Shrinking fermionic modes on the lattice and in the continuum

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Abstract

Recent lattice data indicate that volume occupied by topological fermionic modes shrinks to zero in the continuum limit of vanishing lattice spacing a. The data apparently cannot be accommodated within, say, conventional instanton model. We present field-theoretic arguments which demonstrate that the topological fermionic modes are to shrink to a vanishing submanifold of the whole four dimensional space provided that measurements are performed with high resolution. As for the dimensionality of the submanifold, there are theoretical arguments in favor of three dimensional domain walls.

Introduction

In this talk we will discuss topological defects in YM vacuum. By topological defects we understand regions with large absolute value of the density of the topological charge $Q_{top}(x)$,

$$Q_{top}(x) \; = \; (16\pi^2)^{-1} G^a \tilde{G}^a.$$

Usually one thinks about such regions in terms of instantons. For instantons, [1]

$$\Big(\int d^4x Q_{top}(x)\Big)_{instanton} \; = \; 1 \quad ,$$

which is a large compared to the perturbative noise which contains extra power of α_s. The instanton picture has been challenged since long because of inconsistencies in the large N_c limit [3, 4]. An alternative description could be provided by domain walls [3]. The theory of domain walls is not developed in detail, however.

Note that domain walls are, by definition, 3d defects in vacuum. Thus, one could argue that in the domain-walls picture topological defects would occupy a vanishing fraction of the whole 4d space. As far as we know, however, this point has never been emphasized. Moreover, the first example of low-dimensional vacuum defects was provided by lattice strings, or vortices which are responsible for confinement [2]. references see, e.g., [5, 6]).

The possibility that topological defects could occupy a a submanifold of 4d space was suggested first in Ref [6] in the context of the lattice 3d defects discovered in [7] and closely related to the vortices. Independently, there began to appear data on unusual behavior of fermionic zero modes as function of the lattice spacing [8]. The data does indicate that topological defects shrink to a vanishing subspace of the whole space if measurements are performed with

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[1] The instanton model has been elaborated in great detail, for review see [1, 2].

[2] For review of the role of the vortices in the confinement see [5] while identification of the vortices with strings is discussed, e.g., in [6].

high resolution, that is on the scale of the lattice spacing. However, it is too early to conclude what is the dimension (in physical units) of this submanifold [3].

Low-lying fermionic modes

To uncover topology of the gluonic fields one concentrates on low-lying modes of the Dirac operator. The modes are defined as solutions of the eigenvalue problem

$$D_\mu \gamma_\mu \psi_\lambda \;=\; \lambda \psi_\lambda \;, \tag{1}$$

where the Dirac operator is constructed on the vacuum gluonic field configurations $\{A^a_\mu(x)\}$.

For exactly zero modes, the difference between modes with positive and negative chirality equals to the total topological charge of the lattice volume:

$$n_+ - n_- \;=\; Q_{top} \;. \tag{2}$$

Assuming that topological charge fluctuates by order unit independently on pieces of 4d volumes measured in physical units one derives:

$$\langle Q^2_{top} \rangle \;\sim\; \Lambda^{-4}_{QCD} V_{tot} \;\approx (180 MeV)^4 V_{tot} \quad , \tag{3}$$

The numerical coefficient here is related to the $\eta^{'}$-mass (the Witten-Veneziano relation).

One also considers so called near-zero modes which occupy, roughly speaking the interval

$$0 \;<\; \lambda \;<\; \frac{\pi}{L_{latt}} \quad , \tag{4}$$

where L_{latt} is the linear size of the lattice. Near-zero modes determine the value of the quark condensate via the Banks-Casher relation:

$$\langle \bar{q} q \rangle \;=\; -\,\pi \rho(\lambda \to 0) \;, \tag{5}$$

where $\lambda \to 0$ with the total volume tending to infinity.

Lattice data

While the close relation of the low-lying fermionic modes to the topology of the underlying gluon fields is well known since long, it is only recently that these modes have been measured on the original field configurations, without cooling. This recent progress is due to the advent of the overlap operator [10].

Measurements on original fields [8] confirmed the general relations (3) and (5). However, they also brought an unexpected result that the volume occupied by low-lying modes apparently tends to zero in the continuum limit of vanishing lattice spacing, $a \to 0$. Namely,

$$\lim_{a \to 0} V_{mode} \;\sim\; (a \cdot \Lambda_{QCD})^r \;\to\; 0 \;, \tag{6}$$

where r is a positive number of order unit and the volume occupied by a mode, V_{mode} is defined in terms of the Inverse Participation Ratio (IPR) [4].

A crucial question is then, whether the underlying vacuum structure is the same for the confining fields and fields with non-trivial topology. An attempt to answer this question was

[3]Possible relevance of the domain walls introduced in the dual formulations of gauge theories to lattice measurements on the fermionic modes was pointed out recently in Ref. [9]. Note, however, that vacuum defects introduced in this reference occupy not a 3d submanifold but almost the whole 4d space.

[4]Independent evidence in favor of shrinking of the regions occupied by topologically non-trivial gluon fields was obtained in [11].

undertaken in Ref. [12] through a direct study of correlation between intensities of fermionic modes and of vortices.

In more detail, center vortex is a set of plaquettes $\{D_i\}$ on the dual lattice, for review see [5]. Denote the set of plaquettes dual to $\{D_i\}$ by $\{P_i\}$. Then the correlator in point is defined as:

$$C_\lambda(P) = \frac{\sum_{P_i}\sum_{x\in P_i}(\rho_\lambda(x) - \langle\rho_\lambda(x)\rangle)}{\sum_{P_i}\sum_{x\in P_i}\langle\rho_\lambda(x)\rangle}\,. \tag{7}$$

Since $\sum_x \rho_\lambda(x) = 1$ and $\langle V_{tot}\rho_\lambda(x)\rangle = 1$, Eq (7) can be rewritten as

$$C_\lambda(P) = \frac{\sum_{P_i}\sum_{x\in P_i}(V_{tot}\rho_\lambda(x) - 1)}{\sum_{P_i}\sum_{x\in P_i}1}\,. \tag{8}$$

Results of measurements can be found in the original paper [12]. Here we briefly summarize the finding.

First of all, there is strong positive correlation between intensities of fermionic modes and density of vortices. Second, the value of the correlator depends on the eigenvalue and the correlation is strong only for the topological fermionic modes. Finally and most remarkably, the correlation grows with diminishing larttice spacing. A simple analysis reveals that, indeed, if the 2d defects are entirely responsible for chiral symmetry breaking or constitute a boundary of 3d defects carrying large topological charge, the correlator (7) grows as an inverse power of the lattice spacing. The data does show that the correlator grows for smaller a but does not allow yet to uniquely fix the dimensionality of the chiral defects.

Pieces of theory

The result (6) is in striking contradiction with the instanton model and at first sight seems very difficult to appreciate. A more careful analysis demonstrates, however, that the shrinking of topological fermionic modes could have predicted from field theory [5].

As is explained above the topological fermionic modes just reveal the topological structure of the underlying gluon fields and we can concentrate, therefore, on distribution of $G\tilde{G}(x)$. Consider correlator of the topological density at two points. From general principles alone, one can that for any finite x:

$$\begin{aligned}\langle\, G\tilde{G}(x), G\tilde{G}(0)\,\rangle_{Minkowski} \;&>\; 0\\ \langle\, G\tilde{G}(x), G\tilde{G}(0)\,\rangle_{Euclidean} \;&<\; 0\,.\end{aligned} \tag{9}$$

On the other hand, for a pure instanton, or within a zero mode:

$$\langle\, G\tilde{G}(x), G\tilde{G}(0)\,\rangle_{instanton} \;>\; 0\,. \tag{10}$$

Since the instanton contribution (10) taken alone violates unitarity, compare (9), it cannot dominate and the unitarity is restored at any finite x by perturbative contributions. Somewhat schematically, the correlator can be represented as

$$\langle\, G\tilde{G}(x), G\tilde{G}(0)\,\rangle_{Euclidean} \;\sim\; -\frac{c_1\alpha_s^2}{x^8} \;+\; c_2\Lambda^4_{QCD}\delta(x)\,, \tag{11}$$

where $c_{1,2}$ are positive constants and the $1/x^8$ term is perturbative.

The central point is that by measuring topological modes we filter the perturbative noise away and are left with the local term. In the language of dispersion relations, this is a subtraction term, which has no imaginary part [14].

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[5]The argumentation was worked out by A.I. Vainshtein and the author and outlined in some detail in the talk [13].

It is only natural then that contributions which are described by subtraction constants in dispersion relations appear as vanishing submanifolds once attempt is made to measure their spatial extension, or volume. Moreover, to see that the volume is small we need measurements with high resolution. Hence, dependence on the lattice spacing exhibited by the data (6). In this sense, the lattice spacing a is to be understood as resolution of the measurements rather than as an ultraviolet cut off.

Dimensionality of the chiral defects

Although this type of argument makes observation (6) absolutely natural and predictable, it does not immediately fix the exponent r. Theory seems to favor $r = 1$, or 3d defects but no reliable derivation actually exists. Here we briefly mention the arguments in favor of 3d topological defects.

First, we already mentioned that domain walls appear naturally in dual formulations of Yang-Mills theories with large N_c. Note, however, that there could be non-trivial fractal dimensions so that 3d manifold would percolate through the 4d space and occupy the whole or a finite part of it [9]. From experience with lattice strings, see, e.g. [6], we would still expect zero anomalous fractal dimension and topological defects occupying 3d volume in physical units.

Also, one can invoke analogy with quantum mechanics [6]. In case of quantum mechanics, the phenomenon is that if one tries to measure the time spent by a particle under the barrier, the time turns to be zero. A mnemonic rule is that the particle under the barrier lives in imaginary time and when projected to real time the barrier transition has zero duration (for details and references see [13]). Violation of the unitarity in (10) could also be formally removed by changing one coordinate from real to imaginary values. Thus, one expects that only one coordinate collapses to a vanishing interval.

Clearly, all theoretical arguments in favor 3d defects are suggestive at best.

Note also that while the shrinking of topological modes (6) follows from Yang-Mills theory, explaining the observed correlation of the topological modes with the lattice strings is beyond the scope of field theory. Probably, clues are provided by theory of the defects in the dual, string formulation but there has been no discussion of the issue in the literature.

Protected and unprotected matrix elements

There is a drastic difference between results of measurements of, say, topological susceptibility (3) and of the instanton size. While the value of (3) does not depend on the resolution, or lattice spacing a, the size of topological excitations changes drastically :

$$\big(size\big)_{resolution\ a} \;\sim\; \exp(-const/g^2(a)) \cdot \big(size\big)_{resolution\ \Lambda_{QCD}} \,, \tag{12}$$

where $g^2(a)$ is the bare coupling.

Clearly, one cannot think of deriving (12) in perturbation theory. Rather, one should think in terms of theory of measurements. As a matter of fact, the chiral condensate does not depend on the resolution and can be measurement either without or with cooling while the size of topologically non-trivial regions of gluon field depends power-like on the resolution. One can talk, therefore, about 'protected' and 'unprotected' matrix elements.

The question is then whether we can judge theoretically which matrix element is protected. Without trying to formulate here a general recipe, turn to examples.

The *local* matrix element (5) is given by

$$\langle \bar{q}q\rangle \;\sim\; \lim_{m_q\to 0} m_q \int \rho(\lambda)\,\frac{d\lambda}{(\lambda^2+m_q^2)} \;\;, \tag{13}$$

[6]See footnote 5.

where λ is the eigenvalue, see (1). With lattice spacing $a \to 0$ the number of eigenvalues grows power-like since $\lambda_{max} \sim 1/a$. However, it is clear that all these modes are canceled from (13) in the limit of the vanishing quark mass, $m_q \to 0$. Thus, quark condensate gives an example of a protected matrix element.

To define instanton size in terms of a matrix element one can try a *non-local* generalization of (13). Namely, consider the correlator [7] :

$$\langle \, \bar{q}^{\alpha}(x), q^{\beta}(0) \, \rangle$$

where we omit the spinor indices but keep the color indices, α, β. Because of chirality conservation, the correlator (14) is not contributed by perturbation theory.

True, our correlator is not acceptable yet because it is not gauge invariant. To amend this, one can insert the phase factor:

$$\langle \, \bar{q}^{\alpha}(x)\Phi_{\alpha,\beta}(x,0)q^{\beta}(0) \, \rangle \;\equiv\; f_1(x^2) \quad , \tag{14}$$

where, as usual, $\Phi(x.0) \;=\; P\exp i\int A_{\mu}dx_{\mu}$. In the quasiclassical approximation, instantons do contribute to (14) and produce a non-trivial x^2 dependence with characteristic scale of Λ_{QCD}. This scale could be considered as a general definition of the size of topological excitations in terms of a gauge invariant matrix elements. Another possibility is to introduce a color scalar field ϕ_{α} and consider correlator of colorless spinor currents:

$$\langle \, \bar{q}^{\alpha}(x)\bar{\phi}_{\alpha}(x), q^{\beta}(0)\phi_{\beta}(0) \, \rangle \;\equiv\; f_2(x^2) \quad . \tag{15}$$

The question is now whether the matrix elements (14), (15) are protected or not.

It is obvious that the matrix element (14) changes greatly if measurements are performed with resolution of order a. Indeed the P-exponent entering (14) has ultraviolet divergent action (the same as, say, Wilson line). Therefore,

$$f_1(x) \;\sim\; \exp\big(-const\frac{|x|}{a}\big) \quad , \tag{16}$$

and we conclude that the matrix element (14) is not protected against huge resolution-dependent corrections.

Turn now to another possibility, that is correlator (15). If the scalar field is dynamical and assumed to propagate then the correlator (15) is also hugely suppressed at $|x| \gg a$. Indeed, the scalar particle acquires quadratically divergent mass,

$$M^2_{\phi} \;\sim\; \alpha_s/a^2 \; . \tag{17}$$

Of course one could try to renormalize mass but then many orders of perturbation theory should be taken into account and the correlator (15) is no simpler than (14) discussed above.

An interesting question, what happens if the scalar field–like the fermionic one– is treated in the quenched approximation, i.e. as non-propagating external field. At first sight, then there is no power-like sensitivity of (15) to the resolution. In fact, a power-like dependence on the lattice spacing is still there since the volume occupied by low-lying modes of the scalar field tends to zero in the quenched approximation: [15]

$$V_{mode\ \phi} \;\sim\; a^q \tag{18}$$

with value of q of order unit and varying from one color representation of scalar field to another.

Note that we could evaluate the correlator (15) by inserting a full set of intermediate physical states. In the quenched approximation, this sum is very simple since in pure gluodynamics (also with possible inclusion of a scalar particle) there are no spinor 'hadrons' and the correlator (14) reduces to a local term. In other words, we reproduce again the logic that led us to (11) but this time without the perturbative background.

[7] This definition, in its gauge invariant versions, see below, was suggested to me by V.A. Rubakov.

Conclusions

Lattice measurements on original, hard fields $A^a_\mu \sim 1/a$ brought unexpected, power-like dependences on the lattice spacing [16, 8, 15]. Such dependences have been scarcely discussed theoretically. In this talk we presented a few observations on such divergences not necessarily closely related to each other.

In particular, we argued that the lattice spacing plays the role of resolution. An analogy from quantum mechanics is measurements of instantaneous velocity of a particle: it tends to infinity with space resolution tending to zero.

More specifically, in case of topological fermionic modes (in the quenched approximation) one can utilize dispersion relations to argue that the volume occupied by the modes vanishes indeed in the continuum limit. The physics behind is that large fluctuations of the topological density are due to under-the-barrier transitions in the Euclidean space. Therefore, they do not correspond to any physical intermediate states and are described by local, or subtraction terms in field theory.

Another, and probably most interesting aspect is that dual formulations of YM theories with large N_c predict existence of low-dimensional vacuum defects. One of the earliest of such predictions is existence of domain walls, or 3d defects related to topology [3]. Although the argumentation does not apply literally if N_c is not large, the basic geometrical constructions could survive in the SU(2) case as well. Then, to uncover low-dimensional defects one needs measurements with high resolution since only explicit lattice dependence distinguishes, say, 3d defects from 4d excitations.

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