Operators of observables for neutrino in dense matter and electromagnetic field

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Abstract

The explicit form of kinetic momentum and spin projection operators for neutral particle with anomalous magnetic moment interacting with dense matter and electromagnetic field is found. Possible applications of obtained results for neutrino physics are discussed.

Introduction

In mathematical apparatus of quantum field theory an elementary particle is usually identified with the irreducible unitary representation of the Poincare group.

The irreducible representations are characterized by values of two invariants of the group:

$$P^2 \equiv P^\mu P_\mu = m^2, \tag{1}$$

$$W^2 \equiv W^\mu W_\mu = -m^2 s(s+1). \tag{2}$$

The translation generators P^μ are identified with the particle momentum and Pauli–Lubansky–Bargmann vector

$$W^\mu = -\frac{1}{2}\, e^{\mu\nu\rho\lambda} M_{\nu\rho} P_\lambda \tag{3}$$

characterizes the particle spin.
The invariant m^2 is the particle mass square and s is the value of the particle spin.

The space of unitary representation is marked out by the condition called "wave equation for a particle possessing the mass m and the spin s".

The wave equation for particles with the spin $s = 1/2$ is the Dirac equation

$$(\hat{p} - m)\,\Psi(x) = 0. \tag{4}$$

In this case the realization of momentum and Pauli–Lubansky–Bargmann vector in the coordinate representation is

$$p^\mu = i\partial^\mu, \quad w^\mu = \frac{i}{2}\gamma^5(\gamma^\mu\hat{\partial} - \partial^\mu), \tag{5}$$

Operators p^μ и w^μ:

- commute with the operator of the Dirac equation
- can be identified with observable physical values (since only integrals of the motion can be considered as observables in relativistic quantum mechanics)

  Landau L D and Peierls R 1931 *Zs. f. Phys.* **69** 56

- have self-conjugate expansion on the solution set of the equation (4) with regard to the standard scalar product

$$(\Psi_f, \Psi_i) = \int d\mathbf{x}\, \Psi_f^\dagger(\mathbf{x}, t)\Psi_i(\mathbf{x}, t). \tag{6}$$

Three dimensional particle spin vector **S** is the set of coefficients of the expansion of the W^μ vector in space-like normals n_i^μ $(i = 1, 2, 3)$, $(n_i P) = 0$, $(n_i n_j) = -\delta_{ij}$:

$$S_i = -\frac{1}{\sqrt{P^2}}\,(W n_i)\,. \tag{7}$$

Obviously,

$$\left[S_i, S_j\right] = i e_{ijk} S_k . \tag{8}$$

The choice of normals is not unique and it is possible to construct spin operators determining the spin projection on any direction in an arbitrary Lorentz frame.

Present description of the particle characteristics can not be directly used in the presence of the external fields.

The Dirac equation for the particle in the external field has the form

$$\left(i\hat{\partial} - e\hat{A} - m\right)\Psi(x) = 0. \tag{9}$$

In this case:

- Operators p^μ and w^μ are not always integrals of motion
- Linear combinations of operators p^μ and w^μ with coefficients depending on coordinates are used for the classification of particle states in the external field

Bagrov V. G., Gitman D. M. Exact solutions of relativistic wave equations. Dordrecht/ Boston/ London: Kluwer Academic Publishers, 1990.

- Generally it is not easy to give physics interpretation for these operators that often leads to logical difficulties

Even greater difficulties arise in the consideration of the Dirac–Pauli equation with the phenomenological term describing the interaction of the anomalous magnetic moment μ_0 with the external field:

$$\left(i\hat{\partial} - e\hat{A} - \frac{i}{2}\mu_0 F^{\alpha\beta}\sigma_{\alpha\beta} - m\right)\Psi(x) = 0, \quad (10)$$

or with the axial-vector term describing the propagation of neutrino in the dense matter consisting of fermions:

$$\left(i\hat{\partial} - \frac{1}{2}\hat{f}(1+\gamma^5) - m\right)\Psi(x) = 0. \quad (11)$$

Wolfenstein L 1978 *Phys. Rev.* D **17** 2369

Statement of problem

- The irreducible representation of group is defined accurate up to the equivalence transformation.
- It is reasonable to state a problem of finding such realization of the Lie algebra of Poincare group for which the condition of the representation irreducibility comes to wave equation describing a particle in the given external field.
- To solve this problem it is necessary to find the operator $U(x, x_0)$ which convert solutions of the wave equation for a free particle

$$\big(D_0(x) - m\big)\Psi_0(x) = 0 \tag{12}$$

to solutions of the equation for the particle in the external field

$$\big(D(x) - m\big)\Psi(x) = 0, \tag{13}$$

wherefrom

$$U(x, x_0)\Psi_0(x) = \Psi(x). \tag{14}$$

Specified operator should satisfy the equation

$$D(x)U(x,x_0) - U(x,x_0)D_0(x) = 0. \tag{15}$$

At that generators

$$\tilde{P}^{\mu} = UP^{\mu}U^{-1}, \quad \tilde{M}^{\mu\nu} = UM^{\mu\nu}U^{-1} \tag{16}$$

commute with operator of the wave equation and can be interpreted as initial momentum and angular momentum tensor respectively.

The Pauli–Lubansky–Bargmann vector and the three dimensional spin vector can be constructed in the same way as in the case of a free particle.

Neutrino in homogeneous electromagnetic field

Let us consider the Dirac–Pauli equation for the neutral particle with the anomalous magnetic moment μ_0 in a stationary homogeneous electromagnetic field:

$$\left(i\hat{\partial} - \frac{i}{2}\mu_0 F^{\mu\nu}\sigma_{\mu\nu} - m\right)\Psi(x) = 0. \tag{17}$$

When the second invariant of the electromagnetic field tensor $F^{\mu\nu}$ is equal to zero

$$I_2 = \frac{1}{4}F^{\mu\nu}H_{\mu\nu} = 0, \qquad H^{\mu\nu} = -\frac{1}{2}e^{\mu\nu\rho\lambda}F_{\rho\lambda},$$

special type of the solutions of this equation was found in the paper

Lobanov A E and Pavlova O S 1999 *Vestn. MGU. Fiz. Astron.* **40** (No 4) 3

These solutions possess the following properties

- They can be presented as a result of action of some integral operator on the solutions of the equation for a free particle.
- If the solutions for the free particle are chosen in the plane waves form then the action of given operator reduces to the product on the matrix function depending on the parameter q^μ.
- This parameter satisfies the condition $q^2 = m^2$ and can be interpreted as a kinetic momentum of the particle in the external field.

The explicit form of the wave functions system is defined by the formula

$$\Psi_{q\zeta_0}(x) = \frac{1}{2}\sum_{\zeta=\pm 1} e^{-i(P_\zeta x)}(1-\zeta\gamma^5\hat{S}_{tp}(q))(1-\zeta_0\gamma^5\hat{S}_0(q))(\hat{q}+m)\psi_0. \tag{18}$$

Here

$$P^\mu_\zeta = q^\mu - \zeta H^{\mu\alpha}H_{\alpha\nu}q^\nu/\sqrt{\mathcal{N}}, \qquad S^\mu_{tp}(q) = -H^{\mu\nu}q_\nu/\sqrt{\mathcal{N}}, \tag{19}$$

where

$$\mathcal{N} = q_\mu H^{\mu\nu}H_{\nu\rho}q^\rho.$$

- The four dimensional vector S_0^μ defines the direction of the particle polarization;
- $\zeta_0 = \pm 1$ is the sign of the spin projection on this direction;
- ψ_0 is the constant bispinor normalized by the condition $\bar{\Psi}_0(x)\Psi_0(x) = m/q_0$.

The system of solutions (18)

- describes spin coherent states of neutrino
- is the complete system of solutions of equation (17) characterized by the particle kinetic momentum q^μ and by the quantum number $\zeta_0 = \pm 1$
- is not stationary in general case.

The solutions are stationary when $S_0^\mu(q) = S_{tp}^\mu(q)$.

Stationary case $S_0^\mu(q) = S_{tp}^\mu(q)$

- Wave functions are the eigenfunctions of the spin projection operator to the $S_{tp}^\mu(q)$ direction with eigenvalues $\zeta = \pm 1$ and of the canonical momentum operator $p^\mu = i\partial^\mu$ with eigenvalues P_ζ^μ
- The orthonormalized system of the stationary solutions of equation (17) can be written down as

$$\Psi_{q\zeta}(x) = e^{-i(P_\zeta x)}\sqrt{|J|}(1 - \zeta\gamma^5\hat{S}_{tp}(q))(\hat{q} + m)\psi_0. \tag{20}$$

  $J = 1 - 2\zeta I_1/\sqrt{\mathcal{N}}$ is the transition Jacobian between the variables q^μ and P_ζ^μ, $I_1 = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ is the first invariant of the $F^{\mu\nu}$ tensor.
- The dispersion law is determined by the relation

$$P_\zeta^2 = m^2 - 2I_1 - 2\zeta\Delta\sqrt{P_\zeta^\mu H_{\mu\alpha}H^{\alpha\nu}P_{\zeta\nu}}, \quad \Delta = \mathrm{sign}(J) \tag{21}$$

- It follows from the dispersion law that group velocities for particles with different spin orientations are equal

$$\mathbf{v}_{gr} = \frac{\partial P_\zeta^0}{\partial \mathbf{P}_\zeta} = \frac{\mathbf{q}}{q^0}. \tag{22}$$

The wave functions (20) are the eigenfunctions of both the canonical p^μ and the kinetic $\mathfrak{Q}^\mu$ momentum operators.

Expressing eigenvalues of the kinetic momentum operator q^μ in terms of eigenvalues of the canonical momentum operator P^μ_ζ and replace them by the p^μ operator itself, we get

The explicit form of the kinetic momentum operator $\mathfrak{Q}^\mu$

$$\mathfrak{Q}^\mu = p^\mu + \gamma^5 \frac{H^{\mu\alpha} H_{\alpha\nu} p^\nu H_{\beta\alpha} p^\alpha \gamma^\beta}{p^\beta H_{\beta\alpha} H^{\alpha\rho} p_\rho}. \tag{23}$$

On the solutions of the Dirac–Pauli equation

$$\hat{\mathfrak{Q}} = m, \quad \mathfrak{Q}^2 = m^2. \tag{24}$$

Spin operators

Natural generalization of Pauli–Lubansky–Bargmann vector to the case of the particle movement in the external field is given by the expression

$$W^{\mu} = \frac{1}{2}\gamma^5(\gamma^{\mu}\hat{\mathfrak{Q}} - \mathfrak{Q}^{\mu}).$$

If we normalize spin operators $\mathfrak{S}$ by the condition $\mathfrak{S}^2 = 1$, then a basis in the spin operators space has the form

$$\mathfrak{S}_i = -\gamma^5 \hat{S}_i(q).$$

For our task it is convenient to choose the basis in the following way:

$$\mathfrak{S}_{tp} = -\gamma^5 \hat{S}_{tp}(q), \quad \mathfrak{S}_{1\perp} = -\gamma^5 \hat{S}_{1\perp}(q), \quad \mathfrak{S}_{2\perp} = -\gamma^5 \hat{S}_{2\perp}(q), \tag{25}$$

where the normal $S_{tp}(q)$ is defined by the equation (19) and

$$S^{\mu}_{1\perp}(q) = \frac{S^{\mu}_0(q) + S^{\mu}_{tp}(q)(S_0(q)S_{tp}(q))}{\sqrt{1-(S_0(q)S_{tp}(q))^2}}, \quad S^{\mu}_{2\perp}(q) = \frac{e^{\mu\nu\rho\lambda}q_{\nu}S_{0\rho}(q)S_{tp\lambda}(q)}{m\sqrt{1-(S_0(q)S_{tp}(q))^2}}.$$

Spin operator $\mathfrak{S}_{tp}$ with eigenfunctions (20) is defined by the formula

$$\mathfrak{S}_{tp} = \frac{\gamma^5 \gamma_\mu H^{\mu\nu} \mathfrak{Q}_\nu}{\sqrt{\mathfrak{Q}^\beta H_{\beta\alpha} H^{\alpha\rho} \mathfrak{Q}_\rho}} = \mathrm{sign}\left(1 + \frac{2 I_1 \gamma^5 H^{\mu\nu} p_\nu \gamma_\mu}{p^\beta H_{\beta\alpha} H^{\alpha\rho} p_\rho}\right) \tilde{\mathfrak{S}}_{tp}, \qquad (26)$$

where

$$\tilde{\mathfrak{S}}_{tp} = \frac{\gamma^5 \gamma_\mu H^{\mu\nu} p_\nu}{\sqrt{p^\beta H_{\beta\alpha} H^{\alpha\rho} p_\rho}}. \qquad (27)$$

Operator $\mathfrak{S}_{tp}$

- is the integral of motion
- characterizes a particle spin projection on the magnetic field direction $\mathbf{H}_0$ in the particle rest frame

Spin operator $\mathfrak{S}_0$ with non stationary eigenfunctions (18) is a linear combination of operators (25) with coefficients depending on coordinates:

$$\mathfrak{S}_0 = -(S_0(q)S_{tp}(q))\mathfrak{S}_{tp} + \\ + [\cos 2\theta\, \mathfrak{S}_{1\perp} - \sin 2\theta\, \mathfrak{S}_{2\perp}] \sqrt{1-(S_0(q)S_{tp}(q))^2}, \tag{28}$$

where $\theta = x_\mu H^{\mu\nu} H_{\nu\rho} q^\rho / \sqrt{\mathcal{N}}$.

The form (28) of operator $\mathfrak{S}_0$ points to:

- The solution (18), which is a linear combination of solutions (20), describes a state of a neutral particle moving with a constant velocity $\mathbf{q}/q^0$ and with a spin precessing around $\mathbf{H}_0$ with a frequency $\omega = 2m|\mathbf{H}_0|/q^0$.
- This state is a pure state.

Note

- The existence of plane wave solutions of the Dirac–Pauli equation

$$\left(i\hat{\partial} - \frac{i}{2}\mu_0 F^{\mu\nu}\sigma_{\mu\nu} - m \right) \Psi(x) = 0,$$

  describing a pure state of a neutral particle with non conserving spin projection on the fixed space axis is possible only by choosing the kinetic momentum components as quantum numbers
- This state is a spin coherent one, so solutions (18) do not form the orthogonal basis
- The considering system is not overfill, since a spin operator spectrum is finite, and can be easy orthogonalized

Application of obtained results to the neutrino description

By investigations of influence of the stationary pure magnetic field on neutrino oscillations in a pioneer paper

- Fujikawa K and Shrock R E 1980 *Phys. Rev. Lett.* **45** 963

as well as in others papers stationary solutions $\Psi_{p\tilde{\zeta}}(x)$ first found in

- Ternov I M, Bagrov V G, Khapaev A M 1965 *Zh. Eksp. Teor. Fiz.* **48** 921

were used as wave functions of a particle.

These solutions are the eigenfunctions of the canonical momentum operator p^{μ} and of the spin operator $\tilde{\mathfrak{S}}_{tp}$.

It was supposed:

- The mean value of neutrino helicity is equal to 1 (in absolute value) at the fixed time moment
- Further spin evolution is described by linear combinations of the referred above wave functions $\Psi(x) = \sum_{\tilde{\zeta}=\pm 1} \alpha_{\tilde{\zeta}}(p)\Psi_{p\tilde{\zeta}}(x)$.

Such description is not correct in the general case

- Since a standard helicity operator $(\mathbf{\Sigma p})/|\mathbf{p}|$ does not commute with the operator from the Dirac–Pauli equation (17)

$$\left(i\hat{\partial} - \frac{i}{2}\mu_0 F^{\mu\nu}\sigma_{\mu\nu} - m\right)\Psi(x) = 0,$$

  so a state of the particle with the fixed canonical momentum and with the helicity mean value equal to 1 (at the fixed time moment) can be only mixed spin state
- But in a mixed state the change of a neutrino beam polarization can be caused only by the distinction of group velocities of its components.
- This effect should disappear on large distances since the beam is no longer coherent.

Our results permit to treat a possible effect of the neutrino polarization change in electromagnetic field as a real precession of the particle spin.

Neutrino in dense matter

Generalization of results for the case of the neutrino interaction with a dense matter, i.e. solutions of the equation

$$\left(i\hat{\partial} - \frac{1}{2}\hat{f}(1+\gamma^5) - \frac{i}{2}\mu_0 F^{\mu\nu}\sigma_{\mu\nu} - m \right)\Psi = 0. \qquad (29)$$

was received in papers

- Lobanov A E 2005 *Phys. Lett.* B **619** 136
- Lobanov A E, Murchikova E M 2008 *Vestn. MGU. Fiz. Astron.* (No 2) 11
- Arbuzova E V, Lobanov A E, Murchikova E M arXiv:0711.2649 [hep-ph]

In this case the expression for the wave function takes the form

$$\Psi(x) = \frac{1}{2} \sum_{\zeta=\pm1} e^{-i(P_\zeta x)} (1 - \zeta\gamma^5 \hat{S}_{tp})(1 - \zeta_0\gamma^5 \hat{S}_0)(\hat{q} + m)\psi_0. \tag{30}$$

Here

$$S^\mu_{tp} = \frac{q^\mu(\varphi q)/m - \varphi^\mu m}{\sqrt{(\varphi q)^2 - \varphi^2 m^2}}, \tag{31}$$

$$P^\mu_\zeta = q^\mu \left(1 + \zeta \frac{(f\varphi)}{2\sqrt{(\varphi q)^2 - m^2\varphi^2}}\right) + \frac{1}{2} f^\mu \left(1 - \frac{\zeta\sqrt{(\varphi q)^2 - m^2\varphi^2}}{(\varphi q)}\right) - \varphi^\mu \frac{\zeta (f\varphi) m^2}{2(\varphi q)\sqrt{(\varphi q)^2 - m^2\varphi^2}}, \tag{32}$$

where

$$\varphi^\mu = f^\mu/2 + H^{\mu\nu} q_\nu/m.$$

Conclusions

- In this way we obtained the exact solutions of the Dirac–Pauli equation for neutrino in dense matter and electromagnetic field.
- We get the explicit form of kinetic momentum and spin projection operators.
- It was demonstrated that if the neutrino production occurs in the presence of an external field and a dense matter, then its spin orientation is characterized by the vector S^{μ}_{tp}.
- Using both the stationary and the nonstationary solutions it is possible to calculate the probabilities of various processes with neutrino in the framework of the Furry picture.