Neutrino dispersion in external magnetic field and plasma

A. V. Kuznetsov[a]*, N. V. Mikheev[a]†
[a] *Yaroslavl State (P.G. Demidov) University*
Sovietskaya 14, 150000 Yaroslavl, Russian Federation

Abstract

Neutrino dispersion properties in an active medium consisting of magnetic field and plasma are analysed. We consider in detail the contribution of a magnetic field into the neutrino self-energy operator $\Sigma(p)$. The results for this contribution were contradictory in the previous literature. For the conditions of the early universe where the background medium consists of a charge-symmetric plasma, the pure magnetic field contribution to the neutrino dispersion relation is proportional to $(eB)^2$ and thus comparable to the contribution of the magnetized plasma. We consider one more hypothetical effect of the active medium influence on neutrino properties, the so-called "neutrino spin light" discussed in the literature. We show that this effect has no physical region of realization because of the medium influence on photon dispersion.

1 Introduction

The most important event in neutrino physics of the last decades was undeniably the solving of the Solar neutrino puzzle, made in the unique experiment on the heavy-water detector at the Sudbury Neutrino Observatory. This experiment, together with the atmospheric and the reactor neutrino experiments, has confirmed the key idea by B. Pontecorvo on neutrino oscillations. The existence of non-zero neutrino mass and lepton mixing is thereby established.

In this connection, an enthusiasm has arisen among theorists with respect to the searches of other possible detectable effects in neutrino physics. However, when people are enthusiastic, they could be not enough self-critical. Sometimes, theoretical discoveries are followed by theoretical closings. In this paper, we have to close two recent discoveries in the field of neutrino dispersion in external active medium.

The first one concerned the external magnetic field influence on the neutrino dispersion relation. In the papers by E. Elizalde et al. [1, 2], contrary to previous results obtained by several authors, a gigantic field contribution into the neutrino energy was found. If the result was correct, and if the previous hunters in the field really ignored "the elephant", it would lead to important consequences for neutrino physics in media.

One more promising effect based on using the neutrino dispersion properties in external active medium, the so-called "spin light of neutrino", was proposed in the series of papers [3–7]. However, the medium influence on the photon dispersion was not considered there.

2 Neutrino dispersion in magnetized plasma: a conflict

The presence of matter or electromagnetic fields modifies the dispersion relation of neutrinos only slightly because these particles interact only by the weak force. However, it was recognized

*e-mail: avkuzn@uniyar.ac.ru
†e-mail: mikheev@uniyar.ac.ru

that the feeble matter effect is enough to affect neutrino flavor oscillations in dramatic ways because the neutrino mass differences are very small [8,9], with practical applications in physics and astrophysics whenever neutrino oscillations are important [10].

The presence of external fields will lead to additional modifications of the neutrino dispersion relation. There is a natural scale for the field strength that is required to have a significant impact on quantum processes, i.e. the critical value

$$B_e = m_e^2/e \approx 4.41 \times 10^{13}\ \mathrm{G}\,. \tag{1}$$

Note that we use natural units where $\hbar = c = 1$ and the Lorentz-Heaviside convention where $\alpha = e^2/4\pi \approx 1/137$ so that $e \approx 0.30 > 0$ is the elementary charge, taken to be positive.

There are reasons to expect that fields of such or even larger magnitudes can arise in cataclysmic astrophysical events such as supernova explosions or coalescing neutron stars, situations where a gigantic neutrino outflow should also be expected. There are two classes of stars, i.e. soft gamma-ray repeaters (SGR) and anomalous x-ray pulsars (AXP) that are believed to be remnants of such cataclysms and to be magnetars, neutron stars with magnetic fields 10^{14}–10^{15} G. The possible existence of even larger fields of order 10^{16}–10^{17} G is subject to debate, see e.g. [11] and the references cited therein. The early universe between the QCD phase transition ($\sim 10^{-5}$ s) and the nucleosynthesis epoch ($\sim 10^{-2}$–10^{+2} s) is believed to be yet another natural environment where strong magnetic fields and large neutrino densities could exist simultaneously [12].

The modification of the neutrino dispersion relation in a magnetized astrophysical plasma was studied in the previous literature [13–16]. In particular, a charge-symmetric plasma with $m_e \ll T \ll m_W$ and $B \lesssim T^2$ was considered for the early-universe epoch between the QCD phase transition and big-bang nucleosynthesis. Ignoring the neutrino mass, the dispersion relation for the electron flavor was found to be [15, 16]

$$\begin{aligned}\frac{E}{|\mathbf{p}|} = 1 \;\; &+ \;\; \frac{\sqrt{2}\,G_\mathrm{F}}{3}\left[-\frac{7\pi^2T^4}{15}\left(\frac{1}{m_Z^2}+\frac{2}{m_W^2}\right)+\frac{T^2eB}{m_W^2}\,\cos\phi+\right.\\ &+ \;\; \left.\frac{(eB)^2}{2\pi^2m_W^2}\,\ln\left(\frac{T^2}{m_e^2}\right)\sin^2\phi\right],\end{aligned} \tag{2}$$

where $\mathbf{p}$ is the neutrino momentum and ϕ is the angle between $\mathbf{B}$ and $\mathbf{p}$. The first term proportional to G_F in Eq. (2) is the dominating pure plasma contribution [17], whereas the second term is caused by the common influence of the plasma and magnetic field [15]. The third term is of the second order in $(eB/T^2) \ll 1$ but was included because of the large logarithmic factor $\ln(T/m_e) \gg 1$ [16]. The dispersion relation of Eq. (2) applies to both ν_e and $\bar{\nu}_e$ without sign change in any of the terms.

The B-field induced pure vacuum modification of the neutrino dispersion relation was assumed to be negligible in these papers.

However, this contribution was calculated for the same conditions in Refs. [1, 2] with an absolutely different result:

$$\frac{\Delta E}{|\mathbf{p}|} = \sqrt{2}\,G_\mathrm{F}\,\frac{eB}{8\pi^2}\,\sin^2\phi\,\mathrm{e}^{-p_\perp^2/(2eB)}\,, \tag{3}$$

where $p_\perp$ is the momentum component perpendicular to the B-field. It is easy to check that this would be the dominant B-field induced contribution by far and thus would lead to important consequences for neutrino physics in media.

Because of importance of the question whether the B-field contribution into the neutrino dispersion relation was dominating or negligible, an independent calculation of it was strongly urged.

3 The neutrino self-energy operator $\Sigma(p)$

A literature search reveals that calculations of the neutrino dispersion relation in external B-fields have a long history [18–20]. To compare the different results we introduce the neutrino self-energy operator $\Sigma(p)$ that is defined in terms of the invariant amplitude for the neutrino forward scattering on vacuum fluctuations, $\nu \to \nu$, by the relation

$$\mathcal{M}(\nu \to \nu) = -\bar{\nu}(p)\,\Sigma(p)\,\nu(p)\,, \tag{4}$$

where p is the neutrino four-momentum. Note that we use the signature $(+,-,-,-)$ for the four-metric.

Perturbatively, the matrix element of Eq. (4) corresponds to the one-loop $\nu \to \ell + W \to \nu$ transition with the exact propagators taken for ℓ and W in the external B field. The calculation techniques for loop processes in external electromagnetic fields based on exact propagators started from the classical paper by J. Schwinger [21] and was developed by A. Nikishov, V. Ritus, A. Shabad, V. Skobelev et al. For a recent review see e.g. [22].

It is convenient to express the structure of the $\Sigma(p)$ operator in an external magnetic field in terms of the coefficients $\mathcal{A}_L$, $\mathcal{B}_L$, $\mathcal{C}_L$, $\mathcal{A}_R$, etc.

$$\begin{aligned}
\Sigma(p) &= \left[\mathcal{A}_L\,(p\gamma) + \mathcal{B}_L\,e^2\left(p\tilde{F}\tilde{F}\gamma\right) + \mathcal{C}_L\,e\left(p\tilde{F}\gamma\right)\right]\,L \\
&+ \left[\mathcal{A}_R\,(p\gamma) + \mathcal{B}_R\,e^2\left(p\tilde{F}\tilde{F}\gamma\right) + \mathcal{C}_R\,e\left(p\tilde{F}\gamma\right)\right]\,R \\
&+ m_\nu\,\left[\mathcal{K}_1 + \mathrm{i}\,\mathcal{K}_2\,e\,(\gamma F\gamma)\right]\,,
\end{aligned} \tag{5}$$

where F is the external field tensor, and $\tilde{F}$ is its dual, $L = \frac{1}{2}(1-\gamma_5)$, $R = \frac{1}{2}(1+\gamma_5)$.

Here, the coefficients $\mathcal{A}_L$, $\mathcal{A}_R$ and $\mathcal{K}_1$, being ultraviolet divergent, do not have independent meanings, because they do not give contributions into the real neutrino energy in external field at the one-loop level. They are absorbed by the neutrino wave-function and mass renormalization. The coefficients $\mathcal{B}_R$, $\mathcal{C}_R$ are suppressed by the factor $(m_\nu/m_W)^2$. The coefficient $\mathcal{K}_2$ is suppressed by the factor $(m_\ell/m_W)^2$. Thus, the coefficients $\mathcal{B}_L$, $\mathcal{C}_L$ are of the most interest. The collection of the results for the $\mathcal{B}_L$ and $\mathcal{C}_L$ coefficients of the $\Sigma(p)$ operator (5) is presented in our paper [23]. Our results for relatively weak field $eB \ll m_\ell^2 \ll m_W^2$ are

$$\mathcal{B}_L = -\frac{G_\mathrm{F}}{3\sqrt{2}\,\pi^2\,m_W^2}\left(\ln\frac{m_W^2}{m_\ell^2} + \frac{3}{4}\right), \qquad \mathcal{C}_L = \frac{3G_\mathrm{F}}{4\sqrt{2}\,\pi^2}\,. \tag{6}$$

For moderate field $m_\ell^2 \ll eB \ll m_W^2$ we have obtained

$$\mathcal{B}_L = -\frac{G_\mathrm{F}}{3\sqrt{2}\,\pi^2\,m_W^2}\left(\ln\frac{m_W^2}{eB} + 2.54\right), \tag{7}$$

with the same coefficient $\mathcal{C}_L$.

4 Neutrino energy in a magnetic field

Solving the equation for the neutrino dispersion in a magnetic field ($m_\nu \equiv 0$)

$$\det\left|(p\gamma) - \mathcal{B}_L\,e^2(p\tilde{F}\tilde{F}\gamma)\,L - \mathcal{C}_L\,e(p\tilde{F}\gamma)\,L\right| = 0\,, \tag{8}$$

where the leading terms with $\mathcal{B}_L$, $\mathcal{C}_L$ are only included, one obtains for the neutrino energy in the field:

$$\frac{E}{|\mathbf{p}|} = 1 + \left(\mathcal{B}_L + \frac{\mathcal{C}_L^2}{2}\right)(eB)^2 \sin^2\phi\,. \tag{9}$$

It can be seen that the $\mathcal{B}_L$ coefficient gives the main contribution into the neutrino energy, because the value $\mathcal{C}_L^2/\mathcal{B}_L \sim G_{\rm F} m_W^2$ appears to be of the order of the fine-structure constant $\alpha \simeq 1/137$, thus leading us beyond the frame of the one-loop approximation.

Our results strongly disagree with those by E. Elizalde et al. [1, 2]. We think that the disagreement arises because these authors use only one lowest Landau level in the charged-lepton propagator in the case of moderate field strengths which they call "strong fields." However, the contributions of the next Landau levels appear to be of the same order as the ground-level contribution [23] because in the integration over the virtual lepton four-momentum in the loop the region $q^2 \sim m_W^2 \gg eB$ appears to be essential.

We confirm the assumption [13, 15], that the pure magnetic field contribution into the neutrino energy does not exceed the plasma contribution.

For relatively weak field $eB \ll m_e^2$ we find the pure-field correction to the electron neutrino energy in a magnetic field and plasma, rewriting the last term in Eq. (2) to the form:

$$+\frac{(eB)^2}{2\pi^2 m_W^2} \sin^2\phi \left(\ln\frac{T^2}{m_e^2} - \ln\frac{m_W^2}{m_e^2} - \frac{3}{4}\right). \tag{10}$$

It is seen that the pure magnetic field contribution to the neutrino dispersion is proportional to $(eB)^2$ and thus comparable to the contribution of the magnetized plasma. It is interesting to note that the contributions of plasma and of pure magnetic field into Eq. (10), containing the electron mass singularities $\sim \ln m_e$, exactly cancel each other.

5 No "neutrino spin light" because of photon dispersion in medium

In an astrophysical environment, the main medium influence on neutrino properties is defined by the additional Wolfenstein energy W acquired by a left-handed neutrino [8].

The general expression for this additional energy of a left-handed neutrino with the flavor $i = e, \mu, \tau$ is [17, 24, 25]

$$\begin{aligned} W_i = \sqrt{2}\, G_{\rm F} \Bigg[&\left(\delta_{ie} - \frac{1}{2} + 2\sin^2\theta_{\rm W}\right)\left(N_e - \bar{N}_e\right) \\ &+ \left(\frac{1}{2} - 2\sin^2\theta_{\rm W}\right)\left(N_p - \bar{N}_p\right) - \frac{1}{2}\left(N_n - \bar{N}_n\right) + \frac{1}{2}\sum_{\ell=e,\mu,\tau}\left(N_{\nu_\ell} - \bar{N}_{\nu_\ell}\right)\Bigg], \end{aligned} \tag{11}$$

where the functions $N_e, N_p, N_n, N_{\nu_\ell}$ are the number densities of background electrons, protons, neutrons, and neutrinos, and $\bar{N}_e, \bar{N}_p, \bar{N}_n, \bar{N}_{\nu_\ell}$ are the densities of the antiparticles. To find the additional energy for antineutrinos, one should change the total sign in the right-hand side of Eq. (11).

For a typical astrophysical medium, except for the early Universe and a supernova core, one has $\bar{N}_e \simeq \bar{N}_p \simeq \bar{N}_n \simeq N_{\nu_\ell} \simeq \bar{N}_{\nu_\ell} \simeq 0$, and $N_p \simeq N_e = Y_e\, N_B$, $N_n \simeq (1 - Y_e)\, N_B$, where N_B is the barion density. One obtains

$$W_e = \frac{G_{\rm F}\, N_B}{\sqrt{2}}\,(3\,Y_e - 1)\,, \qquad W_{\mu,\tau} = -\frac{G_{\rm F}\, N_B}{\sqrt{2}}\,(1 - Y_e)\,. \tag{12}$$

As $Y_e < 1$, the additional energy acquired by muon and tau left-handed neutrinos is always negative. At the same time, the additional energy of electron left-handed neutrinos becomes

positive at $Y_e > 1/3$. And vice versa, the additional energy for electron antineutrinos is positive at $Y_e < 1/3$, while it is always positive for the muon and tauon antineutrinos. On the other hand, right-handed neutrinos and their antiparticles, left-handed antineutrinos, being sterile with respect to weak interactions, do not acquire an additional energy.

The additional energy W from Eq. (12) gives an effective mass squared m_L^2 to the left-handed neutrino,

$$m_L^2 = \mathcal{P}^2 = (E + W)^2 - \mathbf{p}^2 \,, \tag{13}$$

where $\mathcal{P}$ is the neutrino four-momentum in medium, while (E, $\mathbf{p}$) would form the neutrino four-momentum in vacuum, $E = \sqrt{\mathbf{p}^2 + m_\nu^2}$.

Given a $\nu\nu\gamma$ interaction, the additional energy of left-handed neutrinos in medium opens new kinematical possibilities for the radiative neutrino transition $\nu \to \nu + \gamma$. It should be self-evident, that the influence of the substance on the photon dispersion must be taken into account, $\omega = |\mathbf{k}|/n$, where $n \neq 1$ is the refractive index.

First, a possibility exists that the medium provides the condition $n > 1$ (the effective photon mass squared is negative, $m_\gamma^2 \equiv q^2 < 0$) which corresponds to the well-known effect [26–28] of "neutrino Cherenkov radiation". In this situation, the neutrino dispersion change under the medium influence is being usually neglected, because the neutrino dispersion is defined by the weak interaction while the photon dispersion is defined by the electromagnetic interaction.

Pure theoretically, one more possibility could be considered when the photon dispersion was absent, and the process of the radiative neutrino transition $\nu \to \nu\gamma$ would be caused by the neutrino dispersion only. As the left-handed neutrino dispersion is only changed, transitions become possible caused by the $\nu\nu\gamma$ interaction with the neutrino chirality change, e.g. due to the neutrino magnetic dipole moment.

Just this situation called the "spin light of neutrino" ($SL\nu$), was first proposed and investigated in detail in an extended series of papers [3–7]. However, in the analysis of this effect the authors overlooked such an important phenomenon as plasma influence on the photon dispersion. As will be shown below, this phenomenon closes the $SL\nu$ effect for all real astrophysical situations.

We have reanalysed the process $\nu_L \to \nu_R\gamma$ taking into account both the neutrino dispersion and the photon dispersion in medium. Having in mind possible astrophysical applications, it is worthwhile to consider the astrophysical plasma as a medium, which transforms the photon into the plasmon, see e.g. Ref. [29] and the papers cited therein.

To perform a kinematical analysis, it is necessary to evaluate the scales of the values of the left-handed neutrino additional energy W and of the photon (plasmon) effective mass squared m_γ^2. One readily obtains from Eq. (12) for the electron neutrino:

$$W \simeq 6 \text{ eV} \left(\frac{N_B}{10^{38}\,\text{cm}^{-3}} \right) (3\,Y_e - 1) \,, \tag{14}$$

where the scale of the barion number density is taken, which is typical e.g. for the interior of a neutron star.

On the other hand, a plasmon acquires in medium an effective mass m_γ which is approximately constant at high energies. For the transversal plasmon, the value m_γ^2 is always positive, and is defined by the so-called plasmon frequency. In the non-relativistic classical plasma (i.e. for the solar interior) one has:

$$m_\gamma \equiv \omega_{\text{pl}} = \sqrt{\frac{4\pi\,\alpha\,N_e}{m_e}} \simeq 4 \times 10^2 \text{ eV} \left(\frac{N_e}{10^{26}\text{cm}^{-3}} \right)^{1/2} . \tag{15}$$

For the ultra-relativistic dense matter one has:

$$m_\gamma = \sqrt{\frac{3}{2}}\,\omega_{\text{pl}} = \left(\frac{2\,\alpha}{\pi} \right)^{1/2} \left(3\,\pi^2\,N_e \right)^{1/3} \simeq 10^7 \text{ eV} \left(\frac{N_e}{10^{37}\,\text{cm}^{-3}} \right)^{1/3} . \tag{16}$$

In the case of hot plasma, when its temperature is the largest physical parameter, the plasmon mass is:

$$m_\gamma = \sqrt{\frac{2\,\pi\,\alpha}{3}}\,T \simeq 1.2 \times 10^7\,\mathrm{eV}\left(\frac{T}{100\,\mathrm{MeV}}\right). \tag{17}$$

One more physical parameter, a great attention was payed to in the $SL\nu$ analysis [3–7], was the neutrino vacuum mass m_ν. As the scale of neutrino vacuum mass could not exceed essentially a few electron-volts, which is much less than typical plasmon mass scales for real astrophysical situations, see Eqs. (15)-(17), it is reasonable to neglect m_ν in our analysis.

Thus, in accordance with (13), a simple condition for the kinematic opening of the process $\nu_L \to \nu_R\gamma$ is:

$$m_L^2 \simeq 2\,E\,W > m_\gamma^2. \tag{18}$$

This means that the process becomes kinematically opened when the neutrino energy exceeds the threshold value,

$$E > E_0 = m_\gamma^2/(2\,W). \tag{19}$$

Let us evaluate these threshold neutrino energies for different astrophysical situations. For the solar interior $N_B \simeq 0.9 \times 10^{26}\,\mathrm{cm}^{-3}$, $Y_e \simeq 0.6$, and the threshold neutrino energy is: $E_0 \simeq 10^{10}\,\mathrm{MeV}$, to be compared with the upper bound ~ 20 MeV for the solar neutrino energies.

For the interior of a neutron star, where $Y_e \ll 1$, the Wolfenstein energy for neutrinos (12) is negative, and the process $\nu_L \to \nu_R\gamma$ is closed. On the other hand, there exists a possibility for opening the antineutrino decay. Taking for the estimation $Y_e \simeq 0.1$, one obtains from (14) and (16) the threshold value: $E_0 \simeq 10^7\,\mathrm{MeV}$, to be compared with the typical energy $\sim$ MeV of neutrinos emitted via the URCA processes.

For the conditions of a supernova core, the additional energy of left-handed electron neutrinos can be obtained from Eq. (11) as follows:

$$W_e = \frac{G_\mathrm{F}\,N_B}{\sqrt{2}}\,(3\,Y_e + Y_{\nu_e} - 1)\,, \tag{20}$$

where Y_{ν_e} describes the fraction of trapped electron neutrinos in the core, $N_{\nu_e} = Y_{\nu_e}\,N_B$. Taking typical parameters of a supernova core, we obtain: $E_0 \simeq 10^7\,\mathrm{MeV}$, to be compared with the averaged energy $\sim 10^2$ MeV of trapped neutrinos.

In the early Universe, when plasma was almost charge symmetric, the Wolfenstein formula (11) giving zero should be changed to a more accurate expression for the additional energy which is identical for both neutrinos and antineutrinos [15, 17]

$$W_i = -\frac{7\sqrt{2}\,\pi^2\,G_\mathrm{F}\,T^4}{45}\left(\frac{1}{m_Z^2} + \frac{2\,\delta_{ie}}{m_W^2}\right)E\,. \tag{21}$$

The minus sign unambiguously shows that in the early Universe, in contrast to the neutron star interior, the decay process is forbidden both for neutrinos and antineutrinos.

Thus, the above analysis shows that the nice effect of the "neutrino spin light", unfortunately, has no place in real astrophysical situations because of the photon dispersion. The sole possibility for the discussed process $\nu_L \to \nu_R\gamma$ to have any significance could be connected only with the situation when an ultra-high energy neutrino threads a star. Obviously it could have only a methodical meaning. The result of a correct calculation of the process width for these purposes will be published elsewhere.

6 Conclusions

- We have calculated the neutrino self-energy operator $\Sigma(p)$ in the presence of a magnetic field B. Our results strongly disagree with those by E. Elizalde et al. [1,2]. We confirm the

assumption by J. C. D'Olivo e.a. [13] and by P. Elmfors e.a. [15], that the pure magnetic field contribution into the neutrino energy does not exceed the plasma contribution.

- We have shown that the effect of "neutrino spin light" [3–7] has no physical region of realization because of the photon dispersion in medium.

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