Practical manual in analytic perturbation theory and Upsilon decay analysis*

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Abstract

Within the ghost-free Analytic Perturbation Theory (APT), devised in the last decade for low energy QCD, simple model approximation is devised for 3-loop analytic couplings and their effective powers, in both the space-like (Euclidean) and time-like (Minkowskian) regions, accurate enough in the large range (1–100 GeV) of current physical interest.

Effectiveness of the new Model is illustrated by the example of $\Upsilon(1S)$ decay where the standard analysis gives $\alpha_s(M_\Upsilon) = 0.170 \pm 0.004$ value that is inconsistent with the bulk of data for α_s. Instead, we obtain $\alpha_s^{Mod}(M_\Upsilon) = 0.185 \pm 0.005$ that corresponds to $\alpha_s^{Mod}(M_Z) = 0.120 \pm 0.002$ that is close to the world average.

The issue of scale uncertainty is also discussed. In the considered case of Υ decay, the scale error dominates over experimental and Model ones.

Introduction

Theoretical expressions for measured quantities of hadron physics contain QCD running coupling α_s . The common formula (eq.(7) in Bethke[1] or eq.(9.5) in Particle Data Group (PDG) review[2]) suffers from the unphysical pole singularity that is an indispensable feature of all renormalization group (RG) sums of ultra-violet(UV) logs obtained from perturbation theory. In QCD, this issue becomes especially troublesome in the few GeV region.

The first model of a ghost-free QED coupling $\alpha_{an}^{QED}(Q^2;\alpha)$ was devised long ago [3] on the basis of the Källen–Lehmann analyticity in the complex Q^2 plane. The price for the absence of ghost is non-analyticity of $\alpha_{an}^{QED}(Q^2;\alpha)$ with respect to coupling constant α at $\alpha = 0$.

The idea to use the Källen–Lehmann imperative to get rid of unphysical singularities (like Landau pole) was transferred to QCD in the mid-90s[4] and named "analyticization". It results in a regular in the low-Q^2 Euclidean domain effective coupling function with a finite IR limit $\alpha_E(Q^2 = 0)$.

On this basis, another ghost-free construction for QCD effective coupling in the Minkowskian domain $\alpha_M(s)$ was introduced [5] via integral transformation. This result turned out to be equivalent to the result of π^2-terms summation derived by similar means[6, 7] in the early 80s (see also, [8]).

Later on, both the constructions were joined [9] by suitable integral transformations into the so-called *Analytic Perturbation Theory* (APT). An essential feature of the APT scheme is that Minkowskian and Euclidean counterparts for powers of usual QCD coupling $(\alpha_s(Q^2))^k$ form nonpower sets $\{\mathfrak{A}_k(s)\}$ and $\{\mathcal{A}_k(Q^2)\}$. For the fresh reviews of APT see [10, 11].

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Meanwhile, ghost free APT expressions for effective couplings in the Euclidean $\alpha_E(Q^2)$ and Minkowskian $\alpha_M(s)$ regions, as well as for their "effective powers" $\mathcal{A}_k(Q^2)$ and $\mathfrak{A}_k(s)$, are presented by simple analytic expressions only in the one-loop case (see eqs.(5)-(8)), which are not accurate enough for practical goals.

Higher-loop APT expressions are more intricate involving a special Lambert function. A few years ago the first three of them $\mathcal{A}_k(Q^2)\,,\,\mathfrak{A}_k(s);\;(k=1,2,3)$, sufficient for most of applications, were tabulated by Magradze and Kourashev [12] in the 3-loop case.

Here, we propose simple model expressions for 3-loop APT couplings $\alpha_E(Q^2)=\mathcal{A}_1$, $\alpha_M(s)=\mathfrak{A}_1$ and for higher expansion functions $\mathcal{A}_k(Q^2)\,,\,\mathfrak{A}_k(s)$ connected by simple iterative relations. The Model depends on one additional parameter and is valid in the region above 1 GeV .

The paper is organized as follows. Sections 1 and 2 contain a brief review of the main ideas, technique and some results of APT. In Section 3, we present our Model. Section 4 contains revised analysis of Υ decay data with α_s value extracted anew by means of APT and our Model. Special attention is paid to scale uncertainty. The last, Section 5, is devoted to the summary of the results.

1 Outline of the Analytic Perturbation Theory

To start this short overview, remind that the cornerstones of APT are *the Q^2 analyticity of coupling functions* and *compatibility with linear integral transformations.*

Here follows compendium of main definitions. The most elegant APT formulation is based on a set of spectral functions $\{\rho_i(\sigma)\}$ defined as

$$\rho_k(z)=\mathrm{Im}([\alpha_s(-z)]^k) \tag{1}$$

The first of them, $\rho_1=\rho(\sigma)$ is just Källen–Lehmann spectral density for the Euclidean APT coupling. Then, higher Euclidean ("analyticized kth power of coupling in the Euclidean domain") and Minkowskian ("effective kth power of coupling in the Minkowskian domain") APT functions will be respectively given by

$$\mathcal{A}_k(Q^2)=\mathbb{A}[\alpha_s^k]=\frac{1}{\pi}\int_0^{+\infty}\frac{\rho_k(\sigma)\,d\sigma}{\sigma+Q^2}\,;\quad \mathfrak{A}_k(s)=\mathbb{R}[\alpha_s]=\frac{1}{\pi}\int_s^{+\infty}\frac{d\sigma}{\sigma}\rho_k(\sigma)\;. \tag{2}$$

These functions are related by integral transformation

$$\mathcal{A}_k(Q^2)=\mathbb{D}[\mathfrak{A}_k]=Q^2\int_0^{+\infty}\frac{\mathfrak{A}_k(s)\,ds}{(s+Q^2)^2}$$

and its reverse. The differential relations connecting higher spectral functions

$$-\frac{1}{k}\frac{d\rho_k}{d\,\ln\sigma}=\beta_0\,\rho_{k+1}+\beta_1\,\rho_{k+2}+\,\ldots\,,\quad \beta_0(n_f)=\frac{33-2n_f}{12\pi}\,,\quad \beta_1=\frac{153-19n_f}{24\pi^2}\,,\,\ldots$$

induce analogous relations for expansion functions

$$\frac{1}{k}\frac{d\mathfrak{A}_k(s)}{d\,\ln s}=-\sum_{n\geq 1}\beta_{n-1}\mathfrak{A}_{k+n}(s)\,,\qquad \frac{1}{k}\frac{d\mathcal{A}_k(Q^2)}{d\,\ln Q^2}=-\sum_{n\geq 1}\beta_{n-1}\mathcal{A}_{k+n}(Q^2), \tag{3}$$

which can be used for iterative definitions. Numerically, these beta-coefficients (defined according to Bethke[1]) and their useful combination $b=\beta_1/\beta_0^2$ are of the order of unity[1]

$$\beta_0(4\mp 1)=0.6631\pm 0.0530\,,\;\beta_1(4\mp 1)=0.3251\pm 0.0802\,,\;B(4\mp 1)=0.7392^{+0.0509}_{-0.0814}\,. \tag{4}$$

[1]There is a misprint in numerical values of $\beta_0(4\mp 1)$ in paper [10].

2 Main Results of APT

One-loop case. In this case, the APT formulae are simple and elegant. Starting with the perturbative RG-improved QCD coupling $\alpha_s^{(1)}(Q^2) = 1/(\beta_0\, l)$, with the help of (1),(2) one arrives at the ghost-free effective Euclidean [2]

$$\mathcal{A}_1^{(1)}(l) = \frac{1}{\beta_0}\left(\frac{1}{l} - \frac{1}{e^l - 1}\right) = \frac{1}{\beta_0 \pi}\left(\frac{1}{l} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right), \quad l = \ln\left(\frac{Q^2}{\Lambda^2}\right) \tag{5}$$

and Minkowskian

$$\mathfrak{A}_1^{(1)}(L) = \frac{1}{\beta_0 \pi} \arccos\left(\frac{L}{\sqrt{L^2 + \pi^2}}\right), \quad L = \ln\left(\frac{s}{\Lambda^2}\right) \tag{6}$$

APT couplings. Here, higher functions $\mathcal{A}_i\,, \mathfrak{A}_i$ can be defined via recursive relations (3) with only one term in the r.h.s. The second and third one-loop functions are

$$\begin{aligned} \mathcal{A}_2^{(1)}(l) &= \frac{1}{\beta_0^2}\left(\frac{1}{l^2} - \frac{e^l}{(e^l - 1)^2}\right), \quad \mathfrak{A}_2^{(1)}(L) = \frac{1}{\beta_0^2}\frac{1}{L^2 + \pi^2}\,;, \\ \mathcal{A}_3^{(1)}(l) &= \frac{1}{\beta_0^3}\left(\frac{1}{l^3} - \frac{1}{2}\frac{e^l + e^{2l}}{(e^l - 1)^3}\right), \quad \mathfrak{A}_3^{(1)}(L) = \frac{1}{\beta_0^3}\frac{L}{(L^2 + \pi^2)^2}\,. \end{aligned} \tag{7}$$

In Section 4.2, we shall also need the fourth Minkowskian function,

$$\mathfrak{A}_4^{(1)}(L) = \frac{1}{\beta_0^4}\frac{L^2 - \pi^2/3}{(L^2 + \pi^2)^3}\,. \tag{8}$$

All APT functions obey important properties that are valid in the higher-loop case:

- Unphysical singularities are absent with no additional parameters introduced.
- In the Euclidean and Minkowskian domains, QCD couplings $\alpha_E(Q^2) = \mathcal{A}_1(Q^2)\,, \alpha_M(s) = \mathfrak{A}_1$ and their "effective powers" $\mathcal{A}_k(Q^2)\,, \mathfrak{A}_k(s)$ are different functions related by integral operations $\mathbb{A}[\;]$ and $\mathbb{R}[\;]$ explicitly defined in eqs. (2).
- Higher functions, e.g., (7),(8) are not equal to powers of the first ones (5), (6).
- Expansion of an observable in coupling powers $(\alpha_s(Q^2))^n$ in the Euclidean or in $(\alpha_s(s))^n$ in the Minkowskian case is substituted by nonpower expansion in sets $\{\mathcal{A}_k\}$, or $\{\mathfrak{A}_k\}$ respectively. The latter expansions exhibit a faster convergence, as compared to common power expansion.

Two particular notes are in order:

— The APT functions $\mathcal{A}_k(Q^2)\,, \mathfrak{A}_k(s)$ essentially differ from common expansion functions $(\alpha_s)^k$ in the low energy region, where they are regular with finite IR limit. In particular, $\alpha_E(0) = \alpha_M(0) = 1/\beta_0$. This behavior provides high stability with respect to variation of the renormalisation scheme [13].

— In the UV limit, all APT functions tend to their usual counterparts $(\alpha_s)^k$. The related small parameters for the measure of deviation are ϵ_E in Euclidean and ϵ_M in Minkowskian case

$$\epsilon_E = \frac{\Lambda^2}{Q^2}\ln\left(\frac{Q^2}{\Lambda^2}\right), \quad \epsilon_M = \frac{\pi^2}{\ln^2(s/\Lambda^2)}\,. \tag{9}$$

Remark here that influence of APT contributions to QCD coupling and to observables is quite different in the Euclidean and Minkowskian regions. In the first case, APT correction $\sim \epsilon_E$ with rise of momentum becomes less than 1% already at $Q \simeq 10\,\mathrm{GeV}$. On the contrary, the influence of Minkowskian correction ϵ_M can be traced up to $100\,\mathrm{GeV}$ scale where the difference between two-loop (NLO) analytic and non-analytic coupling makes roughly 5%. Even in the 3-loop (NNLO) standard Minkowskian case, the effect of π^2 terms $\sim \alpha_s^4 \pi^2 \beta_0^2$ remains essential[3]

[2]Note, we change the notation for arguments of the APT functions: $Q^2 \to l$ and $s \to L$.

[3]For more details see Sections 3.2 and 4 in [10].

up to $10\,\mathrm{GeV}$.

Higher-loop case. The two-loop expressions are more complicated. Here, exact QCD coupling α_s can be expressed explicitly

$$\alpha_s^{(2)}(Q^2) = -\frac{\beta_0}{\beta_1}\frac{1}{1+\mathcal{W}_{-1}(z)} \quad \text{with} \quad z = -\frac{1}{b}\exp\left(-\frac{L}{B}-1\right) \quad \text{and} \quad B = \frac{\beta_1}{\beta_0^2}\,. \qquad (10)$$

in terms of the Lambert function $\mathcal{W}(z)$ defined as a solution[4] of the transcendental equation $\mathcal{W}e^{\mathcal{W}} = z$. This expression yields rather involved formulae for $\mathcal{A}_k$ and $\mathfrak{A}_k$ in terms of $\mathcal{W}_{-1}$.

In the three-loop case, one encounters more complications. Here, only for Padé approximated beta-function, exact solution can be expressed explicitly [12] in terms of the Lambert function and one meets the same situation as with exact two-loop solution[5].

The total picture becomes even more involved after taking into account the matching relation for adjusting kinematic regions with different values of the flavour number n_f . The devised scheme [9], known as "global APT", has been studied by Magradze and Kourashev at the two- and three-loop level. They calculated numerical tables for the first three functions $\mathfrak{A}_k, k = 1,2,3$ and $\mathcal{A}_{1,2,3}$ at three values of $\Lambda^{n_f=3} = 350, 400, 450\,\mathrm{MeV}$ in the interval $1\,\mathrm{GeV} < \sqrt{s}, Q < 100\,\mathrm{GeV}$ [12, 15], and $\mathfrak{A}_{1,2}$, $\mathcal{A}_{1,2}$ in the interval $0.1\,\mathrm{GeV} < \sqrt{s}, Q \lesssim 3\,\mathrm{GeV}$ [16]. Unhappily, their numerical tables, as well as complicated analytic formulae, are not comfortable enough for QCD practitioners. Some other approximations proposed recently [17] are not also of wide use yet.

Physical implications of APT. Re-examination of various processes on the basis of APT during the last decade has been performed in both Minkowskian and Euclidean regions. For a five-years old review, see paper [10]. Here, we list some fresh results.

In Ref.[18], it has been shown that in using the standard perturbation theory for description of the pion electromagnetic form factor, the size of the NLO corrections is quite sensitive to the adopted renormalization scheme and scale setting. Replacing the QCD coupling and its powers by their APT images, both dependences are diminished and the predictions for the pion form factor turn out to be quasi-independent of scheme and scale settings.

Applying appropriate generalization to the fractional powers [19] of APT coupling, the authors of [20] showed that the dependence of APT predictions for the pion form factor on the factorization scale was also diminished.

Inclusive τ-decay on the APT base has been studied by Solovtsov with co-authors. In paper [21], the stability of results with respect to the renormalization scale change was demonstrated. The QCD parameter value, $\Lambda_{f=3} \simeq 400\,\mathrm{MeV}$, close to the world average was obtained in Refs.[22, 23] after due account of non-perturbative values of light quarks masses and summation of threshold singularities.

One more curious application of APT appeared recently from the mass spectrum analysis of ground and first excited quarkonium states by the Milano group. There, it was argued[24, 25] that APT Euclidean coupling $\alpha_E(Q^2)$ at the interval $Q \sim 100 - 400\,\mathrm{MeV}$ should take values corresponding to $\Lambda_{f=3} \simeq 375\,\mathrm{MeV}$.

[4] One should be careful with choosing a proper branch $\mathcal{W}_{-1}$, see, e.g., [14].

[5] For the popular two-loop expression that represents (like eq.(9.5) in [2]) expansion of the iterative solution

$$\alpha_s^{(2,iter)} = \frac{1}{\beta_0(l + b\ln l)} \simeq \alpha_s^{(2,appr)}(Q^2) = \frac{1}{\beta_0}\left(\frac{1}{l} - b\,\frac{\ln l}{l^2}\right)\,,$$

the corresponding Minkowskian counterpart $\mathfrak{A}_1^{(2,appr)}$ is obtained in [6]. However, numerically, it gives a rather crude approximation in the low-energy region.

3 Simple Model for 3-Loop APT Functions

3.1 "One-Loop-Like" Model

Our aim is to construct simple and accurate enough (for practical use) analytic approximations for two sets of functions $\mathfrak{A}_k$ and $\mathcal{A}_k\,, k=1,2,3\,.$ To reduce number of fitting parameters, one should better provide the applicability of the recurrent relations.

To this goal, we suggest that one-loop APT expressions, eqs.(5),(6),(7), with modified logarithmic arguments have to be used

$$\mathcal{A}_k^{mod}(l) = \mathcal{A}_k^{(1)}(l_*)\,; \quad \mathfrak{A}_k^{mod}(L) = \mathfrak{A}_k^{(1)}(L_*)\,, \tag{11}$$

L_* and l_* being some "two-loop RG times".

Model functions (11) are related by the "one-loop-type" recursive relations

$$\mathcal{A}_{n+1}^{mod} = -\frac{1}{n\,\beta_0}\frac{d\mathcal{A}_n^{mod}}{dl_*} = -\frac{1}{n\,\beta_0}\frac{d\mathcal{A}_n^{mod}}{dl}\cdot\frac{dl}{dl_*}\,, \qquad \mathfrak{A}_{n+1}^{mod} = -\frac{1}{n\,\beta_0}\frac{d\mathfrak{A}_n^{mod}}{dL_*}\,.$$

A simple expression for l_* can be borrowed from [26], where a plain approximation for the two-loop effective log in the Euclidean region was used

$$\alpha_E^{mod}(Q^2) = \frac{1}{\beta_0}\left\{\frac{1}{l_2} + \frac{1}{1-\exp(l_2)}\right\}\,, \quad l_2 = l + B\ln\sqrt{l^2+4\pi^2}\,,$$

with B defined in (10). The structure of l_2 was inspired there by an idea of compensation of the first complex branch-cut of the Lambert function arising in the exact two-loop solution. This approximation was shown to combine reasonable accuracy in the low-energy range with the absence of singularities for α_E. We extend this approach to higher functions in both the Euclidean and Minkowskian domains. To this goal, we change square root in "effective logs" $L_2(a)$ and $l_2(a)$: $\sqrt{l^2+4\pi^2} \to \sqrt{l^2+a\pi^2}$ with a, an adjustable parameter. It comes out from thorough numerical analysis that optimal value of the new parameter is $a \approx 2$, while *effective boundaries between the flavor regions have to be chosen on quark masses* $m_c = 1.3\,\mathrm{GeV}$ and $m_b = 4.3\,\mathrm{GeV}$ just as in the $\overline{MS}$ scheme.

That is, we formulate our Model as a set of equations (12) with (6) – (8) and

$$L_* = L_2(a=2) = L + B\ln\sqrt{L^2+2\pi^2}\,, \quad l_* = l_2(2) = l + B\ln\sqrt{l^2+2\pi^2}\,. \tag{12}$$

Here, L and l are defined via common $\Lambda_{\overline{\rm MS}}$ values, like in (5),(6), for each of the flavor region. This choice provides us with overall accuracy of a few per cent – see Table 1.

Advantage of Model (5)–(7),(11), (12) is that it involves only one new parameter, $a=2$ with $\Lambda_{\overline{\rm MS}}$ and n_f taking their usual values.

3.2 Accuracy of the Model vs data errors

In Table 1 we give the maximal errors of our Model expressions (7), (11),(12) in each n_f range obtained by numerical comparison with the above-mentioned Magradze tables in the interval of 3-loop $\Lambda_{\overline{\rm MS}}^{(n_f=3)} \sim 350-400\,\mathrm{MeV}$.

As it follows from the Table, errors of Model for the first three APT functions are small, being of an order of 1-2 per cent for the first functions, of 3-5% for the second and of 6-10% for the third ones in the region above $1.5\,\mathrm{GeV}$, i.e., in the $n_f = 4,5$ ranges. However, its accuracy in the $n_f=3$ region (above 1 GeV) is at the level of 5-10 per cent.

Meanwhile, relative contributions of typical LO, NLO and NNLO terms in APT nonpower expansion for observables are usually something like 60-80%, 30-10%, and 10-1%, respectively (see Table 2 in Ref.[10]). Due to this, the Model accuracy for many cases is defined by that of the first model functions $\mathcal{A}_1^{mod}$, $\mathfrak{A}_1^{mod}$, provided that QCD contribution to an observable starts from one-loop contribution $\sim \alpha_s$. At the same time, for quarkonium decays one meets the case with the leading contribution $\sim \alpha_s^3$. There, the Model error is defined by accuracy of the third Minkowskian function $\mathfrak{A}_3^{mod}$ – see Table 2.

In this Table we compare our Model errors with some data errors in the low energy region. The last ones are taken from a recent Bethke's reviews [1].

With due regard for data errors, we can now set some total margin of accuracy that our Model would satisfy. This margin may be chosen, e.g., as 1/3 of the experimental error bar, which is no less than 10% – see column 5 in Table 2). Then the accuracy limit, imposed upon the first APT functions could be about 3%, for the second function will be at least 3 times weaker than for the first one (say, 10%), and for the third ones, at least 6 times weaker (say, 20%). Analysing Table 1 according to these requirements, one concludes that it is not reasonable to use the Model below 0.5 GeV , whereas above it (or, in part of 3-flavour and in the 4- and 5-flavour regions) it is fully advisable. Now we may proceed to its practical application.

4 Υ(1S) Decay Revisited

To this goal, we take the $\Upsilon(1S)$ non-radiative decay. The main reason of this choice is the troubling disagreement[6], exceeding three standard deviations, between the coupling α_s extracted from this Υ decay and the world average. This can be clearly seen from Fig. 9.2 of the Particle Data Group review [2].

An observable, which could be an apt illustration of the proposed Model should obey the following criteria: first, influence of the π^2 terms upon its value should be large enough, this observable should be measured up to a sufficient precision. The $\Upsilon(1S)$ non-radiative decay satisfies these both. Indeed, on the one hand, the parameter ϵ_M defined in (9) is not small[7] in the region (3-5 GeV) related to this decay: $\epsilon_M^{(2)} = \pi^2/L_2^2 \simeq 0.3 - 0.2$.

On the other hand, the non-radiative decay of $1S$-state provides the best data precision (2.0% for the ratio of hadronic and leptonic widths [2]). Decays of 2S and $3S$ states give a poorer accuracy of this ratio.

At the same time, quarkonium decays are not the best proving ground for the test of model expressions we have devised. Indeed, in the choice of the Model we made accent on the accuracy of first APT functions. Meanwhile, these decays are described by expressions $\sim \alpha_s^3$. Nevertheless, even with 7 per cent error (as it is seen in Table 2) the Model application to $\Upsilon(1S)$ decay will be interesting enough.

4.1 Υ Widths

The paper [7] by Krasnikov and Pivovarov was the predecessor of present work in revisiting Υ decay. 25 years ago, these authors showed the effect of analytical continuation to be significant for Υ decay width, resulting in a considerably larger $\Lambda_{\overline{\rm MS}}$ (1.5 times) than the values obtained earlier by standard RG improved perturbation theory. Later on, the importance of π^2 terms was emphasized by Bjorken [8]. Nevertheless, to our knowledge, extraction of α_s value was performed so far without due regard for the analytic continuation effects, with some exceptions mentioned below.

A few words on other processes involving Υ, like its radiative decays and Υ production. All these have low data precision (typically, radiative widths are measured with the 10% – 40% accuracy). For an extensive review of $q\bar{q}$ decay widths see [28].

[6]This issue is absent in Bethke's reviews [1], because there an α_s estimate is based only upon paper [27], where a reasonable α_s value was extracted from Υ sum rules with Coulomb resummation taken into account.

[7]The importance of the π^2 terms in Υ decay was demonstrated recently. The rough estimate obtained in Section 4 of paper[10] gave $\triangle\alpha_s(M_Y) \simeq +0.012$ that is more than 5 % correction to the NLO result.

The NLO ratio of hadronic and leptonic decay widths of the $\Upsilon\, S$ state was given in [29] (see also [30]) [8]

$$R_\Upsilon = \frac{\Gamma\,(\Upsilon \to hadrons)}{\Gamma\,(\Upsilon \to e^+e^-)} = \frac{10(\pi^2-9)\alpha_s^3(\mu)}{9\,\pi\alpha^2(M_\Upsilon)}\left[1+\frac{\alpha_s(\mu)}{\pi}\left(\tilde{\beta}_0\left(2.78-\frac{3}{2}\ln\frac{M_\Upsilon}{\mu}\right)-14.1\right)\right]. \qquad (13)$$

Here, $\tilde{\beta}_0 = 11 - \frac{2}{3}n_f$ (in PDG normalization) with $n_f = 5$ as $\overline{\rm MS}$ scheme is used.

Meanwhile, in the RG-invariant expression $R(\mu^2)$, scale μ can appear *only* in the argument of QCD coupling $\alpha_s(\mu^2)$. To return eq.(13) to the RG-invariant form, one could set $\mu = M_\Upsilon$

$$R_\Upsilon = \frac{10(\pi^2-9)\alpha_s^3(s_\Upsilon)}{9\,\pi\alpha^2(M_\Upsilon)}\left(1+\frac{\alpha_s(s_\Upsilon)}{\pi}\,7.2\right) = 5360[\alpha_s^3(s_\Upsilon)+2.30\alpha_s^4(s_\Upsilon)]\,; \quad s_\Upsilon = M_\Upsilon^2\,. \qquad (14)$$

Then, the issue of scale should be readdressed to the choice of s_Υ in eq.(14).

It was argued [30] that a proper value for scale μ could be $M_\Upsilon/3$, due to the 3-gluonic mode of the Υ decay. Below, we consider the range close to M_Υ, $\sqrt{s_\Upsilon} = 7-9$ GeV , and discuss the scale uncertainty in the final error estimate. The QED coupling $\alpha(M_\Upsilon)$ is fixed on the Υ mass.

4.2 Reevaluation of Λ_{QCD} from Υ Decay

Calculations. The first attempt to re-evaluate $\Lambda_{\overline{\rm MS}}$ extracted from Υ decay by proper taking into account analytic continuation effects was made[9] in [7].

Analysis analogous to [7] is performed here, with the APT expansion

$$R_\Upsilon^{theor} = 5360\,[\mathfrak{A}_3(L) + 2.30\,\mathfrak{A}_4(L)] \qquad (15)$$

instead of formula (14) which, in turn, within our Model, looks like

$$R_\Upsilon^{Mod} = 5360\left[\mathfrak{A}_3^{Mod}(L) + 2.30\,\mathfrak{A}_4^{Mod}(L)\right]. \qquad (16)$$

By these formulae we extract $\Lambda^{(5)}_{\overline{\rm MS}}$ and α_s values from fresh CLEO III[31] data

$$R = 37.3 \pm 0.75$$

A few words on the fourth APT function $\mathfrak{A}_4(L)$. In the r.h.sides of eq.(15) we use unpublished yet numerical results [32] and in (16)– the Model expressions, eqs. (7), (8), (11) with logarithmic argument (12): $\mathfrak{A}_{3,4}^{Mod}(L) = \mathfrak{A}_{3,4}^{(1)}(L_2)\,; \quad L_2 = L + b\ln\sqrt{L^2+2\pi^2}\,.$

As it follows from a more detailed analysis[10], at $\sqrt{s} \sim 7-9\,$GeV relative error of $\mathfrak{A}_3^{Mod}(L)$ is about 6 %. At the same time, in this interval, the ratio $|\mathfrak{A}_4^{Mod}/\mathfrak{A}_3^{Mod}| \sim 0.16$. This means that the error due to the 2nd term in the r.h.s. of eq.(16), is about 3 % and the total Model error for R_Υ^{Mod} about 9 %. In turn, this gives 3 % Model error for α_s that is equal to $\triangle\alpha_s(M_\Upsilon) = \pm 0.005$.

Now, numerical analysis of the "Exact" case (15) yields $\Lambda_5 = 210\pm 5\,$MeV , that is $\alpha_s(M_Z) = 0.118 \pm 0.0005$. Further on, we shall neglect by this small error. For the "Model" case with $\Lambda_5 = 235, $MeV , that is $\alpha_s(M_Z) = 0.120$, one has the Model error $\triangle\alpha_s(M_Z) = \pm 0.002$.

In line 1 of Part I, "PDG, $1S$", standard PT results on $\Upsilon(1S)$ decay are given. We present them not exactly as they were published in [2] but recalculated along with modern experimental data. Line 2, marked "PDG, Fit" gives the published world average value described by the

[8] This formula was reproduced in review [2] with error, but corrected in the last PDG review version.

[9] One should remark here that some other kind of summation, namely of $1/v-$type terms, for the improvement of $\Lambda_{\overline{\rm MS}}$ extraction from Υ production cross-sections was devised in [27]. Unlike the present case, where we deal with R_Y for Υ decay, the authors of [27] studied sum rules for ratio $R(s) = \frac{\sigma(e^+e^-\to b\bar{b})}{\sigma(e^+e^-\to\mu^+\mu^-)}$. Analytic continuation effects were not taken into account.

[10] Based upon calculation[32] of exact APT formula.

curve on Fig. 9.2 in [2], within the error bars of *all* the processes. Column "$\alpha(M_\Upsilon)$" means "α_s, calculated at the mass of Υ, according to eq.(9.5) of [2]".

Line 1 of Part II, "Exact APT", presents results of Υ1S decay calculated by exact numeric tables for $\mathfrak{A}_3$ and by [32] for $\mathfrak{A}_4$ APT function. Line 2, [Mod], presents values obtained from $\Upsilon(1S)$ decay data by means of the Model eqs.(12),(14). Here, model errors combine Model errors of both the terms in the r.h.s. of eq.(16). Line 3, "Crude APT" gives an earlier result[10] with approximate APT estimate used to correct (Corr) the Bethke-2000 value $\alpha_s(M_\Upsilon) = 0.170$ extracted there from all the Υ decays data.

Everywhere in Part II, we translate our results from APT expressions, by use of related Λ_5 values into standard *non-analytic* coupling α_s at M_Υ and M_Z for ease of comparison with Part I and other standard sources.

The importance of taking into account the analytic effects for proper α_s extraction in the low-energy range is clearly seen from the Table 3. It is also evident that the Model error is small enough even in the non-favorable case of quarkonium decay, in particular when compared with the scale error.

The scale uncertainty. Besides "direct" data error that in our case is 2% for R_Υ , there is a "hidden" uncertainty in theoretical equations (15) and (16) related with a choice of argument $L = \ln(s_\Upsilon/\Lambda^2)$. In given figures, we use $s_\Upsilon = M_\Upsilon^2$. To discuss effect of the s_Υ variation, return to issue mentioned above at the end of Sect.4.1. .

This reference scale issue is actual for QCD analysis of all low energy data. The RG non-invariant term $\ln(M_\Upsilon/\mu)$ in the r.h.s. of eq.(13) just represents an attempt to take into account this effect. The scale effect enlarges quickly with the uncertainty rise. E.g., for its value 1 GeV, i.e., for 8–9 GeV interval, we have $205 \le \Lambda_5 \le 240\,\mathrm{MeV}$, while to the 7–9 GeV case — $180 \le \Lambda_5 \le 240\,\mathrm{MeV}$. In the Table 3, where the APT results are compared with ones of standard PT, for the scale error $[...]_{sc}$ we conditionally give figures related to[11] the 2 GeV interval.

Generally, the scale issue is an intrinsic problem of renormalization group application to observables. On the one hand, the common, *"vulgaris"*, version of RG algorithm corresponds to the UV (massless) case with simplified definition of effective (running) coupling $\bar\alpha(q^2/\mu^2)$. Here, it is tacitly assumed that vertex function $\Gamma(q_1^2, q_2^2, q_3^2)$ entering into the $\bar\alpha$ definition is taken with equal arguments. On the other hand, the condition $q^2 \gg m^2$ is used. Both the assumptions are not valid for the low energy QCD case. On these items, the reader could be addressed to mass-dependent RG formalism [33, 34, 35].

5 Conclusion

1. Our theoretical result is model explicit expressions (11) for the analyticized 3-loop couplings and their effective powers in both the Euclidean and Minkowskian regions. These are just one-loop analytic expressions (5)–(8) of Analytic Perturbation Theory with modified logarithmic arguments (11). The accuracy of the Model is estimated to be sufficient for practical purposes in the region 1–100 GeV, which hosts many important processes.

2. To illustrate the APT and Model application, we consider the α_s value extraction from data for Υ(1S) non-radiative decay measured with 2% error. Fixing the scale at the Υ mass, we got (for the $\alpha_s^{\overline{\mathrm{MS}}}$ coupling) by exact APT numerical calculation and by our APT Model
$$\alpha_s^{\mathrm{APTexact}}(M_Z) = 0.118(1)_{exp}\,, \quad \alpha_s^{\mathrm{APTModel}}(M_Z) = 0.120(2)_{Mod}\,. \qquad (17)$$
The comparison with result of Υ(1S) usual analysis, $\alpha_s(M_Z) = 0.112(2)$, and with the world average, $\alpha_s(M_Z) = 0.1185(20)$, confirms the validity of the devised Model.

[11]To the case $\sqrt{s_\Upsilon} = M_\Upsilon/2 = M_b$ there corresponds $\alpha_s(M_Z) \sim 0.114$. and to $\sqrt{s_\Upsilon} = M_\Upsilon/3$ [30] – $\alpha_s(M_Z) \sim 0.110$.

3. It was established that the scale uncertainty essentaially reduces the value of theoretical analysis of Υ decay. In the considered case, the scale error dominates over experimental and Model ones. This issue is worth urgent further examining.

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Table 1: **Maximal errors of the Model with $a = 2$**

n_f	$Er\mathfrak{A}_1$	$Er\mathfrak{A}_2$	$Er\mathfrak{A}_3$	$Er\mathcal{A}_1$	$Er\mathcal{A}_2$	$Er\mathcal{A}_3$
3 (0.5 - 1.0$\dot{\rm G}$eV)	3%	5	15	3	10	20
3 (1 - 1.5$\dot{\rm G}$eV)	3%	4	8	2	7	10
4	1%	3	8	2	10	20
5	1%	3	6	0.5	4	15

Maximal relative errors $Er\mathfrak{A}_i = \max|\frac{\delta\mathfrak{A}_i}{\mathfrak{A}_i^{tab}}|$, $\delta\mathfrak{A}_i = \mathfrak{A}_i^{tab} - \mathfrak{A}_i^{mod}$ and $Er\mathcal{A}_i$ (defined analogously) are given in per cent. Here, superscript "tab" denotes tabulated functions from papers [15, 16].

Table 2: **Accuracy of the α_s extraction vs Model's errors**

Process	chan.	q GeV	n_f	Err. α_s (q) % data	Err. α_s (q) % **Model**
DIS(Bjork.)	t	1.7	3	16	4
GLS SR	t	1.7	4	25	4
τ decays	s	1.8	4	9	1
Υ decays	s	7-9	5	10	6
$\sigma_{e^+e^-\to hadr}$	s	10	5	30	1

Notation: chan. – t = Euclidean or s = Minkowskian channel; scale $q = \sqrt{s}$ or $q = \sqrt{Q^2}$; $\Delta\alpha/\alpha$ – relative error of α at a given scale; DIS (Bjork.) – Bjorken scaling violation in DIS, GLS — Gross – Llewellyn-Smith sum rule, and * marks combined theor. and exper. errors.

Table 3: **Results of various α_s extraction from Upsilon decays**

Part I. Non-APT treatment			
Source	$\alpha(M_\Upsilon)$	$\alpha(M_Z)$	$\Lambda_{\overline{\rm MS}}^{n_f=5}$
PDG, $\Upsilon, 1S$	0.170(4)	0.112(2)	146^{+18}_{-17}
PDG, global Fit	0.182(5)	0.1185(20)	217^{+25}_{-23}
Part II. APT treatment			
Exact APT, 1S	$0.1805(12)_{exp}$	$0.1179(5)_{exp}$	210(5)
[Mod], $1S$	**0.185**$(5)_{\rm M}$	$0.120(2)_{\rm M}$	235(25)
[10] Crude APT	0.183	0.119	222
Exact±[scale], $1S$	$0.180[7]_{sc}$	$0.118[3]_{sc}$	210[30]