# Fermionic Green Function in 3D-Ball with chirally invariant boundary conditions

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# Spectral Boundary Conditions in the Bag Model

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## 1. Motivation

QCD: Spontaneous chirality breaking (S$\chi$B) vs. confinement.

Relations & interplay – ?

$\Downarrow$

S$\chi$B in finite volume (bag?).

$\Downarrow$

$\chi$-invariant boundary conditions (BC)

Local BC (MIT-bag, cloudy bag) violate $\chi$.

APS (Atiah, Patodi, Singer) = spectral boundary conditions.

- Global (defined on the entire boundary);
- Avoid topological obstructions;
- Physically acceptable? (thick-wall?).

# CHIRAL INVARIANCE

## MASSLESS DIRAC FERMIONS

$$-i \not{\nabla} \psi = -i \nabla_\mu \gamma_\mu \psi = 0;$$

IN EVEN ($d=2n$) dimensions:

$$\gamma_{2n+1} \propto \gamma_1 \ldots \gamma_{2n}\ ^{*)}; \qquad \{\gamma_{2n+1}, \not{\nabla}\} = 0;$$

## LEFT & Right fermions

$$\gamma_5 \psi_L = \psi_L; \qquad \gamma_5 \psi_R = -\psi_R;$$

## Chiral symmetry breaking

$$m \neq 0; \qquad \langle \bar{\psi} \psi \rangle \neq 0;$$

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*) $\gamma_{2n+1} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$ (WEYL REPRESENTATION

## 2. APS BOUNDARY CONDITIONS

MASSLESS FERMIONS IN EUCLIDEAN 4D (EVEN-DIMENSIONAL) CAVITY.

▲ SPECTRAL PROBLEM $-i\not{\nabla}\Psi_\Lambda = \Lambda \Psi_\Lambda$;

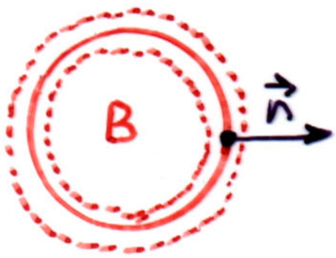
4D-DIRAC OPERATOR

B - CAVITY (4d); $\partial B$ - BOUNDARY. (3d)

▲ APS

$$\boxed{\mathbb{P}\Psi = 0 \Big|_{\partial B} \quad \mathbb{P}\text{- INTEGRAL projector}}$$

- COORDINATES NEAR BOUNDARY $\partial B \otimes R$




$$g_{\mu\nu} = \left(\begin{array}{c|c} 1 & \\ \hline & \eta_{ab} \end{array}\right); \quad dx^2 = dx_n^2 + \eta_{ab}\, dx^a dx^b$$

$\eta_{ab}$ - METRIC ON $\partial B$.

- GAUGE $\quad A_n = A_\mu \cdot n^\mu = 0$;

- $\gamma$-MATRICES $\quad \gamma_n = \begin{pmatrix} 0 & i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix}$; $\gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix}$

(WEYL BASIS). $\quad t \leftrightarrow n$

▲ The DIRAC OPERATOR

$$-i\not{\nabla} = \begin{pmatrix} 0 & \partial_n + i\hat{\nabla} \\ -\partial_n - i\hat{\nabla} & 0 \end{pmatrix};$$

DIRAC OPERATOR ON THE SURFACE $\partial B$

$$\hat{B} = -i\hat{\nabla} = -i\nabla_a\sigma_a = -i(\partial_a - \hat{A}_a)\sigma_a;$$

$\hat{B}$ – BOUNDARY OPERATOR

# „Constructive definition" of APS boundary conditions.

- Spectral problem on $\partial B$

$$-i\hat{\nabla}\,\Omega^{\pm}_{\lambda} = \pm |\lambda|\,\Omega^{\pm}_{\lambda} \,;$$

($d-1$-dimensions, $1/2$ spinor components):

$\Omega^{\pm}_{\lambda}$ - positive(negative) harmonics;

- Fourier expansion near $\partial B$:

$$\Psi_{\lambda}(x) = \begin{pmatrix} f_{\lambda}(x) \\ g_{\lambda}(x) \end{pmatrix} = \begin{pmatrix} \sum_{\lambda} f^{+}_{\lambda}(z)\,\Omega^{+}_{\lambda}(\theta) + f^{-}_{\lambda}(z)\,\Omega^{-}_{\lambda}(\theta) \\ \sum_{\lambda} g^{+}_{\lambda}(z)\,\Omega^{+}_{\lambda}(\theta) + g^{-}_{\lambda}(z)\,\Omega^{-}_{\lambda}(\theta) \end{pmatrix}$$

$\theta$ - coordinates on $\partial B$.

- APS conditions

$$\boxed{f^{+}(x) = g^{-}(x) = 0 \,\Big|_{x\in\partial B}\,.}$$

$$\Rightarrow \begin{cases} \text{positive} \equiv \text{right} \\ \text{negative} \equiv \text{left} \end{cases} \quad \text{on } \partial B.$$

Acceptable:

- particle conservation;
- Hermicity of Lagrangian;

## Physics:

- ON $\partial B$: $f_+(z) = g^-(z) = 0 \,\Big|_{\vec{z} \in \partial B}$

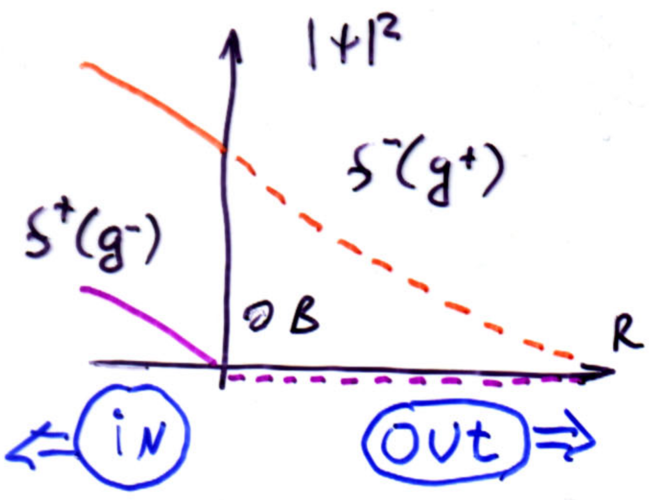
- NEAR $\partial B$:

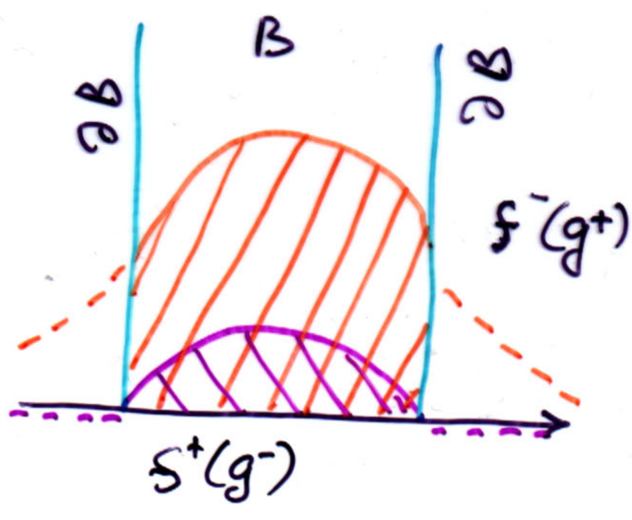
$$\begin{cases} \partial_{\vec{n}}\, g^+ + |\lambda| g^+ = 0\,; \\ -\partial_{\vec{n}}\, f^- - |\lambda| f^- = 0\,; \end{cases}$$

$\vec{n}$ - NORMAL

$$\Rightarrow \quad \frac{\partial_{\vec{n}} f^-_\lambda}{f^-_\lambda}\Big|_{\partial B} = \frac{\partial_{\vec{n}}\, g^+_\lambda}{g^+_\lambda}\Big|_{\partial B} = -\lambda < 0\,;$$

NEAR $\partial B$: $|\psi^2|$ vs. $R$.




RESUMÉ: WAVE FUNCTIONS STAY SQUARE-INTEGRABLE WHEN CONTINUED OUTSIDE OF B.

# INTEGRAL FORM OF APS BOUNDARY CONDITIONS.

$$\boxed{\mathbb{P}\,\psi(\hat{x}) = \oint_{\partial B} \begin{pmatrix} P^+(\hat{x},\hat{y}) & 0 \\ 0 & P^-(\hat{x},\hat{y}) \end{pmatrix} \psi(\hat{y})\, dS_{\hat{y}} = 0 .}$$

$\mathbb{P}(\hat{x},\hat{y})$ - PROJECTOR ONTO APS BOUND. CONDS.

$P^{\pm}(\hat{x},\hat{y})$ - projectors ONTO $(\pm)$-harmonics:

$$P^{\pm}(x,y) = \sum_{\lambda>0} \Omega^{\pm}_{\lambda}(\hat{x})\,[\Omega^{\pm}(\hat{y})]^{\dagger} \;;$$
$$\hat{x},\hat{y} \in \partial B .$$

NB! MANIFEST CHIRAL INVARIANCE:

$$[\mathbb{P}, \gamma_5] = \left[\begin{pmatrix} P^+(x,y) & 0 \\ 0 & P^-(x,y) \end{pmatrix}, \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\right] = 0 .$$

## Explicit FORM: A.A. & A. Wipf (2007)

$$\mathbb{P}(\hat{x},\hat{y}) = \frac{1}{2}\,\delta_{\partial B}(\hat{x}-y) - i\,\gamma_{\vec{n}}\, S(\hat{x}-\hat{y})\Big|_{\hat{x},\hat{y}\in\partial B}$$

$\vec{\gamma}_n = \gamma_i n_i$, $\vec{n}$ - NORMAL to $\partial B$

$S(\hat{x},\hat{y})$ - MASSLES FERMION PROPAGATOR. (FREE!)

$$S(p) = \frac{1}{\not p}$$

# The problem

$$\boxed{\begin{aligned} &-i\not\partial\, S(x,y) = 0; \\ &\mathbb{P} S(x\in\partial B, y) = S\mathbb{P}(x, y\in\partial B) = 0. \end{aligned}}$$

$B$ - 3d Ball; $\partial B = S^2$ - Riemann sphere.

# Solution.

1. The standard way.

- coordinates: spherical;
- $\gamma$-matrices: W-representation;

$$\gamma_W = V_W^+ \gamma_{\text{cartesian}} V; \quad V = \begin{pmatrix} 1 & 0 \\ 0 & i\vec{\sigma}\vec{n} \end{pmatrix}.$$

- Boundary harmonics: spherical spinors;

$$e^+_{l+1} = \Omega_{l+\frac{1}{2},l,k} = \begin{pmatrix} \sqrt{\frac{j+k}{2j}}\, Y_{l,k-\frac{1}{2}} \\ \sqrt{\frac{j-k}{2j}}\, Y_{l,k+\frac{1}{2}} \end{pmatrix};$$

- $\lambda = l+1$

$$e^-_{l} = \Omega_{l-\frac{1}{2},l,k} = \begin{pmatrix} -\sqrt{\frac{j-k+1}{2j+2}}\, Y_{l,k-\frac{1}{2}} \\ \sqrt{\frac{j+k+1}{2j+2}}\, Y_{l,k+\frac{1}{2}} \end{pmatrix};$$

- $\lambda = -l$;

- RADIAL EQUATIONS: DENOTE

- $$S(\vec{x},\vec{y}) = \sum_{\lambda \neq 0} \begin{pmatrix} 0 & S_\lambda(x,y)\,\Omega_\lambda(\hat{x})\,\Omega^\dagger_\lambda(\hat{y}) \\ S^\dagger_\lambda(x,y)\,\Omega_\lambda(\hat{x})\,\Omega^\dagger_\lambda(\hat{y}) & 0 \end{pmatrix};$$

- $\hat{x} = \dfrac{\vec{x}}{x}$, $\hat{y} = \vec{y}/y$;

then:

$$-\frac{\partial S_\lambda(x,y)}{\partial x} - \frac{1-\lambda}{x} S_\lambda(x,y) = \frac{1}{x\,y}\,\delta(x-y);$$

$$\frac{\partial S_\lambda(x,y)}{\partial y} + \frac{1+\lambda}{y} S_\lambda(x,y) = \frac{1}{x\,y}\,\delta(x-y).$$

- SUMMATION:

$$S(x,y) = \frac{1}{x\,y} \sum_{\lambda=1}^{\infty} \theta(y-x)\left(\frac{x}{y}\right)^\lambda \Omega_\lambda(\hat{x})\,\Omega^\dagger_\lambda(\hat{y}) - \theta(x-y)\left(\frac{y}{x}\right)^\lambda \Omega_{-\lambda}(\hat{x})\,\Omega^\dagger_{-\lambda}(\hat{y})\,;$$

- RESULT (AFTER BACK ROTATION)

$$S(\vec{x},\vec{y}) = \frac{i\,(\vec{\gamma}\cdot(\vec{x}-\vec{y}))}{4\pi\,|\vec{x}-\vec{y}|^3}.$$

THE FREE PROPAGATOR!

# BOUNDARY CONDITIONS

## (HEURISTIC SOLUTION).

- INTEGRAL BOUNDARY CONDITIONS:

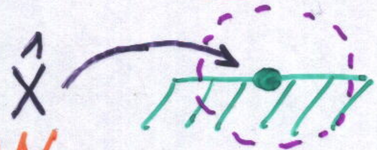
$$\mathbb{P} S(x,y)\Big|_{x\in S^2} = \oint_{S^2} P(\hat{x},\hat{z})\, S(\hat{z},y)\, dS_z = \qquad \hat{x},\hat{z}\in S^2$$

$$= \oint_{S^2}\left[\frac{1}{2}\delta_{S^2}(\hat{x}-\hat{z}) + i S(\hat{x},\hat{z})\gamma_z\right] S(\hat{z},y)\, dS_z =$$

$$= \frac{1}{2} S(\hat{x},y) + \int \underbrace{\left[i\frac{\partial}{\partial z_k} S(\hat{x},z)\right]\gamma_k}_{\delta^3(\hat{x}-z)} S(z,y)\, dV_z +$$

$$+ \int S(\hat{x},z)\underbrace{\left[i \not{\partial}_z S(z,y)\right]}_{-\delta^3(z-y)} dV_z =$$

$$= \frac{1}{2} S(\hat{x},y) + \frac{1}{2}^{*} S(\hat{x},y) - S(\hat{x},y) = 0\,!$$

*) 

BOUNDARY

EXPLANATION

FREE PROPAGATOR SATISFIES

a) DIRAC EQUATION;

b) APS BOUNDARY CONDITIONS.

# SUMMARY

- $$S(x,y) = \frac{i(\cancel{x}-\cancel{y})}{4\pi(\vec{x}-\vec{y})^3};$$

- „MINIMALITY" OF APS BOUNDARY CONDITIONS:
  - DON'T CHANGE THE PROPAGATOR;
  - — " — " — " — DIAGRAM TECHNIQUE;
  - VOLUME ≡ THE ONLY RESTRICTION.

- $R \to \infty$ : the „GEOGRAPHYCAL" LIMIT.