Thresholds in FAPT: Euclid vs Minkowski

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OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT**
- Resolution — **FAPT**: Completed set $\{\mathcal{A}_\nu; \mathfrak{A}_\nu\}_{\nu\in\mathbb{R}}$ and its properties
- Technical development of **FAPT**: higher loops, convergence, accuracy, thresholds
- Euclidean **FAPT**: Pion form factor
- Minkowskian **FAPT**: Higgs boson decay $H^0 \to b\bar{b}$
- Conclusions

Recent Related Publications

- A. B., Passek-Kumerički, Schroers, Stefanis — **PRD 70 (2004) 033014**
- A. B., Stefanis — **NPB 721 (2005) 50**
- A. B., Mikhailov, Stefanis — **PRD 72 (2005) 074014**
- A. B., Karanikas, Stefanis — **PRD 72 (2005) 074015**
- A. B., Mikhailov, Stefanis — **PRD 75 (2007) 056005**
- A. B.&Mikhailov — “Resummation in (F)APT”, **arXiv:0803.3013 [hep-ph]**
- A. B. — “Global FAPT in QCD with Selected Applications”, **arXiv:0805.0829 [hep-ph]**

Analytic Perturbation Theory

in

QCD

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- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$

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- RG evolution: $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\, B(\mu^2)$ reduces in 1-loop approximation to

$$Z \sim a^{\nu}[L]\Big|_{\nu\, =\, \nu_0\, \equiv\, \gamma_0/(2b_0)}$$

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- $\boxed{\text{PT}}$ $\displaystyle\sum_m d_m a_s^m(Q^2) \quad \Rightarrow \quad \sum_m d_m \mathcal{A}_m(Q^2)$ $\boxed{\text{APT}}$

  m – power $\quad \Rightarrow \quad$ m – index

Spectral representation

By **analytization** we mean "Källen–Lehman" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

with spectral density $\rho_f(\sigma) = \textbf{Im}\left[f(-\sigma)\right]/\pi$.

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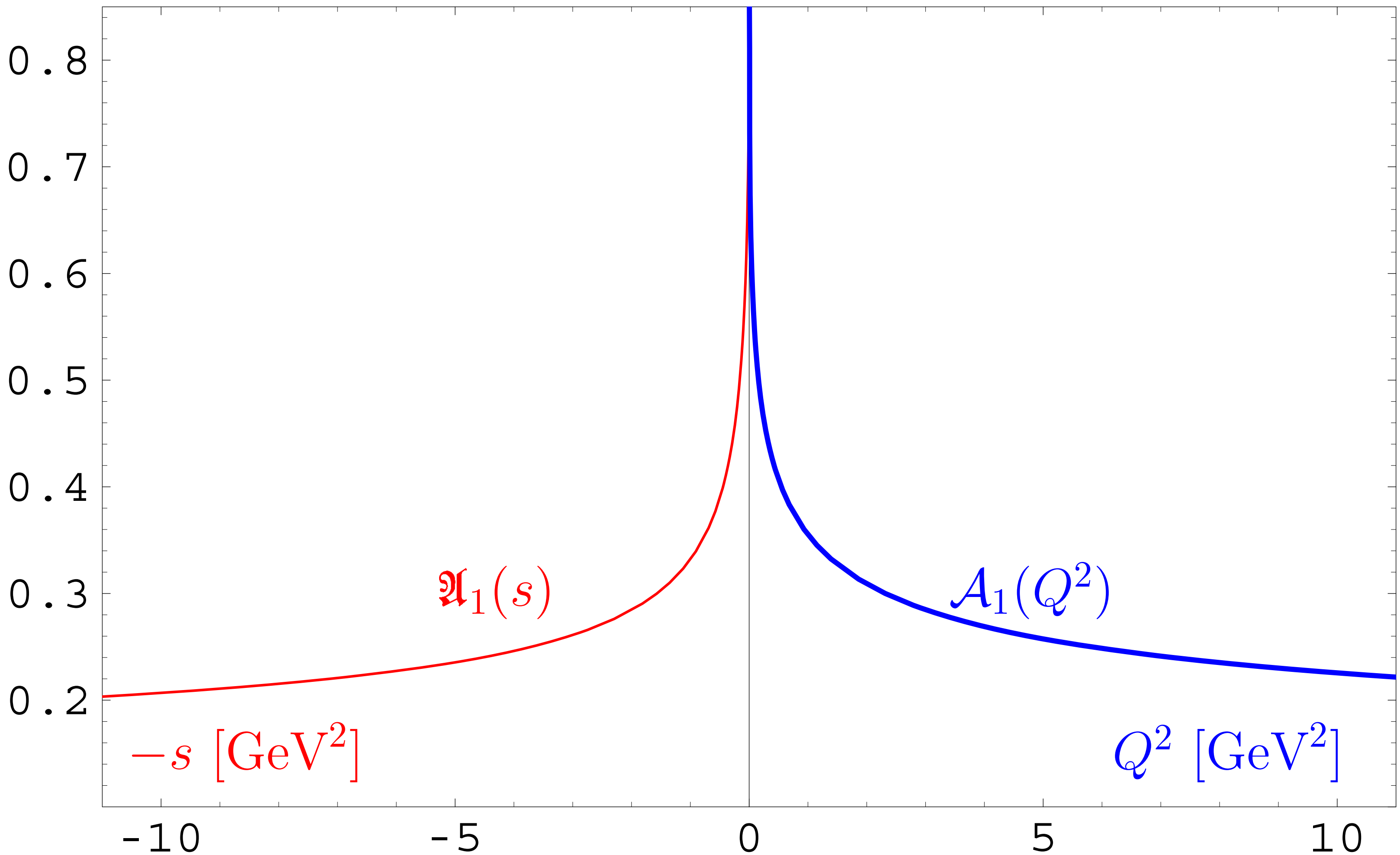
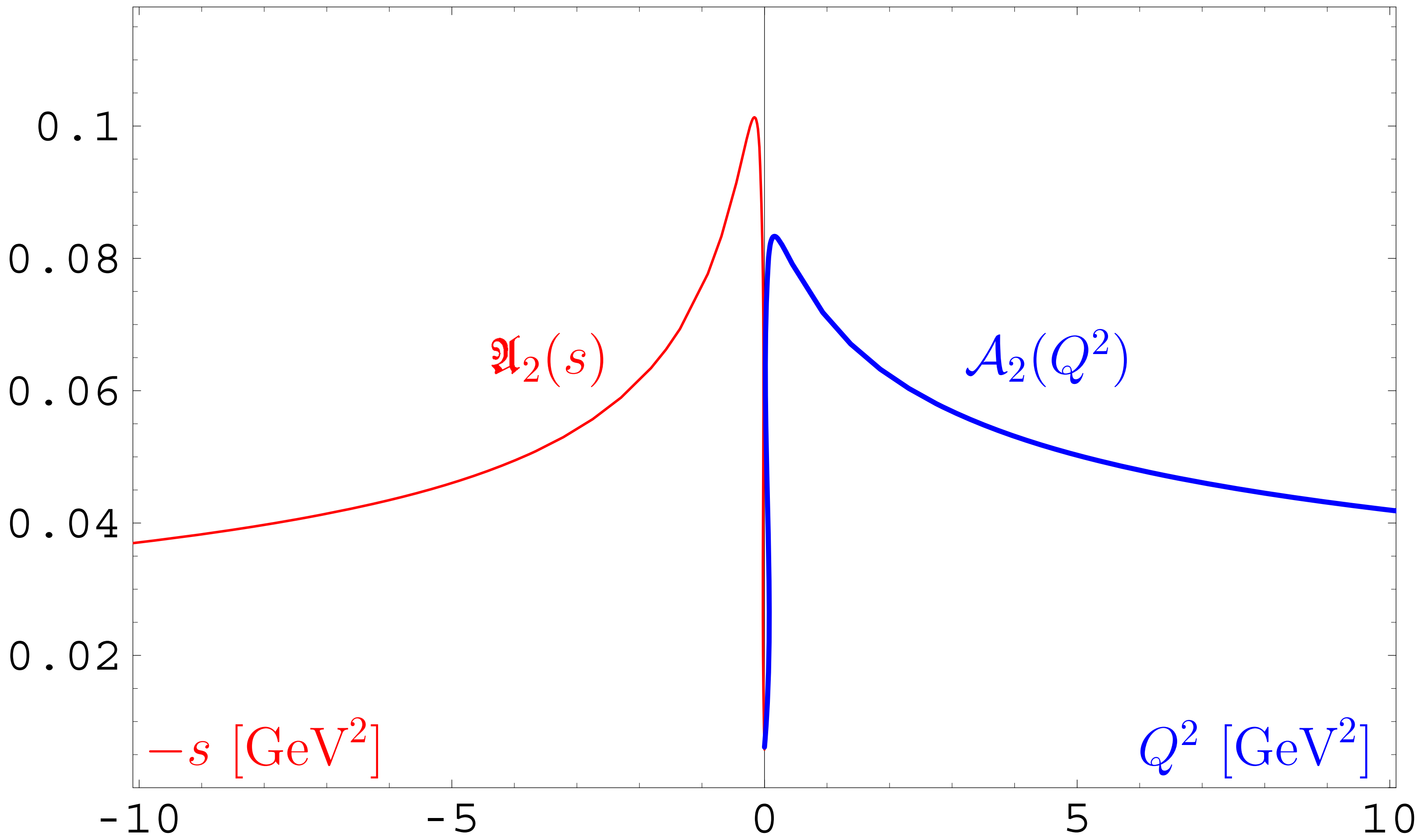
APT *graphics: Distorting mirror*

First, couplings: $\mathfrak{A}_1(s)$ and $\mathcal{A}_1(Q^2)$



APT graphics: Distorting mirror

Second, square-images: $\mathfrak{A}_2(s)$ and $\mathcal{A}_2(Q^2)$



Problems

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Open Questions

- “Analytization” of multi-scale amplitudes beyond LO of pQCD: additional logs depending on scale that serves as factorization or renormalization scale **[Karanikas&Stefanis – PLB 504 (2001) 225]**

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- "Analytization" of multi-scale amplitudes beyond LO of pQCD: additional logs depending on scale that serves as factorization or renormalization scale **[Karanikas&Stefanis – PLB 504 (2001) 225]**
- Evolution induces some non-integer, fractional, powers of coupling constant
- Resummation of gluonic corrections, giving rise to Sudakov factors, under "Analytization" difficult task **[Stefanis, Schroers, Kim – PLB 449 (1999) 299; EPJC 18 (2000) 137]**

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- Factorization $\longrightarrow$ $[a_s[L]]^n \, L^m$

Fractional

APT

Constructing one-loop FAPT

In one-loop **APT** we have a very nice recursive relation

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We can use it to construct **FAPT**.

FAPT(E): Properties of $\mathcal{A}_\nu[L]$

First, Euclidean coupling ($L = L(Q^2)$):

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

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- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
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- $\mathfrak{A}_{-2}[L] = L^2 - \dfrac{\pi^2}{3}, \quad \mathfrak{A}_{-3}[L] = L\left(L^2 - \pi^2\right), \ \ldots$;
- $\mathfrak{A}_m[L] = (-1)^m \mathfrak{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
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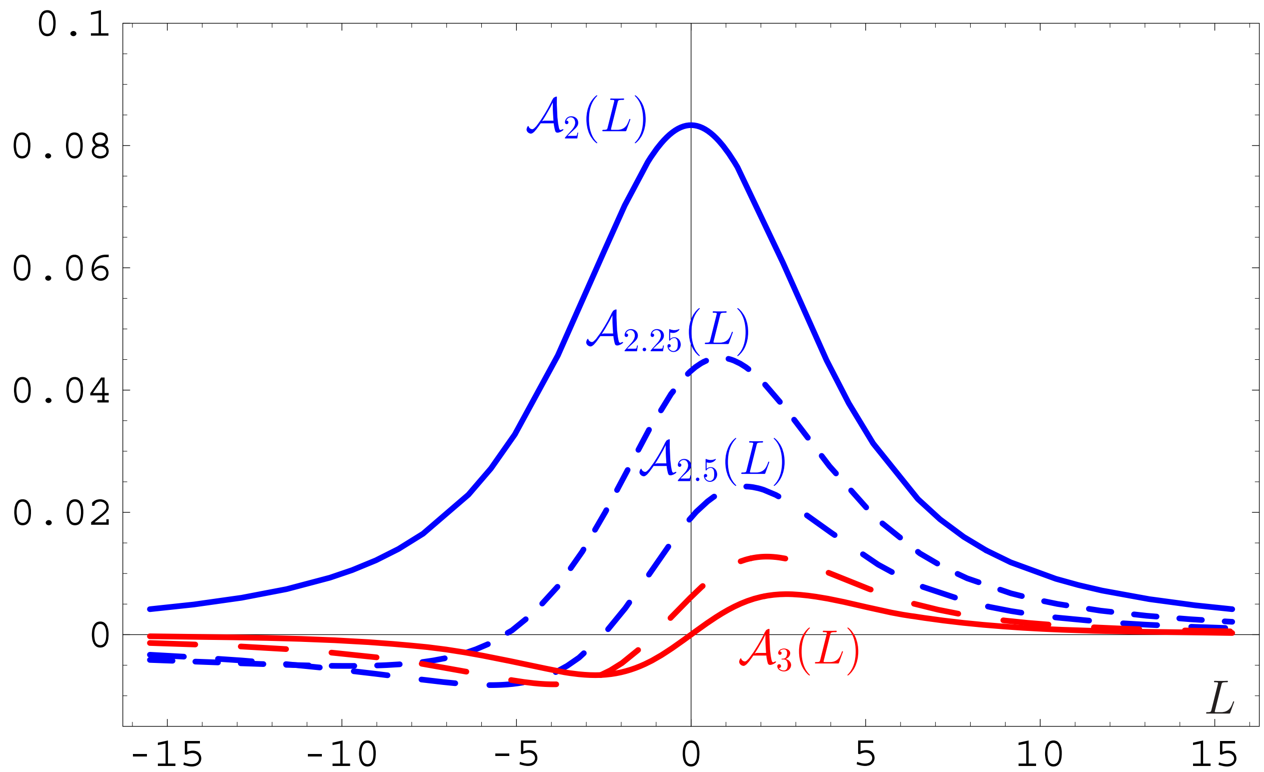
FAPT(E): Graphics of $\mathcal{A}_\nu[L]$ vs. L

$$\boxed{\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}}$$

First, graphics for fractional $\nu \in [2, 3]$:



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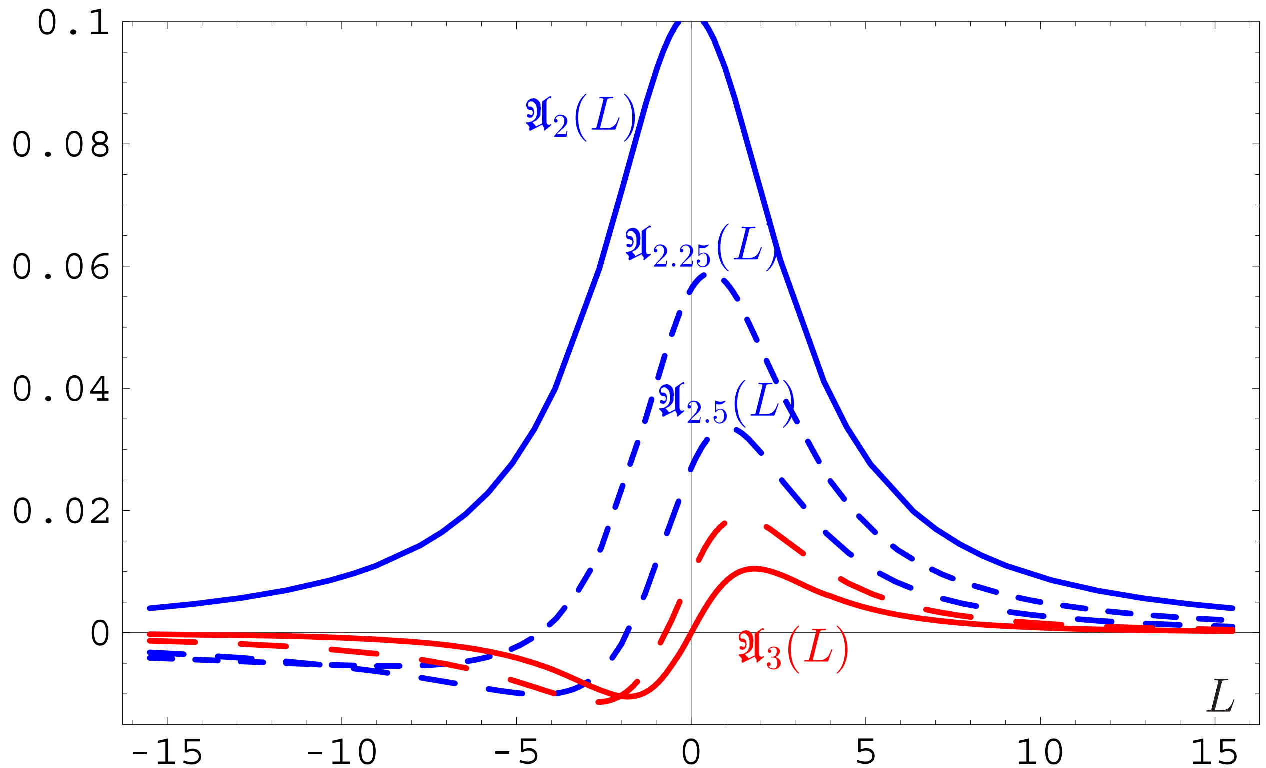
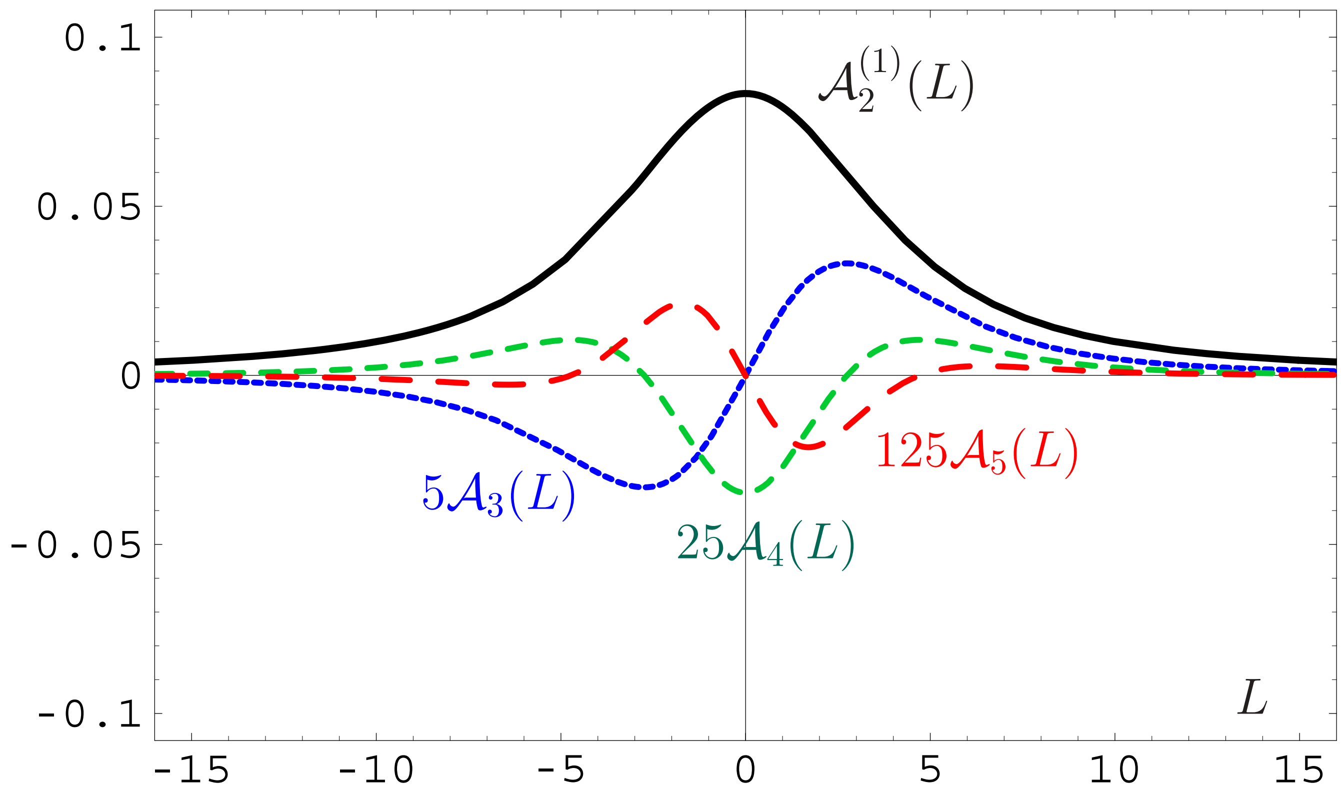
Compare with graphics in Minkowskian region :



FAPT(E): Graphics of $\mathcal{A}_\nu[L]$ vs. L

$$\mathcal{A}_1[0] = \frac{1}{2}\,,\ \mathcal{A}_2[0] = \frac{1}{12}\,,\ \mathcal{A}_4[0] = \frac{-1}{720}\,,\ \mathcal{A}_3[0] = \mathcal{A}_5[0] = 0$$

Next, graphics for $\nu = 2, 3, 4, 5$:



FAPT(M): Graphics of $\mathfrak{A}_\nu[L]$ vs. L

$$\mathfrak{A}_1[0] = \frac{1}{2}\,,\ \mathfrak{A}_2[0] = \frac{1}{\pi^2}\,,\ \mathfrak{A}_4[0] = -\frac{1}{3\pi^4}\,,\ \mathfrak{A}_3[0] = \mathfrak{A}_5[0] = 0$$

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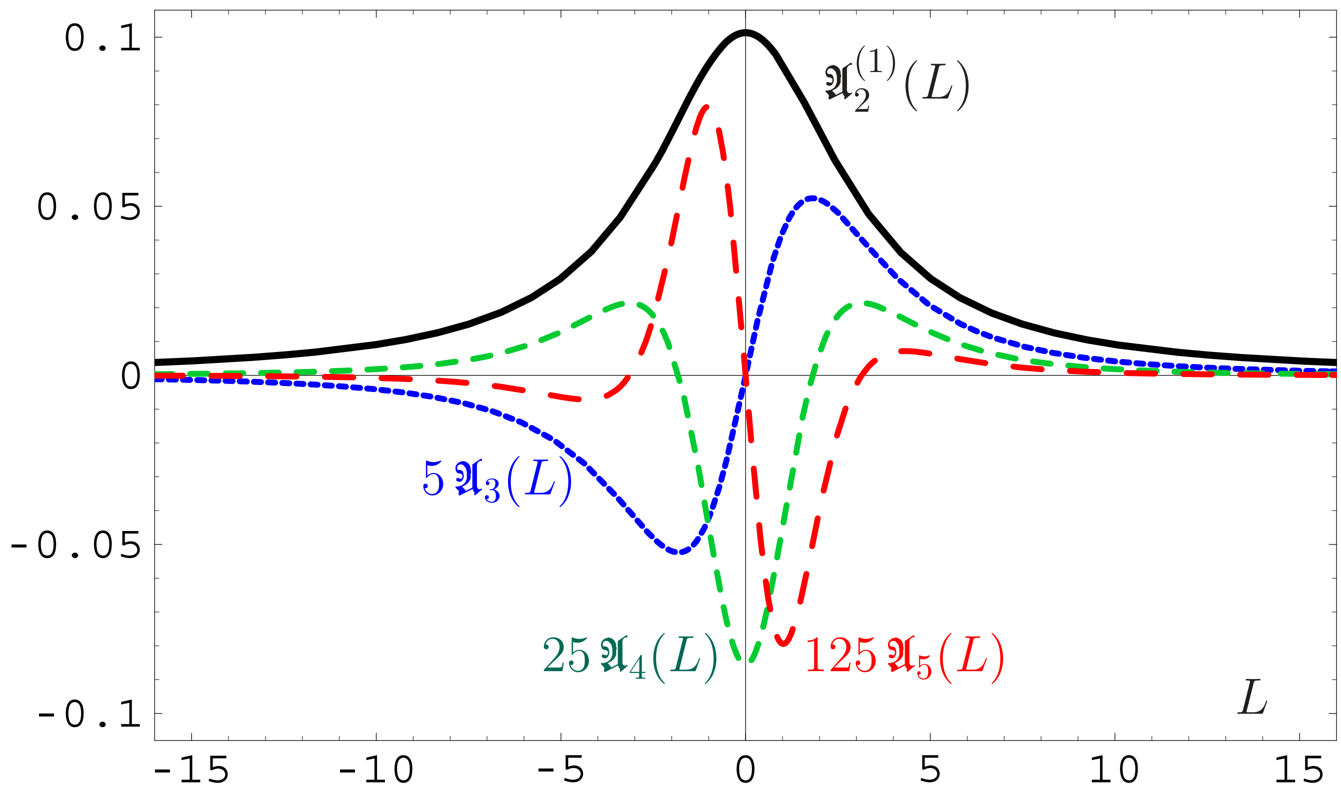

*Comparison of **PT**, **APT**, and **FAPT***

Theory	PT	APT	FAPT
Set	$\left\{a^{\nu}\right\}_{\nu\in\mathbb{R}}$	$\left\{\mathcal{A}_m, \mathfrak{A}_m\right\}_{m\in\mathbb{N}}$	$\left\{\mathcal{A}_\nu, \mathfrak{A}_\nu\right\}_{\nu\in\mathbb{R}}$

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Logarithms	$a^{\nu} L^k$	—	$\mathcal{A}_{\nu-k}$

Development of FAPT:

Higher Loops and Logs

Development of FAPT: Two-loop coupling

Two-loop equation for normalized coupling $a = b_0\,\alpha/(4\pi)$ reads

$$\frac{da_{(2)}}{dL} = -a_{(2)}^2[L]\left[1 + c_1\,a_{(2)}[L]\right] \quad \textbf{with } c_1 \equiv \frac{b_1}{b_0^2}$$

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RG solution of this equation assumes form:

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Expansion of $a_{(2)}[L]$ in terms of $a_{(1)}[L] = 1/L$ with inclusion of terms $O(a^3_{(1)})$:

$$a_{(2)} = a_{(1)} + c_1\, a^2_{(1)} \ln a_{(1)} + c_1^2\, a^3_{(1)} \left(\ln^2 a_{(1)} + \ln a_{(1)} - 1\right) + \ldots$$

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$$\frac{1}{a_{(2)}[L]} + c_1 \ln\left[\frac{a_{(2)}[L]}{1 + c_1 a_{(2)}[L]}\right] = L = \frac{1}{a_{(1)}[L]}$$

Expansion of $a_{(2)}[L]$ in terms of $a_{(1)}[L] = 1/L$ with inclusion of terms $O(a_{(1)}^3)$:

$$a_{(2)} = a_{(1)} + c_1\, a_{(1)}^2 \ln a_{(1)} + c_1^2\, a_{(1)}^3 \left(\ln^2 a_{(1)} + \ln a_{(1)} - 1\right) + \ldots$$

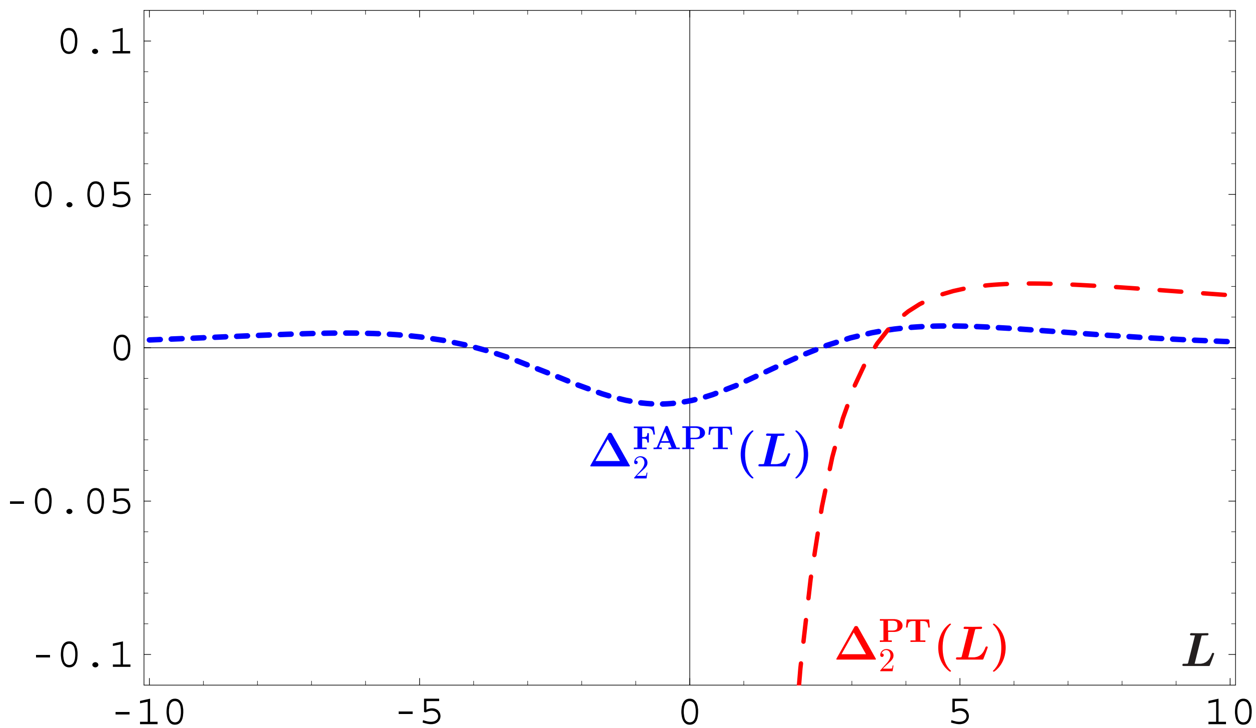
Analytic version of this expansion:

$$\mathcal{A}_1^{(2)}[L] = \mathcal{A}_1^{(1)} + c_1\, \mathcal{D}\, \mathcal{A}_{\nu=2}^{(1)} + c_1^2 \left(\mathcal{D}^2 + \mathcal{D}^1 - 1\right) \mathcal{A}_{\nu=3}^{(1)} + \ldots$$

Development of FAPT: Two-loop coupling

Nice convergence of this expansion for $\mathcal{A}_1^{(2)}[L]$:

$$\Delta_2^{\text{FAPT}}[L] = 1 - \frac{\mathcal{A}_1^{(1)}[L] + c_1\,\mathcal{D}\mathcal{A}_{\nu=2}^{(1)}[L]}{\mathcal{A}_1^{(2)}[L]}$$



Development of FAPT: Two-loop coupling

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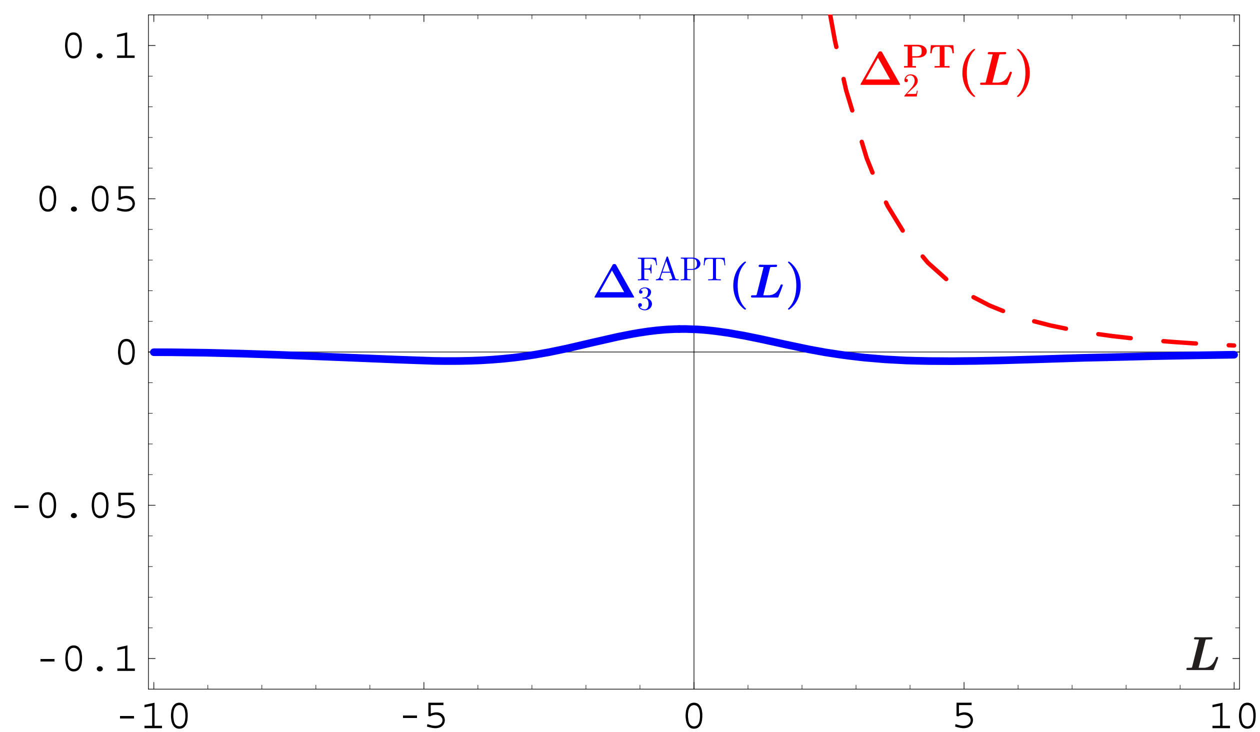
$$\Delta_3^{\text{FAPT}}[L] = \Delta_2^{\text{FAPT}}[L] - \frac{c_1^2\ \left(\mathcal{D}^2 + \mathcal{D}^1 - 1\right)\ \mathcal{A}_{\nu=3}^{(1)}[L]}{\mathcal{A}_1^{(2)}[L]}$$



FAPT: Two-loop coupling with log $\mathcal{L}^{(2)}_{\nu,m}[L]$

Now, coupling with log:

$$\mathcal{L}^{(2)}_{\nu,1}[L] \;=\; \mathbf{A_E}\left[\left(a_{(2)}\right)^{\nu} L\right]\left[\left(a_{(2)}\right)^{\nu} L\right]$$

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Now, coupling with log:

$$\begin{aligned}
\mathcal{L}^{(2)}_{\nu,1}[L] &= \mathbf{A_E}\left[\left(a_{(2)}\right)^{\nu} L\right]\left[\left(a_{(2)}\right)^{\nu} L\right] \\
&= \mathcal{A}^{(2)}_{\nu-1} + c_1\,\mathcal{D}\,\mathcal{A}^{(2)}_{\nu} - c_1^2\,\mathcal{A}^{(2)}_{\nu+1} + \frac{c_1^3}{2}\,\mathcal{A}^{(2)}_{\nu+2} + \ldots
\end{aligned}$$

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Exact spectral density can be easily found:

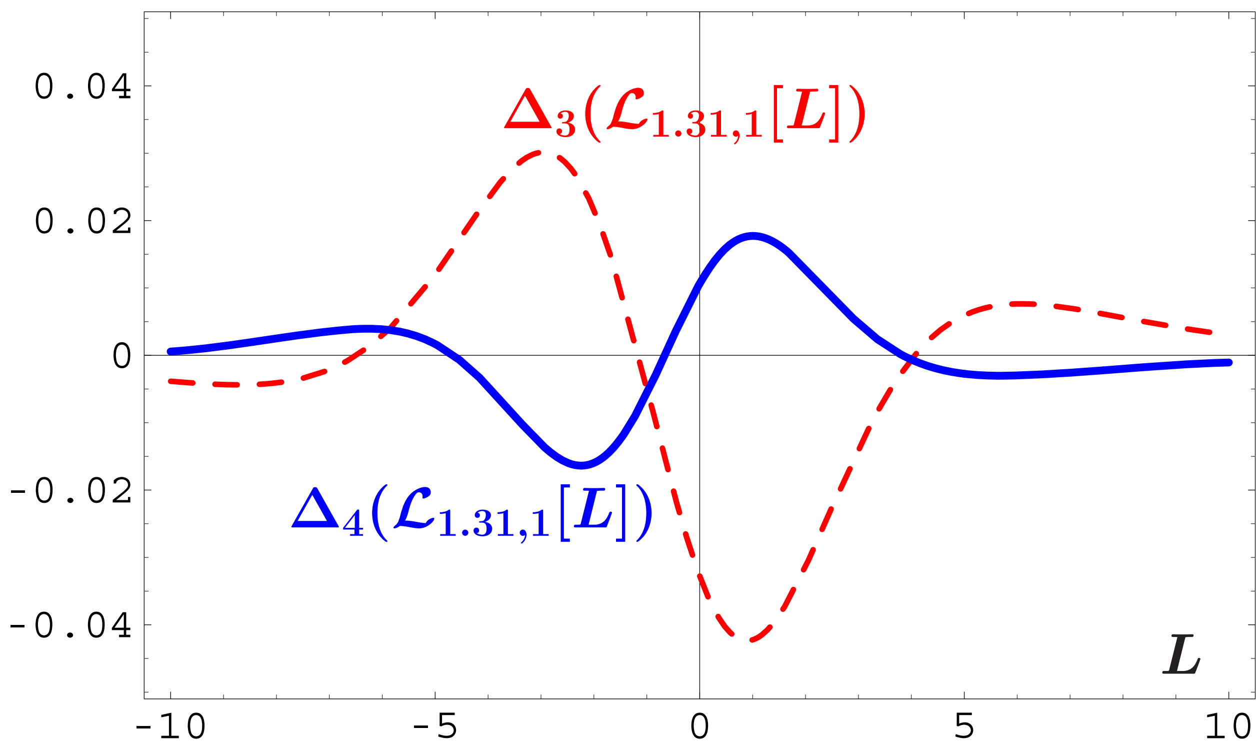
$$\rho^{(2)}_{\mathcal{L}_{\nu,1}}[L] = \frac{R_{(1)}[L]}{R^{\nu}_{(2)}[L]}\,\mathbf{sin}\left[\nu\varphi_{(2)}[L] - \varphi_{(1)}[L]\right]$$

with $R_{(1,2)}[L]$ and $\varphi_{(1,2)}[L]$ being inverse modula and phases of corresponding 1- and 2-loop densities.

FAPT: Two-loop coupling with log $\mathcal{L}^{(2)}_{\nu,m}[L]$

Relative deviations:

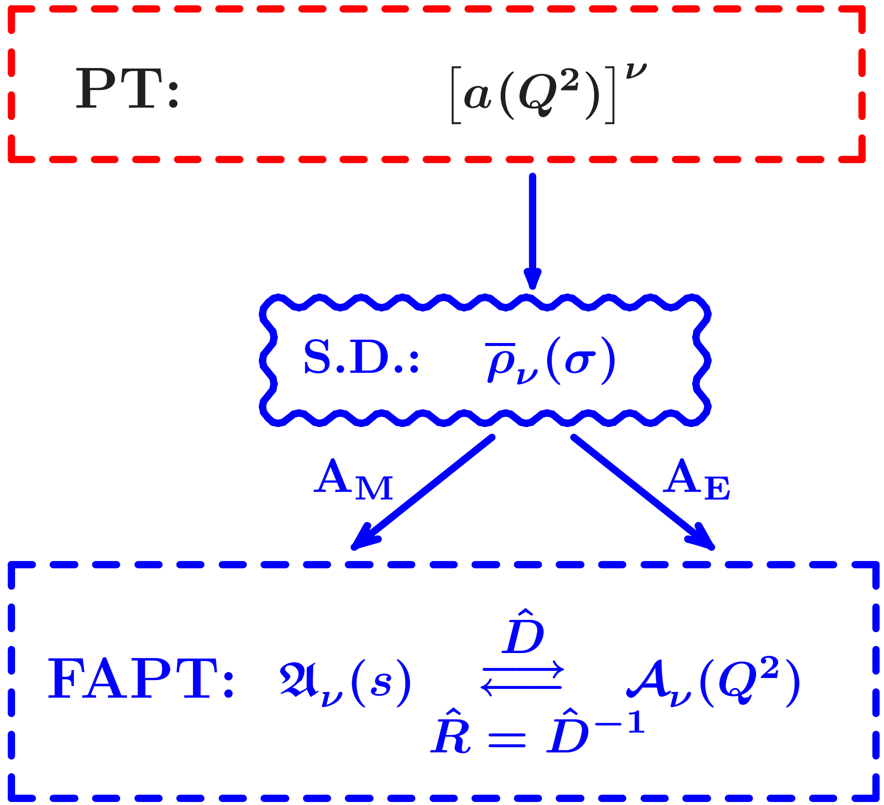
$$\Delta_{3,4}(\mathcal{L}_{1.31,1}) = \frac{\mathcal{L}^{(1)}_{1.31,1} + O(c_1) + O(c_1^2) + O(c_1^3) + O(c_1^4)}{\mathcal{L}^{(2)}_{1.31,1}} - 1$$



Development of FAPT:

Heavy-Quark Thresholds

Conceptual scheme of ***FAPT***



Here N_f is fixed and factorized out.

Conceptual scheme of ***FAPT***

$$
\begin{tikzcd}
& \text{PT:}\quad \left[\alpha_s(Q^2;N_f)\right]^{\nu} \arrow[d] & \\
& \text{S.D.:}\quad \overline{\rho}_{\nu}(\sigma;N_f) \arrow[dl, "\mathbf{A_M}"'] \arrow[dr, "\mathbf{A_E}"] & \\
\text{FAPT:}\ \overline{\mathfrak{A}}_{\nu}(s;N_f) \arrow[rr, shift left, "\hat{D}"] & & \overline{\mathcal{A}}_{\nu}(Q^2;N_f) \arrow[ll, shift left, "\hat{R}=\hat{D}^{-1}"]
\end{tikzcd}
$$

Here N_f is fixed, but not factorized out.

Conceptual scheme of ***FAPT***

PT: $\left[\alpha_s^{\mathbf{glob}}(Q^2)\right]^\nu$

S.D.: $\overline{\rho}_\nu^{\mathbf{glob}}(\sigma)$

$\mathbf{A_M}$ $\mathbf{A_E}$

FAPT: $\mathfrak{A}_\nu^{\mathbf{glob}}(s) \underset{\hat{R}=\hat{D}^{-1}}{\overset{\hat{D}}{\rightleftarrows}} \mathcal{A}_\nu^{\mathbf{glob}}(Q^2)$

Here we see how “analytization” takes into account N_f-dependence.

Global FAPT: Single threshold case

- Consider for simplicity only one threshold at $s = m_c^2$ with transition $N_f = 3 \to N_f = 4$.

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Then:

$$\mathfrak{A}_\nu^{\textbf{glob}}[L] = \theta\left(L < L_4\right)\left[\overline{\mathfrak{A}}_\nu[L;3] - \overline{\mathfrak{A}}_\nu[L_4;3] + \overline{\mathfrak{A}}_\nu[L_4+\lambda_4;4]\right]$$
$$+\,\theta\left(L \geq L_4\right)\overline{\mathfrak{A}}_\nu[L+\lambda_4;4]$$

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and

$$\mathcal{A}_\nu^{\text{glob}}[L] = \overline{\mathcal{A}}_\nu[L+\lambda_4;4] + \int_{-\infty}^{L_4} \frac{\overline{\rho}_\nu\left[L_\sigma;3\right] - \overline{\rho}_\nu\left[L_\sigma+\lambda_4;4\right]}{1+e^{L-L_\sigma}} dL_\sigma$$

Global FAPT: Deviation from fixed N_f-case

In Euclidean domain:

$$\mathcal{A}_{\nu}^{\text{glob}}[L] = \overline{\mathcal{A}}_{\nu}[L + \lambda_4; 4] + \Delta\overline{\mathcal{A}}_{\nu}[L]$$

with:

$$\Delta\overline{\mathcal{A}}_{\nu}[L] = \int_{-\infty}^{L_4} \frac{\overline{\rho}_{\nu}\left[L_{\sigma}; 3\right] - \overline{\rho}_{\nu}\left[L_{\sigma} + \lambda_4; 4\right]}{1 + e^{L - L_{\sigma}}} \, dL_{\sigma}$$

Global FAPT: Deviation from fixed N_f-case

In Minkowskian domain:

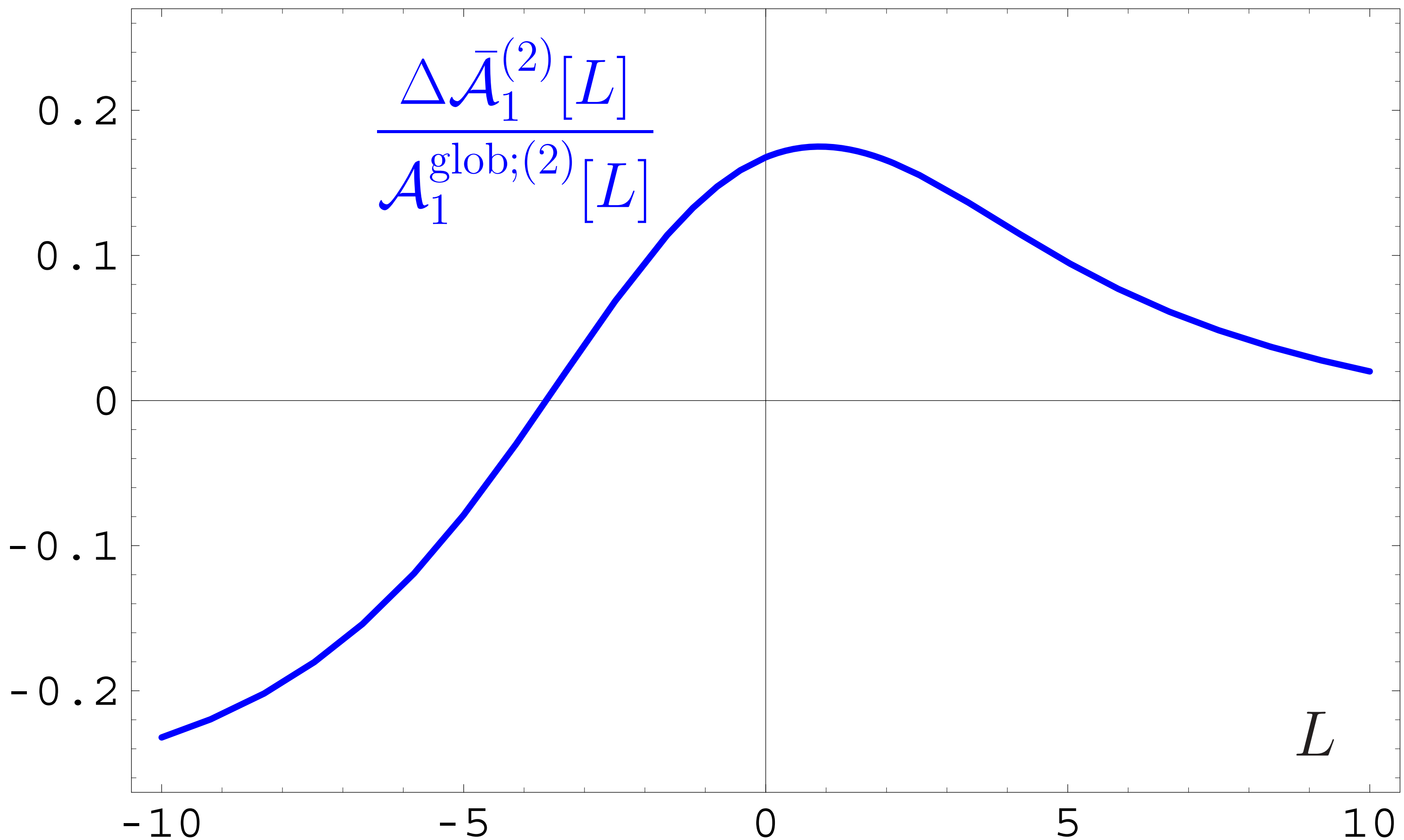
$$\mathfrak{A}_\nu^{\textbf{glob}}[L] = \overline{\mathfrak{A}}_\nu[L+\lambda_4;4] + \Delta\overline{\mathfrak{A}}_\nu[L]$$

with:

$$\Delta\overline{\mathfrak{A}}_\nu[L] = \theta\,(L < L_4)\,\Big[\,\overline{\mathfrak{A}}_\nu[L;3] - \overline{\mathfrak{A}}_\nu[L_4;3]$$
$$+\overline{\mathfrak{A}}_\nu[L_4+\lambda_4;4] - \overline{\mathfrak{A}}_\nu[L+\lambda_4;4]\Big]$$

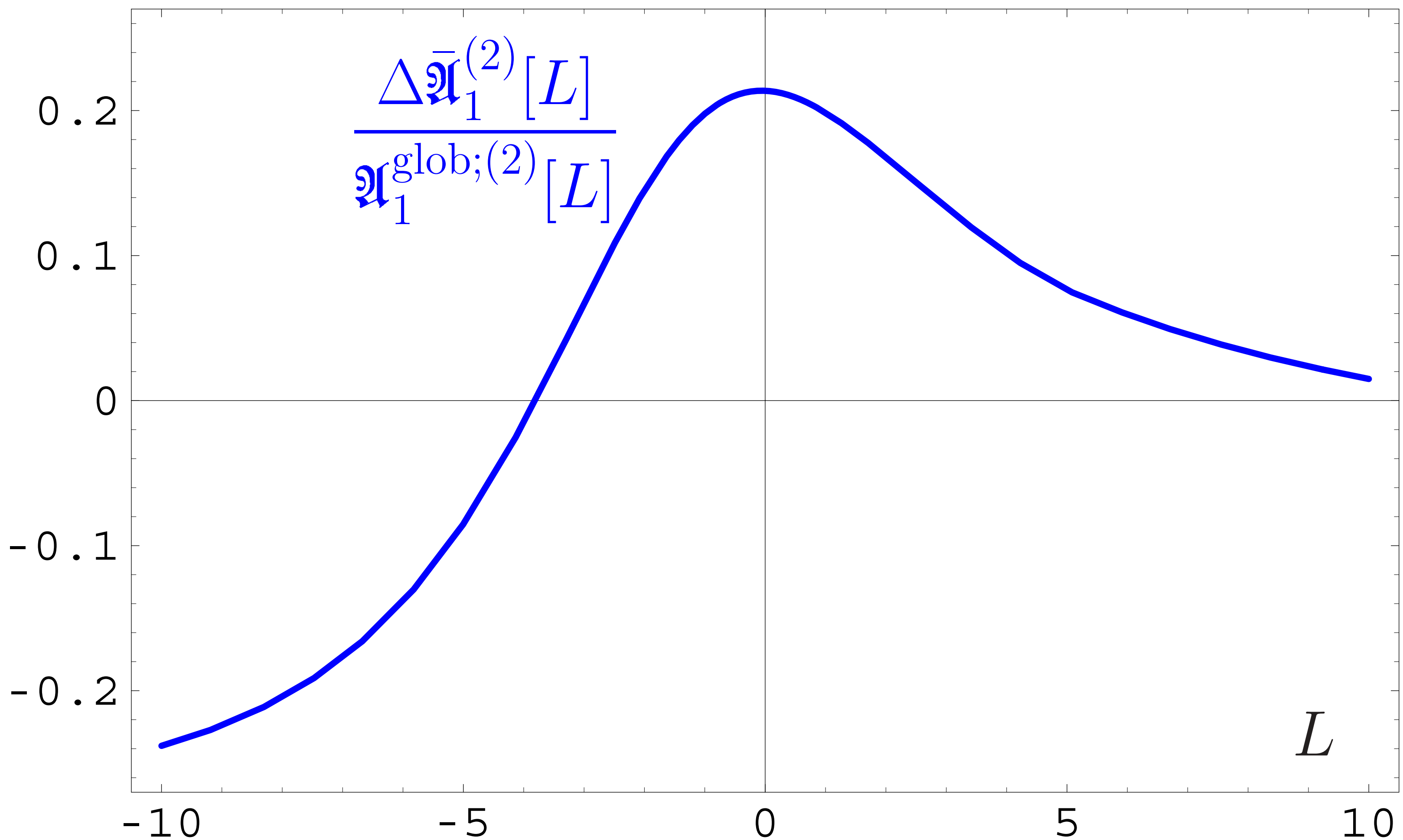
Global FAPT: Deviation from fixed N_f-case

Euclidean deviation: $\mathcal{A}_\nu^{\text{glob}}[L] = \overline{\mathcal{A}}_\nu[L + \lambda_4; 4] + \Delta\overline{\mathcal{A}}_\nu[L]$



Global FAPT: Deviation from fixed N_f-case

Minkowskian deviation: $\mathfrak{A}_{\nu}^{\textbf{glob}}[L] = \overline{\mathfrak{A}}_{\nu}[L+\lambda_4;4] + \Delta\overline{\mathfrak{A}}_{\nu}[L]$



Application:

Pion FF in FAPT

Factorizable part of pion FF at NLO

Scaled hard-scattering amplitude truncated at NLO and evaluated at renormalization scale $\mu_R^2 = \lambda_{\text{R}} Q^2$ reads

$$Q^2 T_{\text{H}}^{\text{NLO}}\left(x, y, Q^2; \mu_F^2, \lambda_{\text{R}} Q^2\right) = \alpha_s\left(\lambda_{\text{R}} Q^2\right) t_{\text{H}}^{(0)}(x, y)$$

$$+ \frac{\alpha_s^2\left(\lambda_{\text{R}} Q^2\right)}{4\pi} C_{\text{F}}\, t_{\text{H},2}^{(1,\text{F})}\left(x, y; \frac{\mu_F^2}{Q^2}\right)$$

$$+ \frac{\alpha_s^2\left(\lambda_{\text{R}} Q^2\right)}{4\pi} \left\{ b_0\, t_{\text{H}}^{(1,\beta)}(x, y; \lambda_{\text{R}}) + t_{\text{H}}^{(\text{FG})}(x, y) \right\}$$

with shorthand notation

$$t_{\text{H},2}^{(1,\text{F})}\left(x, y; \frac{\mu_F^2}{Q^2}\right) = t_{\text{H}}^{(0)}(x, y)\left[2\Big(3 + \ln\left(\bar{x}\,\bar{y}\right)\Big) \ln\frac{Q^2}{\mu_F^2}\right]$$

Pion Distribution Amplitude

Leading twist 2 pion DA at normalization scale μ_{F}^2 is given by

$$\varphi_\pi(x,\mu_{\mathsf{F}}^2) = 6\,x\,(1-x)\left[1 + a_2(\mu_{\mathsf{F}}^2)\,C_2^{3/2}(2x-1) \right.$$
$$\left. + a_4(\mu_{\mathsf{F}}^2)\,C_4^{3/2}(2x-1) + \ldots\right]$$

All nonperturbative information encapsulated in Gegenbauer coefficients $a_n(\mu_0^2)$ enters to $a_n(\mu_{\mathsf{F}}^2)$ through ERBL evolution.
To obtain factorized part of pion FF $\Rightarrow$ convolute pion DA with hard-scattering amplitude:

$$F_\pi^{\mathsf{Fact}}(Q^2) = \varphi_\pi(x,\mu_{\mathsf{F}}^2) \underset{x}{\otimes} T_{\mathsf{H}}^{\mathsf{NLO}}\left(x,y,Q^2;\mu_{\mathsf{F}}^2,\lambda_{\mathsf{R}}Q^2\right) \underset{y}{\otimes} \varphi_\pi(y,\mu_{\mathsf{F}}^2)$$

Analyticity of Pion FF at NLO

Naive "analytization" **[Stefanis, Schroers, Kim – PLB 449 (1999) 299; EPJC 18 (2000) 137]**

$$\left[Q^2 T_{\mathsf{H}}\left(x, y, Q^2; \mu_{\mathsf{F}}^2, \lambda_{\mathsf{R}} Q^2\right)\right]_{\text{Nai-An}} =$$

$$\mathcal{A}_1^{(2)}(\lambda_{\mathsf{R}} Q^2)\, t_{\mathsf{H}}^{(0)}(x, y) + \frac{\left(\mathcal{A}_1^{(2)}(\lambda_{\mathsf{R}} Q^2)\right)^2}{4\pi}\, t_{\mathsf{H}}^{(1)}\left(x, y; \lambda_{\mathsf{R}}, \frac{\mu_{\mathsf{F}}^2}{Q^2}\right)$$

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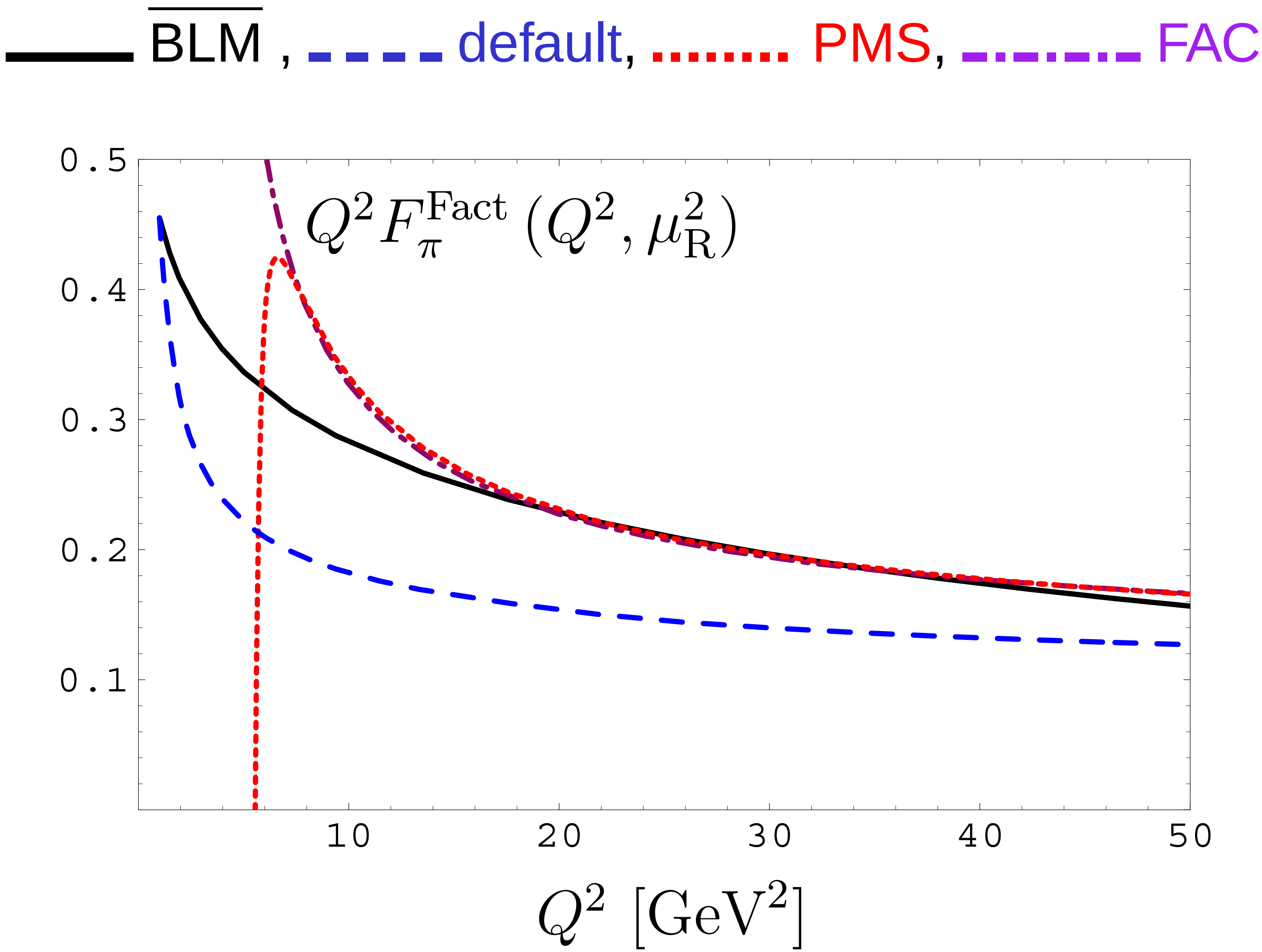
Maximal "analytization" **[A. B., Passek, Schroers, Stefanis – PRD 70 (2004) 033014]**

$$\left[Q^2 T_{\mathsf{H}}\left(x, y, Q^2; \mu_{\mathsf{F}}^2, \lambda_{\mathsf{R}} Q^2\right)\right]_{\textsf{Max-An}} =$$

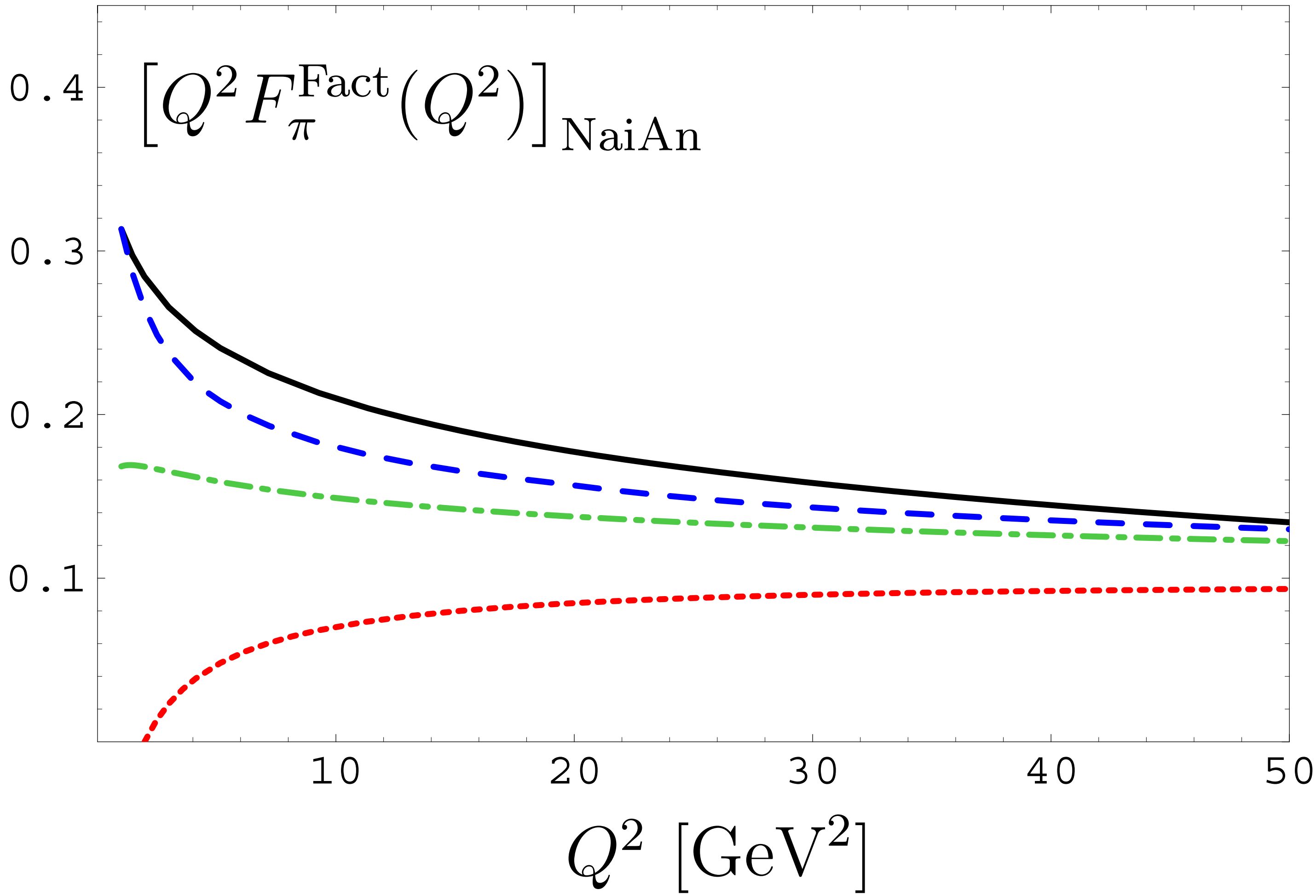
$$\mathcal{A}_1^{(2)}(\lambda_{\mathsf{R}} Q^2)\, t_{\mathsf{H}}^{(0)}(x, y) + \frac{\mathcal{A}_2^{(2)}(\lambda_{\mathsf{R}} Q^2)}{4\pi}\, t_{\mathsf{H}}^{(1)}\left(x, y; \lambda_{\mathsf{R}}, \frac{\mu_{\mathsf{F}}^2}{Q^2}\right)$$

Factorized Pion FF in Standard $\overline{\textbf{MS}}$ scheme

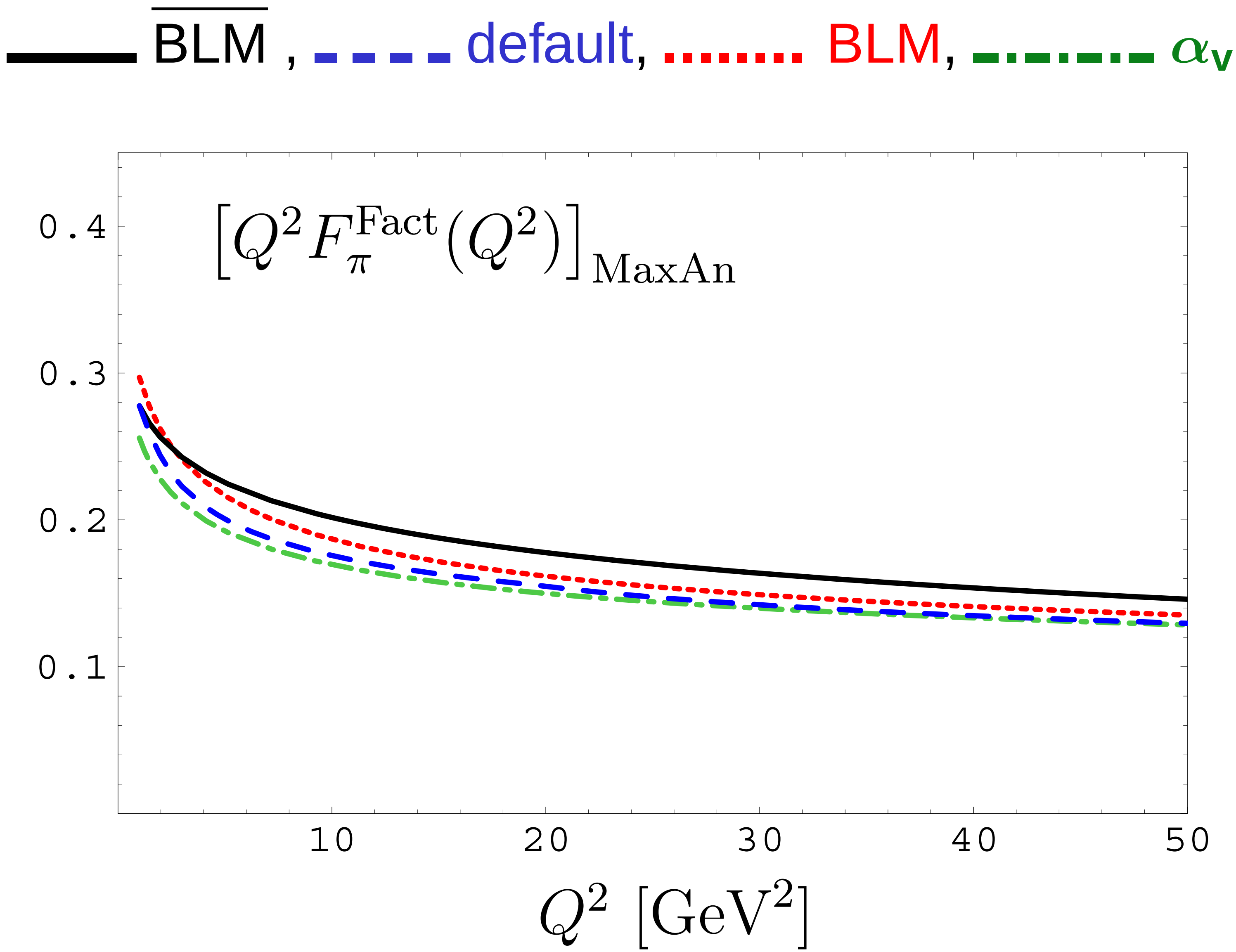


Factorized Pion FF in Naive APT

$\overline{\text{BLM}}$, default, BLM, α_V



Factorized Pion FF in Maximal APT



Factorized Pion FF in FAPT(E)

If we put $\mu_{\mathsf{F}}^2 = Q^2$ — then we obtain in pion FF convolutions with $\varphi_\pi(x, Q^2)$ which contains ERBL evolution factors

$$a_{2n}(Q^2) = a_{2n}(\mu_0^2)\left[\frac{\alpha_s(Q^2)}{\alpha_s(\mu_0^2)}\right]^{\nu_{2n}} \text{ with } \nu_{2n}(N_f) = \frac{\gamma_0(2n)}{2b_0(N_f)}$$

Numerically $\nu_2 = 0.62 - 0.72$ and $\nu_4 = 0.90 - 1.06$.

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In $T_{\text{H}}^{\text{NLO}}(x, y, Q^2)$ we have three types of contributions:

$$\boxed{\alpha_s(Q^2) \quad \alpha_s^2(Q^2) \quad b_0(N_f)\alpha_s^2(Q^2)}$$

Factorized Pion FF in FAPT(E)

Scheme of “analytization” of N_f-dependent quantity:

$$\textbf{PT:} \qquad b_0(N_f)\left[\alpha_s^{\mathbf{glob}}(Q^2)\right]^{\nu(N_f)}$$

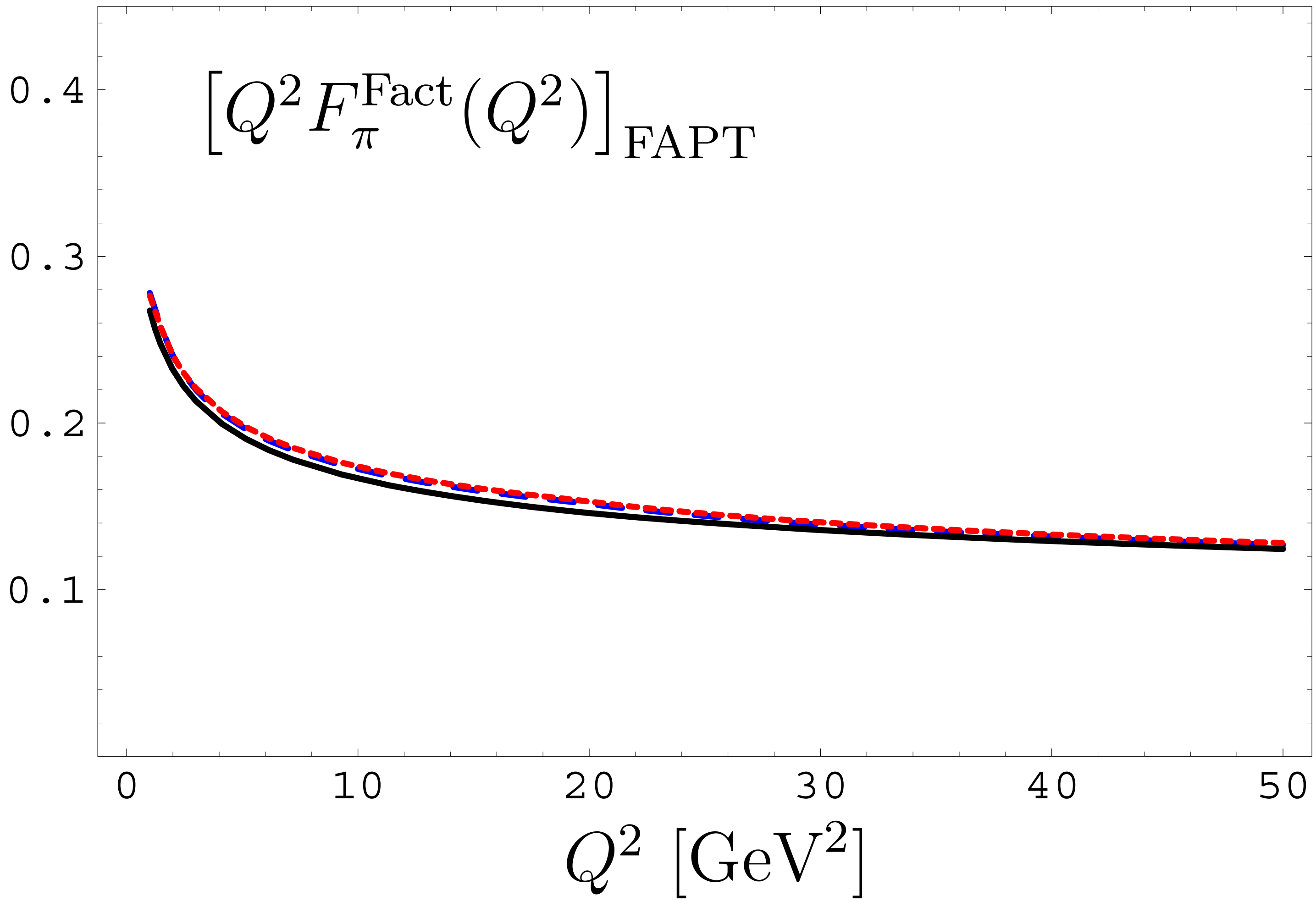
$$\textbf{S.D.:} \qquad \overline{\rho}_{\nu;b_0}^{\mathbf{glob}}(\sigma)$$

$$\mathbf{A_M} \qquad\qquad \mathbf{A_E}$$

$$\textbf{FAPT:} \quad \mathfrak{A}_{\nu;b_0}^{\mathbf{glob}}(s) \underset{\hat{R}=\hat{D}^{-1}}{\overset{\hat{D}}{\rightleftarrows}} \mathcal{A}_{\nu;b_0}^{\mathbf{glob}}(Q^2)$$

Factorized Pion FF in FAPT(E)

APT, FAPT $(\mu_F^2 = Q^2)$, FAPT $(\mu_F^2 \approx 6\,\mathbf{GeV}^2)$



Factorized Pion FF in FAPT(E)

Conclusion: In Euclidean problem taking **thresholds** and N_f**-dependence** of coefficients into account generates tiny correction!

Main advantage: No problem with thresholds!

Pion FF automatically appears to be analytic function out of Minkowski cut. **A. B. [arXiv:0805.0829 (hep-ph)]**

Application:

Higgs decay in FAPT(M)

Higgs boson decay into $b\bar{b}$-pair

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_{\mathsf{S}}(x) = :\bar{b}(x)b(x):$

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_{\mathsf{S}}(x) J_{\mathsf{S}}(0)\,]\, |0\rangle$$

in terms of discontinuity of its imaginary part

$$R_{\mathsf{S}}(s) = \textbf{Im}\,\Pi(-s - i\epsilon)/(2\pi\, s)\,,$$

so that

$$\Gamma(\mathsf{H} \to b\bar{b}) = \frac{G_F}{4\sqrt{2}\pi} M_{\mathsf{H}}\, m_b^2(M_{\mathsf{H}})\, R_{\mathsf{S}}(s = M_{\mathsf{H}}^2)\,.$$

Standard PT analysis of R_{S}

Direct multi-loop calculations are usually performed in the Euclidean region for the corresponding Adler function D_{S}, where QCD perturbation theory works:

$$\widetilde{D}_{\mathsf{S}}(Q^2) = 3\, m_b^2(Q^2) \left[1 + \sum_{n>0} d_n\, \alpha_s^n(Q^2)\right] .$$

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Functions D and R can be related to each other via a dispersion relation without any reference to perturbation theory. This generates relations between r_n and d_n

$$\widetilde{R}_{\mathbf{S}}(s) = 3 m_b^2(s) \left[1 + \sum_{n>0} r_n\, \alpha_s^n(s)\right] .$$

Standard PT analysis of R_S

Coefficients r_n contain 'π^2 terms' due to integral transformation of $\ln^k(Q^2/\mu^2)$ in d_n:

$$\mathfrak{A}_{-2}[L] = L^2 - \frac{\pi^2}{3}, \quad \mathfrak{A}_{-3}[L] = L\left(L^2 - \pi^2\right), \; \ldots$$

Standard PT analysis of R_S

Coefficients r_n contain 'π^2 terms' due to integral transformation of $\ln^k(Q^2/\mu^2)$ in d_n.

Influence of these π^2 terms can be substantial, see **[Baikov, Chetyrkin, and Kühn, PRL 96 (2006) 012003]**

$$\frac{\widetilde{R}_S}{3m_b^2} = 1 + 5.6668\,a_s + 29.147\,a_s^2 + 41.758\,a_s^3 - 825.7\,a_s^4$$

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$$= 1 + 0.2075 + 0.0391 + 0.0020 - 0.00148\,.$$

Here $a_s = \alpha_s(M_\mathsf{H}^2)/\pi = 0.0366$ corresponds to Higgs boson mass $M_\mathsf{H} = 120$ GeV.

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2\left[\alpha_s(Q^2)\right]^{\nu_0}\left[1+\delta_1\,\alpha_s(Q^2)\right]^{\nu_1}$$

with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 8$ **GeV**) and $\nu_0 = 1.04$, $\nu_1 = 1.86$, and $\delta_1 = c_1 b_0/(4\pi)$.

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This gives us

$$\frac{\widetilde{D}_{\mathbf{S}}(Q^2)}{3\,\hat{m}_b^2} = \sum_{n\geq 0} d_n\, \frac{\alpha_s^{n+\nu_0}(Q^2)}{\pi^n} \left[1 + \delta_1\, \alpha_s(Q^2)\right]^{\nu_1}\,.$$

We define analytic images of $\alpha_s^{n+\nu_0}(Q^2) \left[1 + \delta_1\, \alpha_s(Q^2)\right]^{\nu_1}$ in Minkowski region as $\mathfrak{B}_{n+\nu_0}(s)$.

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Following the FAPT(M;5) procedure we obtain

$$\widetilde{R}_{\text{S}}^{(l)\text{FAPT(M;5)}}(s)=3\hat{m}_b^2\left[\mathfrak{B}_{\nu_0}^{(l);\text{glob}}(s)+\sum_{n\geq 1}^{l}d_n(5)\frac{\mathfrak{B}_{n+\nu_0}^{(l);\text{glob}}(s)}{\pi^n}\right]$$

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Following the complete FAPT(M) procedure we obtain

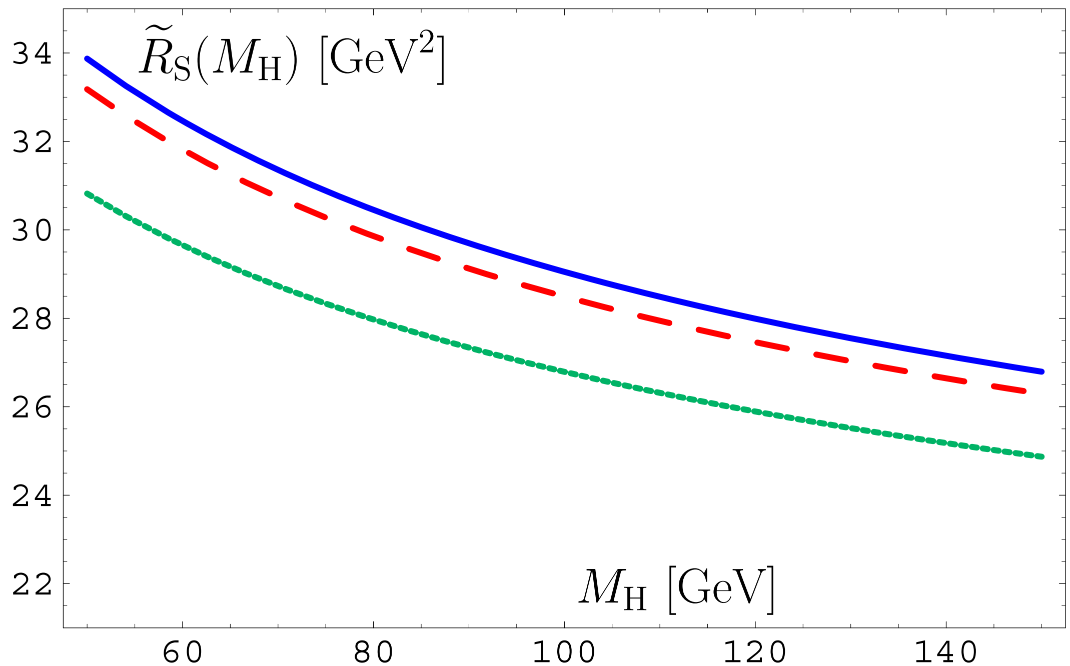
$$\widetilde{R}_{\mathbf{S}}^{(l)\,\textbf{FAPT(M)}}(s) = 3\hat{m}_b^2\left[\mathfrak{B}_{\nu_0}^{(l);\textbf{glob}}(s) + \sum_{n\geq 1}^{l} \frac{\mathfrak{B}_{n+\nu_0;d_n}^{(l);\textbf{glob}}(s)}{\pi^n}\right]$$

with analytic images $\mathfrak{B}_{n+\nu_0;d_n}(s)$ absorbing N_f-dependence of d_n coefficients.

Graphics for R_S in three loops

Illustration of $\widetilde{R}_S(M_H^2)$ calculation in different schemes:

—— = **3L FAPT**(M;5), - - - - = **4L PT**, ... = **1L FAPT**(M)



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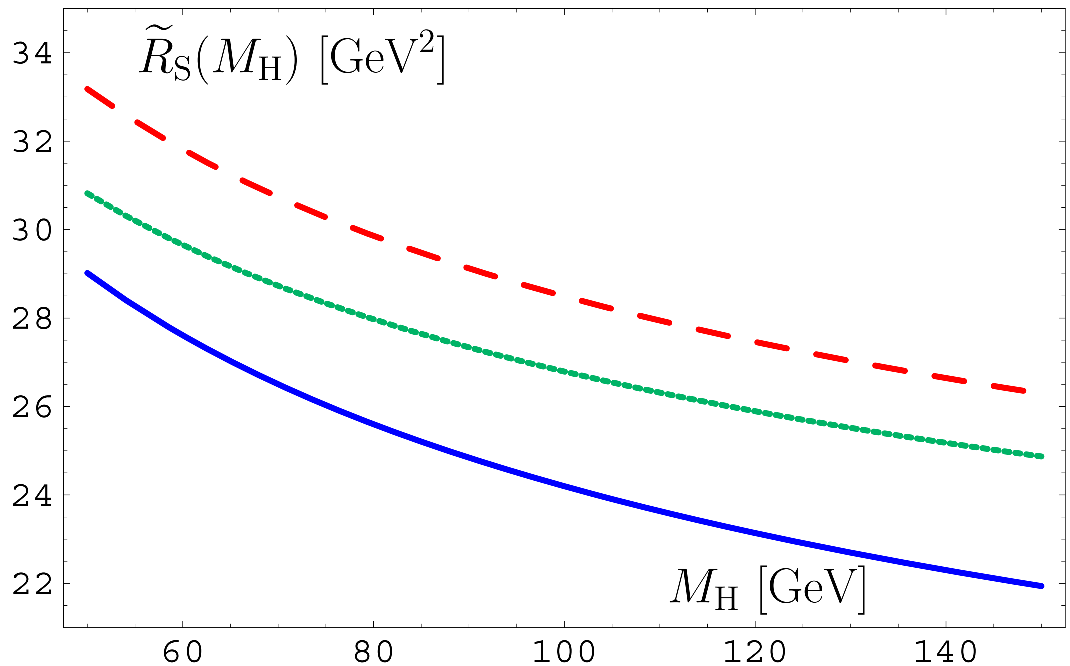
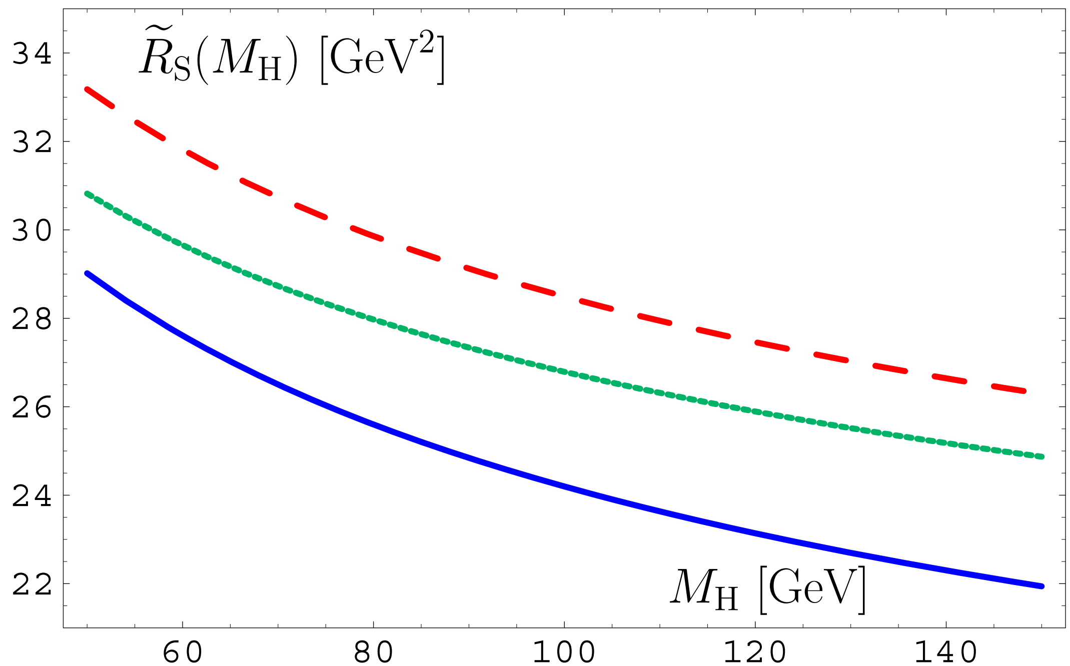


Graphics for R_S in three loops

Illustration of $\widetilde{R}_S(M_H^2)$ calculation in different schemes:

▬▬▬ = **3L FAPT**(M), ▬ ▬ ▬ ▬ = **4L PT**, ... = **1L FAPT**(M)



Conclusion: In Minkowskian problem taking **N_f-dependence** of coefficients into account generates 14% correction!

Reminding about pion FF in FAPT(E)

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What is the reason for this **asymmetry** between **Minkowskian** and **Euclidean** regions ?

Reminding about pion FF in FAPT(E)

> **Conclusion:** In Euclidean problem taking **thresholds** and N_f**-dependence** of coefficients into account generates tiny correction!

What is the reason for this **asymmetry** between **Minkowskian** and **Euclidean** regions ?

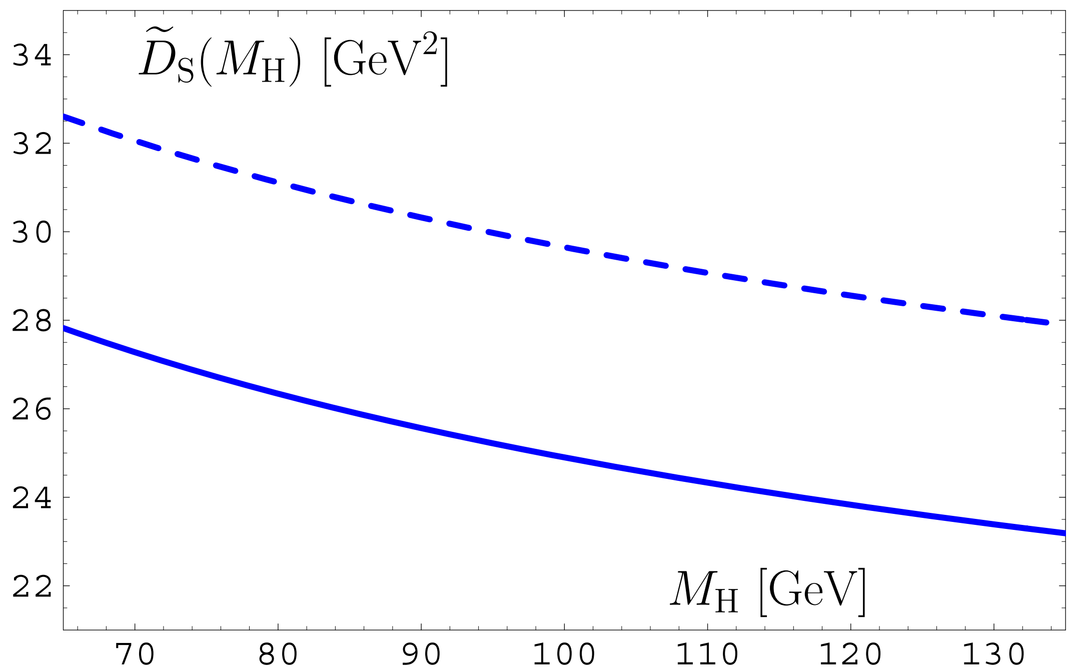
Answer: Large dependence of coefficients on N_f in the problem considered in Minkowski region.

Next slide: **Euclidean** analog of $R_{\mathbf{S}}$ with the same coefficients $d_n(N_f)$ demonstrates reduction of the same order.

Graphics for D_S in three loops

Compare analogous quantities in Euclidean domain:

—— = **3L FAPT**(M), - - - - = **3L FAPT**(M;5)



Conclusion: In Euclidean problem taking N_f**-dependence** of coefficients into account generates 20% correction!

Concluding Remarks

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- Application to decay $H^0 \to b\bar{b}$: Taking into account t-quark effects in virtual loops via **FAPT**(M) **reduces** the result by 14%.

Sergiev Posad photos