On the $sp(2)$ invariant formulation of quadratic HS Lagrangians

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HS FIELDS

- Metric-like and frame-like massless fields (Fronsdal'78, Vasiliev'80)

$$\varphi_{\mu_1\ldots\mu_s} \equiv \varphi_{\mu(s)} \qquad \delta\varphi_{\mu(s)} = \mathcal{D}_\mu \varepsilon_{\mu(s-1)}$$

$$\Omega^{A(s-1),\,B(s-1)} \qquad \delta\Omega^{A(s-1),\,B(s-1)} = D_0 \varepsilon^{A(s-1),\,B(s-1)}$$

- Partially-massless fields on the $(A)dS_d$ (Deser, Waldron'01, Zinoviev'03, Skvortsov, Vasiliev'06)

$$\varphi_{\mu_1\ldots\mu_s} \equiv \varphi_{\mu(s)} \qquad \delta\varphi_{\mu(s)} = \underbrace{\mathcal{D}_\mu \ldots \mathcal{D}_\mu}_{s-t} \varepsilon_{\mu(t)}$$

$$\Omega^{A(s-1),\,B(t)} \qquad \delta\Omega^{A(s-1),\,B(t)} = D_0 \varepsilon^{A(s-1),\,B(t)}$$

- All symmetric fields

$$\Omega^{A(s-1),\,B(t)} = \begin{cases} t = s-1\,, & \text{massless} \\ t < s-1\,, & \text{partially} - \text{massless} \end{cases}$$

MOTIVATION

The action functional for a set of free fields

$$\mathcal{S}_2[\Omega] = \sum_n \sum_{s,t} \chi_{s,t;n}\, \mathcal{S}_2[\Omega_{s,t;n}]$$

HS interactions: all fields must belong to a HS algebra multiplet!

The $sp(2)$ invariant formulation is a framework

that allows one to consider non-linear actions

(at least in the cubic approximation)

THE $sp(2)$ SYMMETRY

Let Y_α^A be auxiliary commuting variables with $A = 0 \div d$ and $\alpha = 1, 2$. Let us consider polynomials (Howe'89, Vasiliev'03)

$$F(Y) = F_{A_1 \ldots A_{m_1};\, B_1 \ldots B_{m_2}}\, Y_1^{A_1} \ldots Y_1^{A_{m_1}}\, Y_2^{B_1} \ldots Y_2^{B_{m_2}}$$

Operators

$$L_\alpha{}^\beta = Y_\alpha^A \frac{\partial}{\partial Y_\beta^A}$$

form the $gl(2)$ algebra

$$[L_\alpha{}^\beta, L_\gamma{}^\rho] = \delta_\alpha{}^\rho L_\gamma{}^\beta - \delta_\gamma{}^\beta L_\alpha{}^\rho$$

Young symmetry conditions $\Leftrightarrow$ $F(Y)$ belongs to HWR of $gl(2)$

$$L_\alpha{}^\beta F(Y)\Big|_{\alpha<\beta} = 0 \qquad L_\alpha{}^\beta F(Y)\Big|_{\alpha=\beta} = m_\alpha F(Y)$$

In components

$$F^{(A_1 \ldots A_{m_1}; A_{m_1+1}) B_2 \ldots B_{m_2}} = 0$$

Blocks are $sl(2)$ invariants! Let us introduce $sl(2)$ generators

$$\tilde{L}_\alpha{}^\beta = L_\alpha{}^\beta - \frac{1}{2}\delta_\alpha{}^\beta N$$

where $N = L_\gamma{}^\gamma$ is a central element of $gl(2)$.

The $sl(2)$ invariance

$$\tilde{L}_\alpha{}^\beta F(Y) = 0$$

Recall that $sl(2) \sim sp(2)$! Then $sp(2)$ generators

$$T_{\alpha\beta} = \epsilon_{\alpha\gamma}\tilde{L}_\beta{}^\gamma + \epsilon_{\beta\gamma}\tilde{L}_\alpha{}^\gamma \equiv \epsilon_{\alpha\gamma}L_\beta{}^\gamma + \epsilon_{\beta\gamma}L_\alpha{}^\gamma$$

form the $sp(2)$ algebra and the equivalent form of $sl(2)$ invariance condition reads now

$$T_{\alpha\beta}F(Y) \equiv \Big(\epsilon_{\alpha\gamma}L_\beta{}^\gamma + \epsilon_{\beta\gamma}L_\alpha{}^\gamma\Big)F(Y) = 0$$

TRACE DECOMPOSITION

For an arbitrary two-row rectangular traceful tensor there is a two-parametric family of components

$$F^{A(m),\,B(m)} = \bigoplus_{l=0}^{[m/2]} \bigoplus_{k=0}^{[m/2]-l} F^{A(m-2l),\,B(m-2l-2k)}$$

Here the parameter $2l + k = n$ counts a number of removed traces and two-row tensors on the right-hand-side are traceless

$$\eta_{A(2)} F^{A(p),\,B(t)} = 0 \quad \eta_{AB} F^{A(p),\,B(t)} = 0 \quad \eta_{B(2)} F^{A(p),\,B(t)} = 0$$

$sp(2)$ FORM OF TRACE DECOMPOSITION

Let us introduce trace creation and annihilation operators

$$t_{\alpha\beta} = \eta_{AB} Y^A_\alpha Y^B_\beta \quad \text{and} \quad \bar{s}^{\alpha\beta} = \eta^{AB} \frac{\partial^2}{\partial Y^A_\alpha \partial Y^B_\beta}$$

Traceless $o(d-1,2)$ tensors satisfy the constraint

$$\bar{s}^{\alpha\beta} F(Y) = 0$$

Traceful $o(d-1,2)$ tensors are

$$\bar{s}^{\alpha\beta} F(Y) \neq 0$$

The general solution reads

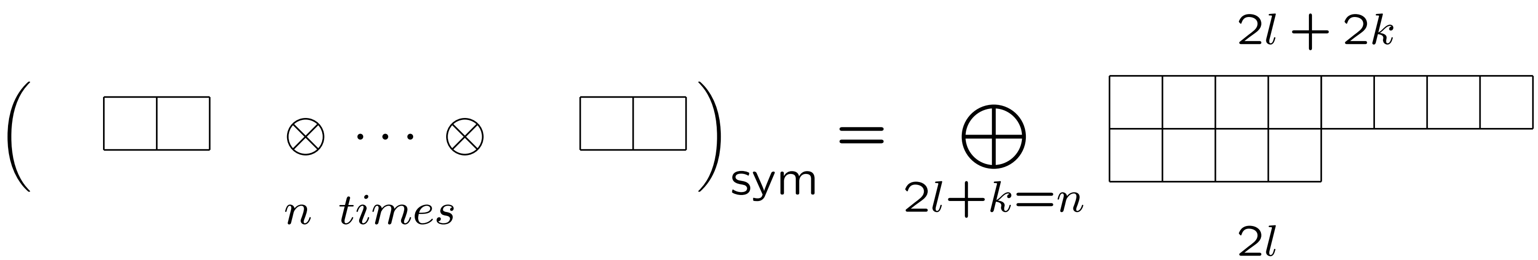
$$F(Y) = F_0(Y) + t_{\alpha\beta} F_1^{\alpha\beta}(Y)$$

where $F_0(Y)$ satisfies the tracelessness condition, while $F_1^{\alpha\beta}(Y)$ is a symmetric $sp(2)$ tensor.

A general decomposition of order $2m$ polynomial $F(Y)$ to traceless parts

$$F(Y) = \sum_{n=0}^{2[m/2]} t_{\alpha_1\beta_1} \cdots t_{\alpha_n\beta_n} F_n^{\alpha_1\beta_1;\,\cdots;\,\alpha_n\beta_n}(Y)\,, \qquad \bar{s}^{\gamma\rho} F_n^{\alpha_1\beta_1;\,\cdots;\,\alpha_n\beta_n}(Y) = 0$$

It is described by



The decomposition can be cast into the form

$$F(Y) = \sum_{l=0}^{[m/2]} \sum_{k=0}^{[m/2]-l} t_{\alpha_1\ldots\alpha_{2k}} Z_+^l F_{2l+k}^{\alpha_1\ldots\alpha_{2k}}(Y)\,, \qquad \bar{s}^{\gamma\rho} F_{2l+k}^{\alpha_1\ldots\alpha_{2k}}(Y) = 0$$

where

$$t_{\alpha_1\ldots\alpha_{2k}} = t_{(\alpha_1\alpha_2} \cdots t_{\alpha_{2k-1}\alpha_{2k})} \quad \text{and} \quad Z_+ = t_{\alpha\beta}t^{\alpha\beta}$$

The $F_{2l+k}^{\alpha(2k)}(Y)$ gives $F^{A(m-2l),\,B(m-2l-2k)}$ with the difference between lengths of first and second rows equal to $2k$.

Let us change notation and introduce

$$F_{p,t}(Y) = t_{\alpha_1 \dots \alpha_{2t}} F_{p,t}^{\alpha_1 \dots \alpha_{2t}}(Y)\,, \qquad \bar{s}^{\gamma\rho} F_{p,t}^{\alpha_1 \dots \alpha_{2t}}(Y) = 0$$

where functions $F_{p,t}^{\alpha(2t)}(Y)$ describe traceless $o(d-1,2)$ tensors $F^{A(p),\,B(p-2t)}$.

Functions $F_{p,t}(Y)$ are $sp(2)$ invariant

$$T_{\alpha\beta}\, F_{p,t}(Y) = 0$$

and subject to a generalized traceless condition

$$(\bar{s}^{\alpha\beta})^{t+1} F_{p,t}(Y) = 0$$

The trace decomposition becomes now manifestly $sp(2)$ invariant

$$F(Y) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m}{2}]-k} Z_+^l\, F_{m-2l,\,k}(Y)$$

The trace decomposition of a single traceful tensor admits a direct generalization to the case when an $sp(2)$ singlet $F(Y)$ is an infinite power series in the auxiliary variables

$$F(Y) = \sum_{m=1}^{\infty} F^{(m)}(Y)$$

where $F^{(m)}$ is a polynomial of $2m-2$ order in variables Y^A_α

$$F^{(m)}(tY) = t^{2m-2} F^{(m)}(Y)$$

By making appropriate field redefinitions and resummations we obtain

$$F(Y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \rho(k,m,n)\, Z_+^n\, F_{k,m;\,n}(Y)$$

It follows that traceless tensors of various symmetry types $F^{A(k),\,B(k-2m)}$ originated from $F(Y)$ are arranged into an infinite sequence enumerated by a degree of the quantity Z_+.

THE ACTION FUNCTIONAL: PRELIMINARIES

The action functional for a HS bosonic AdS_d field
(Lopatin, Vasiliev'88, Vasiliev'01, Skvortsov, Vasiliev'06)

$$\mathcal{S}_2[\Omega] = \int H^{\cdots}(V)\, \epsilon^{\cdots}{}_{M_1 \ldots M_{d-4} N} E_0^{M_1} \wedge \cdots \wedge E_0^{M_{d-4}} V^N \wedge R^{\cdots} \wedge R^{\cdots}$$

- E_0 is the background frame field: it defines the AdS_d geometry
- $R = D_0 \Omega$ is the linearized curvature, $D_0 D_0 = 0$
- $V^A : V^2 = 1$ is the compensator
- The action is manifestly gauge invariant under $\delta\Omega = D_0 \varepsilon$
- The extra field decoupling condition

A useful reformulation with HS fields being functions of X and Y

$$\mathcal{S}_2 = \int \tilde{H}\Big(\frac{\partial}{\partial Y}, \frac{\partial}{\partial X}\Big)\, R(Y)\, R(X)\Big|_{X=Y=0}$$

It is similar to SFT!

TRIPLE SYSTEM OF AUXILIARY VARIABLES

Let us supplement undotted variables by dotted ones and define a set $Y_i^A = (Y_\alpha^A, Y_{\dot\alpha}^B)$, with $\alpha, \dot\alpha = 1, 2$, and $A, B = 0, \ldots, d$. Also, we introduce an additional auxiliary anticommuting variable θ^A that transforms as $o(d-1,2)$ vector.

The following differential operators

$$\bar{s}^{ij} = \eta^{AB} \frac{\partial^2}{\partial Y_i^A \partial Y_j^B} \qquad \bar{v}^i = V^A \frac{\partial}{\partial Y_i^A}$$

and

$$\bar{\eta}^i = \eta^{AB} \frac{\partial^2}{\partial Y_i^A \partial \theta^B} \qquad \chi = V^A \frac{\partial}{\partial \theta^A} \qquad E_0 = E_0^A \frac{\partial}{\partial \theta^A}$$

The combination

$$\Gamma = \frac{1}{(d+1)!}\, \epsilon_{A_1 \cdots A_{d+1}}\, \theta^{A_1} \ldots \theta^{A_{d+1}}$$

is built with the help of the $(d+1)$-dimensional Levi-Civita symbol.

BILINEAR SYMMETRIC FORM

Let us introduce the bilinear form

$$\mathcal{A}(F,G) = \int_{\mathcal{M}^d} \mathcal{H}(\bar{s},\bar{\eta},\bar{v})\,(\wedge E_0)^{d-4}\,\chi\,\Gamma \wedge F(x|\,Y) \wedge G(x|\,\dot{Y})\Big|_{Y=\dot{Y}=\theta=0}$$

The function $\mathcal{H}$ depends on

$$\mathcal{H}(\bar{s},\bar{\eta},\bar{v}) \equiv \mathcal{H}(\bar{s}^{\alpha\beta},\bar{s}^{\dot{\alpha}\dot{\beta}},\bar{s}^{\alpha\dot{\alpha}},\bar{v}^{\alpha},\bar{v}^{\dot{\alpha}},\bar{\eta}^{\alpha},\bar{\eta}^{\dot{\alpha}})$$

We require the bilinear form to be symmetric

$$\mathcal{A}(F,G) = \mathcal{A}(G,F)$$

This property imposes the constraints

$$\bar{s}^{\alpha\beta}\frac{\partial\mathcal{H}}{\partial\bar{s}^{\alpha\beta}} = \bar{s}^{\dot{\alpha}\dot{\beta}}\frac{\partial\mathcal{H}}{\partial\bar{s}^{\dot{\alpha}\dot{\beta}}} \qquad \bar{v}^{\alpha}\frac{\partial\mathcal{H}}{\partial\bar{v}^{\alpha}} = \bar{v}^{\dot{\alpha}}\frac{\partial\mathcal{H}}{\partial\bar{v}^{\dot{\alpha}}} \qquad \bar{\eta}^{\alpha}\frac{\partial\mathcal{H}}{\partial\bar{\eta}^{\alpha}} = \bar{\eta}^{\dot{\alpha}}\frac{\partial\mathcal{H}}{\partial\bar{\eta}^{\dot{\alpha}}}$$

and

$$(\bar{\eta}_{\alpha}\frac{\partial}{\partial\bar{\eta}_{\alpha}} + \bar{\eta}_{\dot{\alpha}}\frac{\partial}{\partial\bar{\eta}_{\dot{\alpha}}})\mathcal{H} = 4\,\mathcal{H}$$

AUXILIARY $sp(2)$ INVARIANT VARIABLES

The variables can be rearranged inside the function $\mathcal{H}$ into

$$\mathrm{c}_1 = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,\bar{s}^{\alpha\dot\alpha}\,\bar{v}^{\beta}\bar{v}^{\dot\beta} \qquad \mathrm{c}_2 = \frac{1}{4}\,\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,\bar{s}^{\alpha\dot\alpha}\,\bar{s}^{\beta\dot\beta}$$

and four more involving the trace annihilation operators $\bar{s}^{\alpha\beta}$ and $\bar{s}^{\dot\alpha\dot\beta}$

$$\mathrm{c}_3 = (\epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\bar{s}^{\alpha\gamma}\bar{v}^{\beta}\bar{v}^{\rho})(\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\rho}\,\bar{s}^{\dot\alpha\dot\gamma}\bar{v}^{\dot\beta}\bar{v}^{\dot\rho})$$
$$\mathrm{c}_4 = \epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\rho}\,\bar{s}^{\alpha\dot\alpha}\bar{s}^{\gamma\dot\gamma}\bar{s}^{\beta\rho}\bar{s}^{\dot\beta\dot\rho}$$
$$\mathrm{c}_5 = \epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\rho}\,\bar{s}^{\alpha\dot\alpha}\bar{s}^{\beta\rho}\bar{s}^{\dot\beta\dot\rho}\bar{v}^{\gamma}\bar{v}^{\dot\gamma}$$
$$\mathrm{c}_6 = (\epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\bar{s}^{\alpha\gamma}\bar{s}^{\beta\rho})(\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\rho}\,\bar{s}^{\dot\alpha\dot\gamma}\bar{s}^{\dot\beta\dot\rho})$$

and

$$\bar\eta = (\epsilon_{\alpha\beta}\bar\eta^{\alpha}\bar\eta^{\beta})(\epsilon_{\dot\alpha\dot\beta}\bar\eta^{\dot\alpha}\bar\eta^{\dot\beta})\,, \qquad \bar\eta\bar\eta = 0$$

The function $\mathcal{H}$ is written now as

$$\mathcal{H} = H(\mathrm{c})\,\bar\eta \qquad H(\mathrm{c}) = \sum_{k_i \geq 0}^{\infty} \xi(k_i\,;d)\,\prod_{i=1}^{6}(\mathrm{c}_i)^{k_i}$$

ACTIONS FOR SYMMETRIC FIELDS: GENERAL ANALYSIS

Let us introduce the $sp(2)$ invariant 1-form gauge field

$$\Omega(x|\,Y) = \mathrm{d}x^{\underline{n}}\,\Omega_{\underline{n}}(x|\,Y) \qquad T_{\alpha\beta}\Omega(x|\,Y) = 0$$

Frame-like higher spin gauge fields are then identified with the expansion coefficients of $\Omega(x|\,Y)$ with respect to the auxiliary variables. An irreducible field (massless or partially massless) of a given spin $s' = s$ and depth $t' = s - 2t$ appears in $\Omega(x|\,Y)$ in infinitely many copies

$$\Omega(Y|\,x) = \sum_{n,\,s,\,t=0}^{\infty} \rho(s,t,n)\,Z_{+}^{n}\,\Omega_{s,\,t;\,n}(Y|\,x)$$

The HS action functional

$$\mathcal{S}_2[\Omega] = \frac{1}{2}\,\mathcal{A}(R,R)$$

The action should satisfy

- **The diagonalization condition**: no cross-terms containing products of fields $\Omega_{s,\,t;\,m}(x)$ and $\Omega_{s,\,t;\,n}(x)$ for $m \neq n$

$$\mathcal{S}_2[\Omega] = \sum_n \sum_{s,\,t} \chi(s,\ t;\ n)\,\mathcal{S}_2[\Omega_{s,\,t;\,n}]$$

- **The extra field decoupling condition**: no more than two derivatives in the action

$$\frac{\delta \mathcal{S}_2[\Omega]}{\delta \Omega^{extra}} \equiv 0$$

A NON-DEGENERATE SET OF FIELDS

Each field in a single copy!

The function $H(\mathsf{c})$ takes the form

$$H = H_1(\mathsf{c}) + \mathsf{c}_5\, H_2(\mathsf{c})$$

and

$$\frac{\partial H_1(\mathsf{c})}{\partial \mathsf{c}_5} = \frac{\partial H_2(\mathsf{c})}{\partial \mathsf{c}_5} = 0$$

$$\frac{\partial H_1(\mathsf{c})}{\partial \mathsf{c}_6} = \frac{\partial H_2(\mathsf{c})}{\partial \mathsf{c}_6} = 0$$

In general case we have four numbers

$$s,\, t \qquad \text{and} \qquad m,\, l$$

For massless fields:

$$H = H(\mathsf{c}_1, \mathsf{c}_2)$$

ACTION FOR MASSLESS FIELDS

For a given spin-s field we have

$$H(\mathsf{c}_1, \mathsf{c}_2) = \frac{1}{4} \int_0^1 \mathrm{d}t \; t^{(d-5)/2} \exp\left(\frac{1-t}{t} \, \mathsf{c}_2 \frac{\partial}{\partial \mathsf{c}_1} \right) (t\mathsf{c}_1)^{s-2}$$

Equivalently,

$$H(\mathsf{c}_1, \mathsf{c}_2) = \frac{1}{(s-1)} \sum_{m=0}^{s-2} \xi(m; d, s) \; \mathsf{c}_1^{s-m-2} \, \mathsf{c}_2^{m}$$

where

$$\xi(m; d, s) = \frac{B(m+1, s-m-1+(d-5)/2)}{B(m+1, s-m-1)}$$

This function reproduces original coefficients found in (Vasiliev'01)

$$\zeta(m; d, s) = \zeta(d, s) \, \frac{(s-m-1)(d-5+2(s-m-2))!!}{(s-m-2)!}$$

CONCLUSIONS AND OUTLOOKS

- Higher rank tensors appear as HS connections of the $*$-product algebra generated by $Y^A_\alpha * Y^B_\beta - Y^B_\beta * Y^A_\alpha = \epsilon_{\alpha\beta}\eta^{AB}$ (Vasiliev'03).

- The bilinear form $\mathcal{A}(F, G)$ defined on arbitrary $sp(2)$ singlet fields $F(Y\,|x)$ and $G(\dot{Y}\,|x)$.

- In summary, our main result is that we have brought together formulations used previously for the HS algebra (Vasiliev'03) and the HS action functionals (Vasiliev'01, Alkalaev, Shaynkman, Vasiliev'05) and provided for them a unified framework.