Entanglement entropy in lattice gauge theories [ArXiv:0802.4247]

Pavel Buividovich (speaker) and Mikhail Polikarpov

ITEP, Moscow, and JIPNR, Minsk

Quarks'08

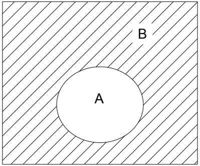
Outline

Quantum entanglement and effective IR degrees of freedom

- Quantum entanglement has been recently discussed as a probe of IR behavior of lattice field theories [Levin'04, Cardy'05, Velitsky'08]
- Quantum entanglement is a "universal order parameter" for quantum phase transitions [Kitaev *et al.*'04, Levin, Wen'04]
- For lattice theories near critical couplings, ground state is a strongly entangled quantum superposition of elementary lattice degrees of freedom
- Away from continuum limit (e.g. strong coupling) ground state is almost a direct product
- Thus strong quantum entanglement of the ground state is a signature of the emergence of new collective degrees of freedom

Entanglement entropy

- What is the measure of a quantum entanglement of the vacuum of some QFT?
- For field theory in $(D-1)+1$-dimensional space-time divide the $D-1$-dimensional space into the region A and its complement B
- How much information is lost if all measurements are restricted to A?



Entanglement entropy

- This is the entanglement entropy [Srednicki'93, Bernstein'96]:

$$\boxed{S[A] = -\mathrm{Tr}_A\,(\hat{\rho}_A \ln \hat{\rho}_A)}$$

- Here $\hat{\rho}_A = \mathrm{Tr}_B\,|0\rangle\langle 0|$ is the reduced density matrix for the fields within the region A

Counting degrees of freedom

- Entanglement entropy $S[A]$ typically behaves as:

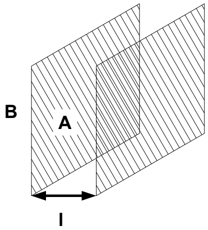
$$\boxed{S[A] = \kappa \Lambda_{UV}^2 |\partial A| + S_f[A]}$$

- S_f is proportional to the number of effective degrees of freedom (central charge for CFT's) [Calabrese,Cardy'05]
  at the scale determined by the size of A

Counting degrees of freedom

If A is a slab of width l, one can define the entropic C-function, which is a counter of d.o.f.'s at scale $\sim l^{-1}$ [Takayanagi'06]:

$$\boxed{C(l) = l^3 \frac{\partial}{\partial l} S(l)}$$



Quantum entanglement and gravity

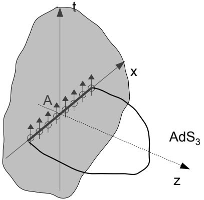
- Reduction procedure is implemented in nature for black holes
- Entanglement entropy = black hole entropy [Srednicki'93, Fursaev'06]
- Black hole entropy is just the area of the horizon:

$$\boxed{S_{BH} = c m_{pl}^2 \Sigma}$$

- where Σ is the horizon area and m_{pl} is the effective cutoff scale

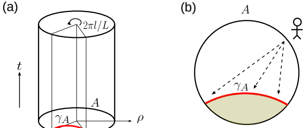
Synthesis of field theory and gravity approaches

- Generalization of Bekenstein-Hawking formula in AdS/CFT:



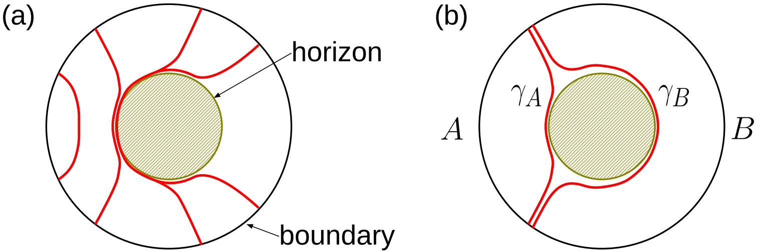
$$S[A] = c' m_{pl}^2 \Sigma_{bulk}[A]$$

- $\Sigma_{bulk}[A]$ is the minimal area of the hypersurface in the bulk bounded by ∂A
- This result is exact for 2D CFT and AdS_3 [Ryu and Takayanagi'06]



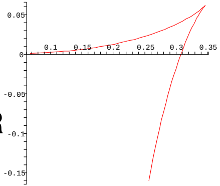
Entanglement entropy for confining gauge theories

- Confining gauge theories in 4D are dual to certain 5D black hole backgrounds [Witten'98, Klebanov'02]
- There are two extremal surfaces for such backgrounds [Klebanov'07, Ryu, Takayanagi'06]
- As the minimal surface changes from γ_A to γ_B, entanglement entropy is nonanalytic



Phase transition for the entanglement entropy

- At some critical size l_c of A γ_A becomes larger than γ_B
- At l_c the derivative of the entanglement entropy and the entropic C-function are discontinuous
- Effective number of degrees of freedom rapidly drops down in the IR
- Qualitatively this is the transition from asymptotic freedom to confinement
- Rapid transition implies that in some sense quarks and hadrons do not coexist



Entanglement entropy in lattice gauge theories

- We have tried to detect this stepwise change for entanglement entropy in lattice simulations
- The concept of entanglement entropy is intrinsic for Hamiltonian formulation
- For lattices we need path-integral formulation

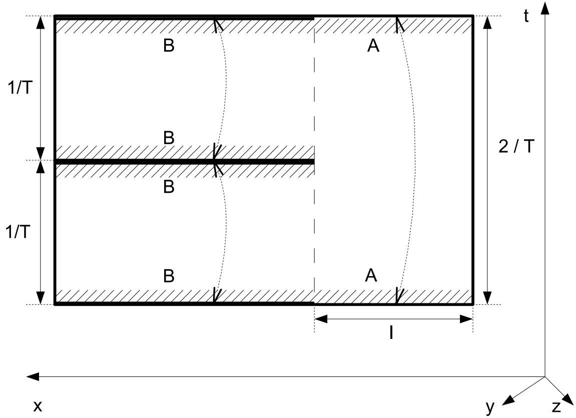
Path integral representation for entanglement entropy

- Entanglement entropy is related to a set of free energies on spaces of special topology [Cardy, Calabrese'04]:

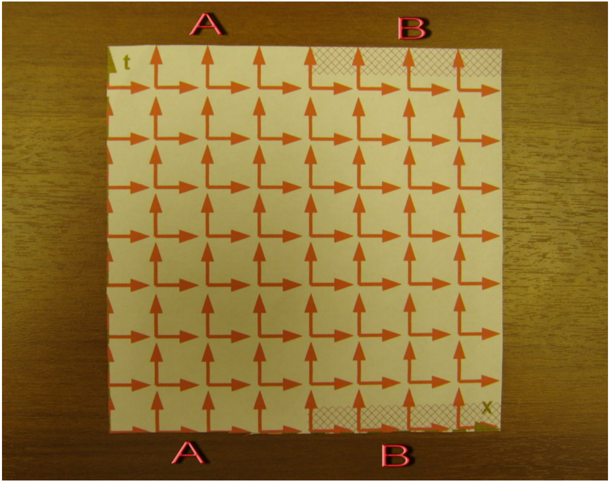
$$\boxed{S[A] = \lim_{T \to 0} \left(\lim_{s \to 1} \frac{\partial}{\partial s} F[A, s, T] - F(T) \right)}$$

- Boundary conditions for fields in $F[A, s, T]$:
  - Periodic in time with period sT^{-1}, $\vec{x} \in A$
  - Periodic in time with period T^{-1} otherwise
- Such spaces can be realized as *s*-sheeted Riemann surfaces [Cardy, Calabrese'04]
- The derivation is straightforward once Feynman-Kac formula is remembered

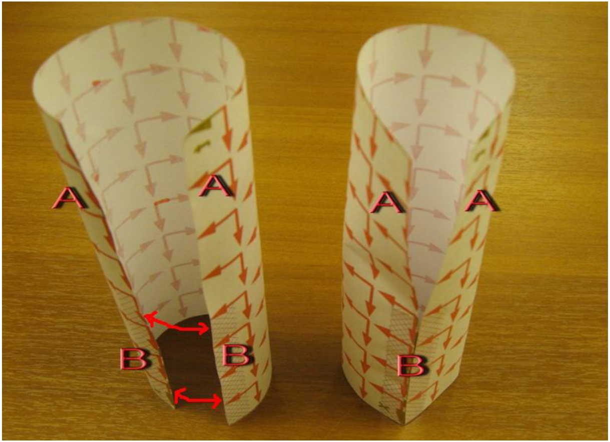
Boundary conditions for $s = 2$



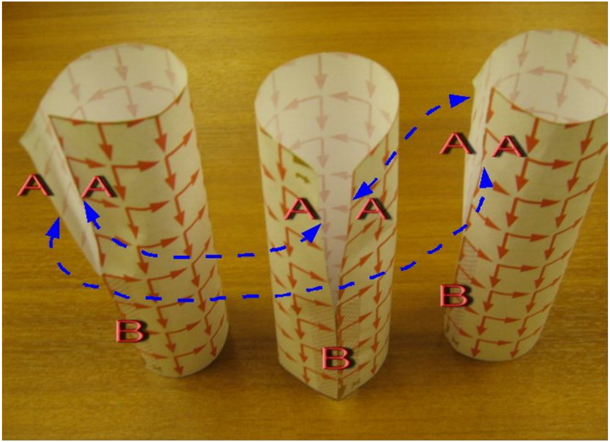
Constructing lattices with cuts I



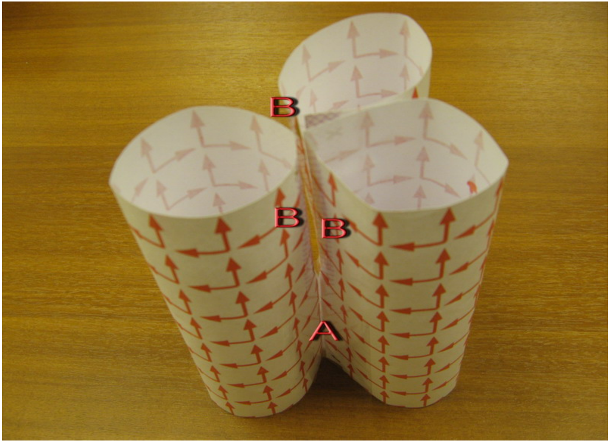
Constructing lattices with cuts II



Constructing lattices with cuts III



The result: three-sheeted Riemann surface, $s = 3$



Measuring entanglement entropy

- Derivative over s is estimated from finite difference

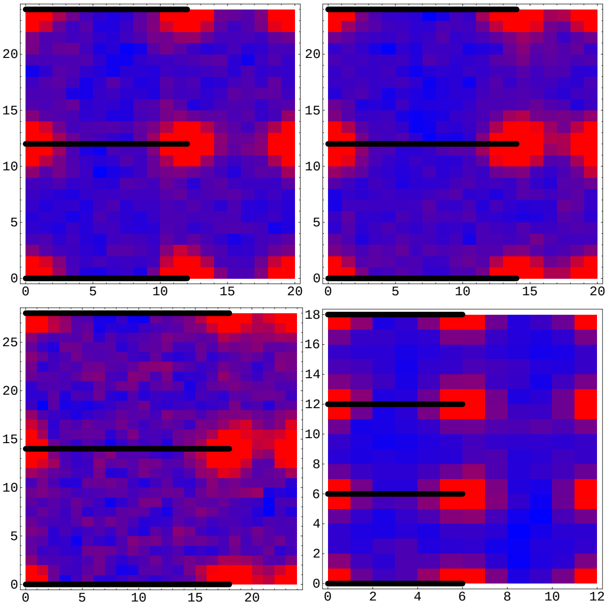
$$\boxed{\begin{aligned}&\lim_{T\to 0}\left(\lim_{s\to 1}\frac{\partial}{\partial s}F\left[A,s,T\right]-F\left(T\right)\right)\approx\\ &\approx\lim_{T\to 0}\left(F\left[A,2,T\right]-2F\left(T\right)\right)\end{aligned}}$$

- Measurements of free energy:
  - Indirect method:

$$\boxed{\sum_{p}\langle\,\beta\left(1-\frac{1}{2}\,\mathrm{Tr}\,g_p\right)\rangle=\frac{\partial}{\partial\ln\beta}F\left(\beta\right)}$$

  - Direct method [Fodor *et al.*'07]: small differences of free energies

Average action density on lattices with cuts



UV divergent part of entanglement entropy

- Branching points cover the surface with total area

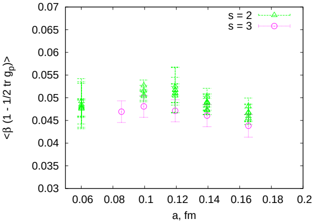
$$\Sigma = s N_L^2 a^2$$

- Total action excess is:

$$\delta S \sim s a^{-2} |\partial A|$$

- Hence entanglement entropy is

$$S[A] \sim a^{-2} |\partial A|$$



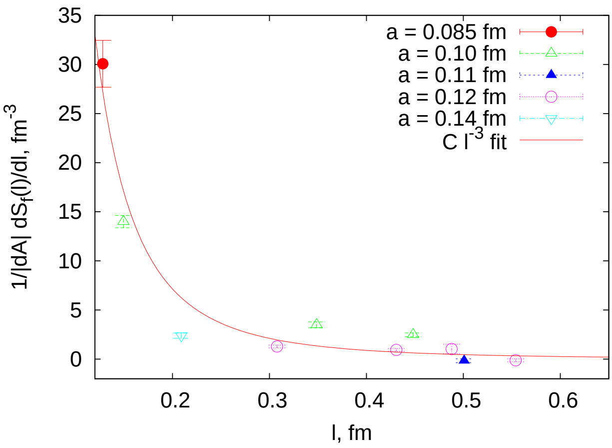
UV finite part of entanglement entropy

- We have extracted the derivative of entanglement entropy from finite differences:

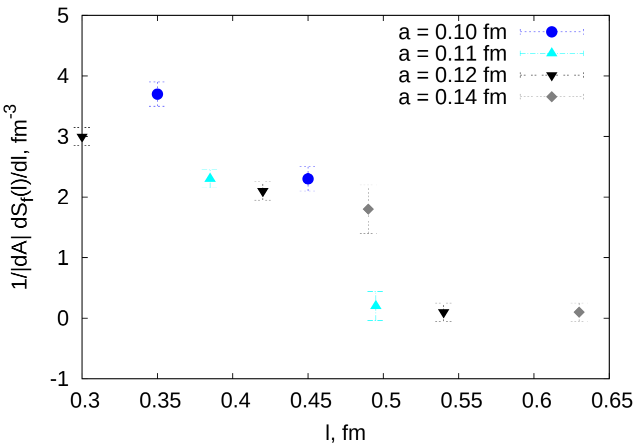
$$\boxed{\begin{aligned}&(F(l+1,2,T)-2F(T))-(F(l,2,T)-2F(T))=\\&=F(l+1,2,T)-F(l,2,T)\end{aligned}}$$

- l was varied both by changing a and $a^{-1}l$

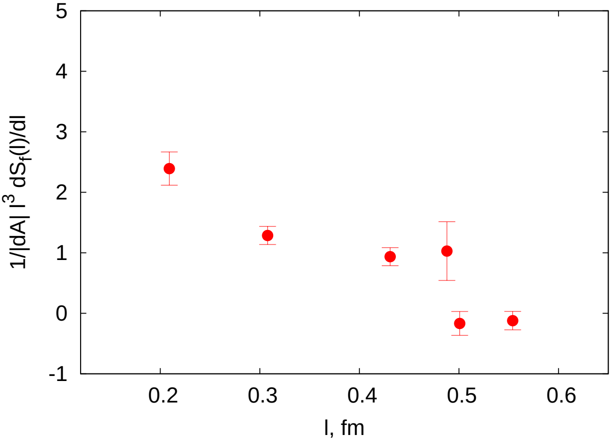
Derivative of entanglement entropy



Discontinuity of derivative



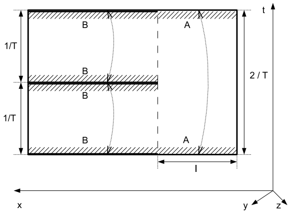
Entropic *C*-function



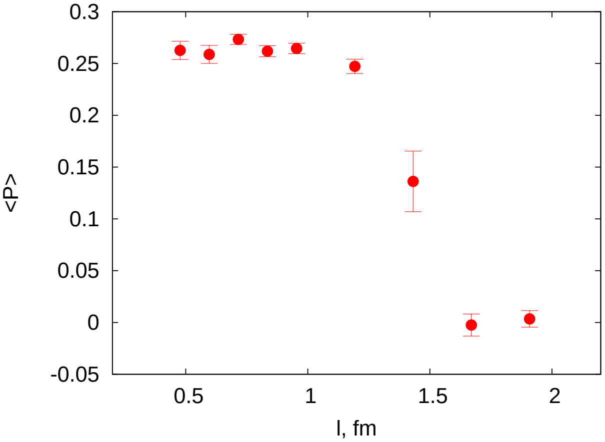
This is a signature of second-order phase transition

Entanglement entropy at finite temperatures

- Consider theory at $T > T_c$ but $T/2 < T_c$
  - $l = 0$ - two systems at temperature T, deconfinement
  - $l \to \infty$ - theory at temperature $T/2$, confinement
- How sharp is the transition between two phases?

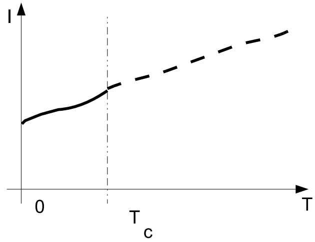


Expectation values of Polyakov line vs l



Line of phase transitions in the $l - T$ plane

- Above T_c different measures of entanglement give different transition points
- This looks like a crossover
- Below T_c l_c slowly increases with T



Conclusions

- We confirm basic properties of entanglement entropy, valid for all local field theories
  - Quadratic divergence
  - $\sim l^{-2}$ behavior at small distances
- Highly nontrivial property: discontinuity of derivative
- This prediction was first made from AdS/CFT [Klebanov *et al.*'07, Takayanagi'06]
- For nonzero temperatures there are still no theoretical predictions!