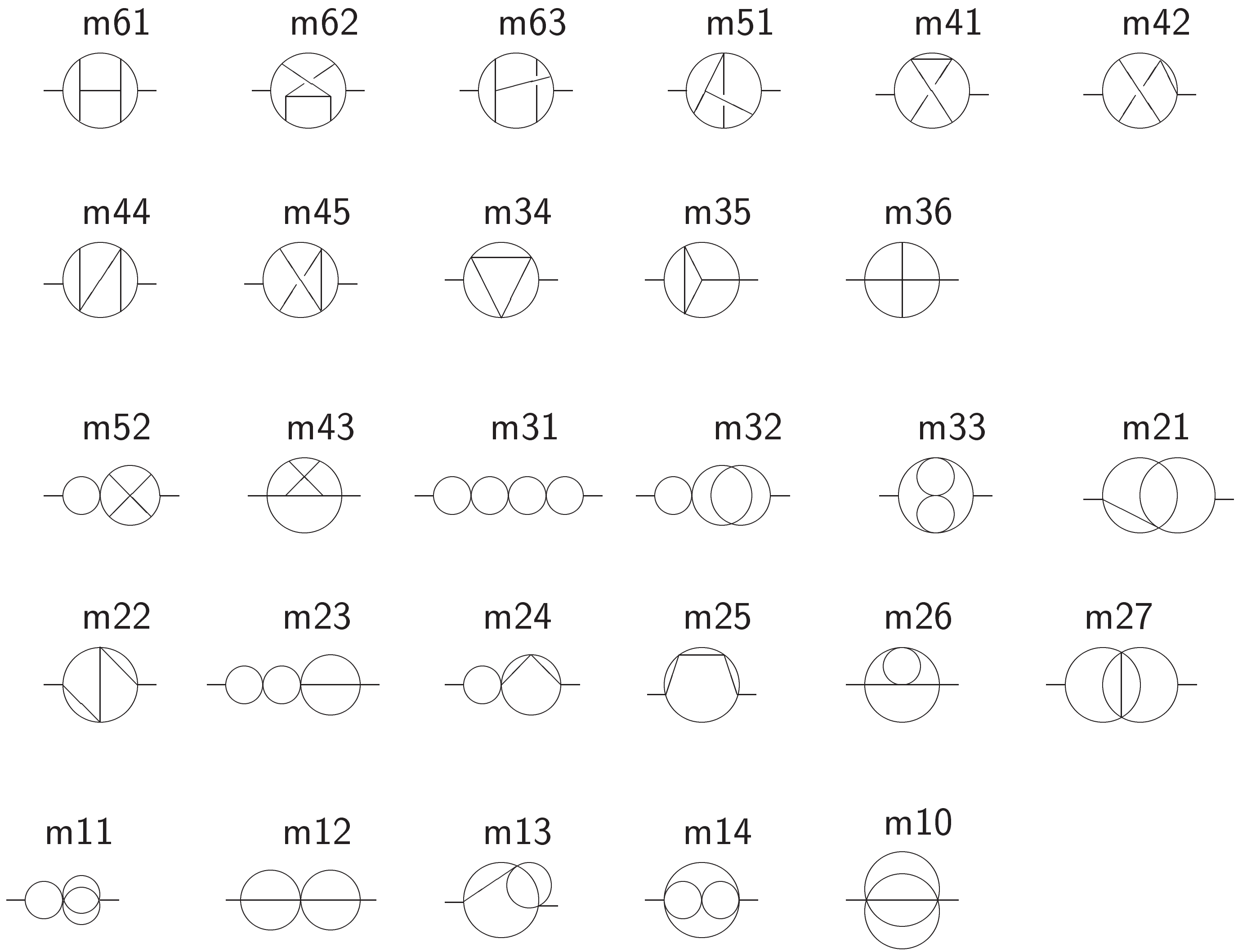
[1] Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

All relevant Master Integrals solved analytically (2004)

(method: "glue and cut" (Chetyrkin, Tkachov, (1981)) + BAICER)



Tool Box *

- **IRR / Vladimirov, (78)/ + IR R^* -operation /K. Ch., Smirnov (1984)/ + resolved combinatorics /K. Ch., (1997)/**
- **reduction to Masters: "direct and automatic" construction of CF's through $1/D$ expansion—made with BAICER—within the Baikov's representation for Feynman integrals**[1]
- **all 4-loop master p-integrals are known analytically /P. Baikov and K.Ch. (2004)/**
- **computing time and required resources: could be huge (the price for full automatization)**; to cope with it we use parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – ...) **and** HP XC4000 supercomputer of the Karlsruhe University

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* NO IBP identities are ever used at any step!

[1] Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

$$d_4 = n_f^3 \left[-\tfrac{6131}{5832} + \tfrac{203}{324}\,\zeta_3 + \tfrac{5}{18}\,\zeta_5\right]$$ (“renormalon” chain /M. Beneke 1993/)

$$+\, n_f^2 \left[\tfrac{1045381}{15552} - \tfrac{40655}{864}\,\zeta_3 + \tfrac{5}{6}\,\zeta_3^2 - \tfrac{260}{27}\,\zeta_5\right]$$ /Baikov, Kühn, K.Ch. (2002)/

$$+\, n_f \left[-\frac{13044007}{10368} + \frac{12205}{12}\,\zeta_3 - 55\,\zeta_3^2 + \frac{29675}{432}\,\zeta_5 + \frac{665}{72}\,\zeta_7\right]$$

$$+ \left[\frac{144939499}{20736} - \frac{5693495}{864}\,\zeta_3 + \frac{5445}{8}\,\zeta_3^2 + \frac{65945}{288}\,\zeta_5 - \frac{7315}{48}\,\zeta_7\right]$$

Interesting features:

1. irrationals up to ζ_7 (understandable from the structure of the masters)

2. no ζ_4 and/or ζ_6 (expected but mysterious!)

Result for the very R(s)

$$\boldsymbol{R = 1 + a_s + (1.9857 - 0.1152 n_f)\, a_s^2 + (-6.6369 - 1.2001 n_f - 0.0052 n_f^2)\, a_s^3 +}$$

$$\boxed{\boldsymbol{+(-156.61 + 18.77\, n_f - 0.7974\, n_f^2 + 0.02152\, n_f^3)\, a_s^4}}$$

and after separating dynamical from kinematical terms:

$$R = R = 1 + \ldots \big(\underline{18.24} - 24.88 + (\underline{0.086} - 0.091)\, n_f^2 + (\underline{-4.22} + 3.02)\, n_f^3\big)\ a_s^3$$

$$+ \big((\underline{135.8} - 292.4 + (\underline{-34.4} + 53.2)\, n_f + (\underline{1.88} - 2.67)\, n_f^2 + (\underline{-0.010} + 0.031)\, n_f^2\big)\ a_s^4$$

note: the π^2-dominance (Radyushkin, Pivovarov, Kataev, Shirkov, ...) is not well pronounced

FAC/PMS predictions* versus exact results

$n_f = 3:$

$$r_4^{\mathrm{FAC/PMS}} = -129 \pm 16 \quad \longleftrightarrow \quad r_4^{\mathrm{exact}} = -106.88 = \underline{48.08} - 155$$

$n_f = 4:$

$$r_4^{\mathrm{FAC/PMS}} = -112 \pm 30 \quad \longleftrightarrow \quad r_4^{\mathrm{exact}} = -92.89 = \underline{27.34} - 120.28$$

$n_f = 5:$

$$r_4^{\mathrm{FAC/PMS}} = -97 \pm 44 \quad \longleftrightarrow \quad r_4^{\mathrm{exact}} = -79.98 = \underline{9.21} - 89.191$$

* **(Kataev, Starshenko (95); Baikov, K.Ch., Kühn (2002))**

impact on α_s from Z-decays

$$R(s) = D(s) - \pi^2\beta_0^2 \left\{ \frac{d_1}{3} a_s^3 + \left(d_2 + \frac{5\beta_1}{6\beta_0} d_1 \right) a_s^4 \right\}$$

$$\Rightarrow \delta\alpha_s(M_Z) = 0.0005$$

$$\alpha_s(M_Z)^{\mathrm{NNNLO}} = 0.1190 \pm 0.0026$$

The theory error gets less by a factor 5 – 10!

impact on α_s from τ-decays

$$\frac{\Gamma(\tau\to h_{s=0}\nu)}{\Gamma(\tau\to l\overline{\nu}\nu)} = \mid V_{ud}\mid^2 S_{\mathrm{EW}} 3\,(1 + \boxed{\delta_P} + \underbrace{\delta_{\mathrm{EW}}}_{\text{small}} + \underbrace{\delta_{\mathrm{NP}}}_{0.003\pm 0.003})$$

$R_\tau = 3.471 \pm 0.011$

(Davier, Höcker, Zhang; ALEPH, OPAL, CLEO,. . .)

$\delta_P = 0.1998 \pm 0.043$ (exp) scale $\mu^2/M_\tau^2 = 0.4 - 2$

	$\alpha_s^{FO}(M_\tau)$	$\alpha_s^{CI}(M_\tau)$
no α_s^4	$0.337 \pm 0.004 \pm 0.03$	$0.354 \pm 0.006 \pm 0.02$
$d_4 = 25$	$0.325 \pm 0.004 \pm 0.02$	$0.347 \pm 0.006 \pm 0.009$
$d_4 = 49.08$	$0.322 \pm 0.004 \pm 0.02$	$0.342 \pm 0.005 \pm 0.01$

use mean value between FOPT and CIPT*

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*A.A. Pivovarov (1991,1992); F. Le Diberder and A. Pich (1992)/

$$\alpha_s(M_\tau) = 0.332 \pm 0.005_{\mathrm{exp}} \pm 0.015_{\mathrm{theo}}$$

four-loop running[1] + four-loop matching at quark thresholds[2]
($m_c(m_c) = 1.286(13)$ GeV, $m_b(m_b) = 4.164(25)$ GeV)

$$\begin{aligned} \alpha_s(M_Z) &= 0.1202 \pm 0.0006_{\mathrm{exp}} \pm 0.0018_{\mathrm{theo}} \pm 0.0003_{\mathrm{evol}} \\ &= 0.1202 \pm 0.0019 \end{aligned}$$

consistent with α_s from Z

$\delta\alpha_s$ from τ dominated by theory.
$\delta\alpha_s$ from Z dominated by statistics.

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[1] T. van Ritbergen, J.A.M. Vermaseren, and S.A. Larin (1997);M. Czakon (2005)
[2] Y. Schröder and M. Steinhauser (2006); K.G. Ch., J.H. Kühn, and C. Sturm (2006)

Summary

- Adler function, $R(s)$, R_τ available to $\mathcal{O}(\alpha_s^4)$
- First and only N^{3}LO results

$$\alpha_s(M_z) = \begin{cases} 0.1190 \pm 0.0026 & \text{from } Z \\ 0.1202 \pm 0.0019 & \text{from } \tau \end{cases}$$

- α_s^4 terms move Z and τ closer together

combined

$$\boxed{\alpha_s(M_Z) = 0.1198 \pm 0.0015}$$

Hystory: # The long march towards α_s^4

1-loop: 1 diagram /BC?/

2-loop: 3 diagrams /1951/

$\}$ **textbook/student problems these days**

3-loop: 37 diagrams /1979/ (completely by hand)

4-loop: 738 diagrams /1991/ (the first semi-manual calculation /correct from the second try only/)

1997 (the first completely automatic calculation)/

$\Downarrow$

5-loop: 19832! diagrams

1991+ 1997 - 1979 = 2008

The march started exactly 30 years ago at the INR (Moscow) AND JINR (Dubna):

Volume 85B, number 2,3 PHYSICS LETTERS 13 August 1979

HIGHER-ORDER CORRECTIONS TO $\boldsymbol{\sigma}_{\rm tot}(\boldsymbol{e}^+\boldsymbol{e}^- \longrightarrow$ HADRONS) IN QUANTUM CHROMODYNAMICS

K.G. CHETYRKIN, A.L. KATAEV and F.V. TKACHOV
Institute for Nuclear Research of the Academy of Sciences of the USSR, Moscow, USSR

Received 24 May 1979

We present the α_s^2 corrections to $\sigma_{\rm tot}(e^+e^- \longrightarrow$ hadrons) in massless QCD . . .

Volume 93B, number 4 PHYSICS LETTERS 30 June 1980

THE GELL-MANN-LOW FUNCTION OF QCD IN THE THREE-LOOP APPROXIMATION

O.V. TARASOV, A.A. VLADIMIROV and A.Yu. ZHARKOV
Joint Institute for Nuclear Research, Dubna, USSR

Received 28 March 1980

Both works share a number of spectacular features:

- All authors were well under 30 and some of them were, in fact, MSU diploma students when calculations started (and finished)
- each work has collected by now $\approx$ 500 citations

- two new mightly tools were discovered, developed and effectively used to get the work done:

  Gegenbauer Polynomial Technique in x-space /Moscow group/

  IRR: (Infrared Rearrangement) /Dubna group/

- the works would never appear without continuous exchange of ideas and methods (well before their official publications) between Moscow and Dubna groups

- **Last but not the least:**

Dubna-Moscow cooperation was forcefully encouraged by our scientific leaders:

D.V. Shirkov, A.N.Tavkhelidze and V.A. Matveev

whose role could not be overestimated! BIG THANKS!