Mass Terms in the Skyrme Model*

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Abstract

We consider various forms of the mass term for the Skyrme model and their implications on the properties of baryonic states. We show that modifications of the mass term, without changing the asymptotic behaviour of the profile function at large r, can considerably alter the mass term contributions.

We argue that if one considers the Skyrme model as an effective model for nuclei, then the new mass terms should be taken to obtain a better description of the existing nuclei. In particular, we find that for some choices of the mass term, the formation of a shell like structure for large baryon number is energetically unfavourable for low baryon numbers and for the physical values of the pion mass.

1 The Skyrme Model

The Skyrme model[1] was constructed by T. Skyrme as an attempt to describe nucleons. The idea was to "build" baryons out of pion fields through a nonlinear constraint. In the original model, the field U took values in $SU(2)$ (later generalised to $SU(N)$), and was described by the Lagrangian

$$L = \frac{F_\pi^2}{8}\int_{R^3} (\frac{1}{2}\mathrm{Tr}(U_\mu^\dagger U_\mu)^2 + \frac{1}{a}\mathrm{Tr}[U^\dagger U_\mu,\ U^\dagger U_\nu]^2) d^3\vec{x}, \tag{1}$$

where $U_\mu = \frac{\partial U}{\partial x^\mu}$ and where F_π is the pion decay constant and a is a free parameter. The connection to the pion field was done by observing that U can be written in full generality as

$$U = \sigma 1 + i\vec{\pi}.\vec{\tau} \tag{2}$$

where τ are the Pauli matrices and $\vec{\pi}$ is the pion field. The field σ is an auxiliary field related to the pion field by the constraint that U must be unitary: $\sigma^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = 1$.

For the energy to be finite, one has to impose the condition that U is a constant (usually taken to be the unit matrix) at infinity. Mathematically, this is equivalent to compactifying the 3 dimensional space into the 3 dimensional sphere, and as $SU(2)$ is topologically equivalent to S^3, each Skyrme field U is characterised by a topological charge $Q(U)$ that counts the number of times the spatial 3 dimensional space is mapped into the target space $SU(2)$. Skyrme's main idea was to interpret this topological charge as the baryon charge.

Using the pion decay constant and the proton mass as input parameters, Skyrme was able to predict with his model the radius of a nucleon with a reasonable accuracy.

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The Skyrme model was then forgotten for more than 20 years until Witten [2] showed that, at low energies and in the limit of infinite number of colours, it is equivalent to Quantum Chromodynamics, proposed and confirmed several years earlier as the fundamental theory of strong interactions.

The advantage of the Skyrme model is that it is a semi-classical field theory for which one can compute static solutions for any baryon number, something that is virtually impossible to do for a quantum theory like QCD. Because of this, one can then take the Skyrme model as an effective theory of nuclei and see if one can improve it by adding or modifying some terms in its Lagrangian.

An example of this is the model studied by [3] where a mass term is added to the Lagrangian (1) to make the pion massive:

$$L = \frac{F_\pi^2}{8}\int_{R^3}(\frac{1}{2}\mathrm{Tr}(U_\mu^\dagger U_\mu)^2 + \frac{1}{a}\mathrm{Tr}[U^\dagger U_\mu,\ U^\dagger U_\nu]^2 + m_\pi^2 Tr(U-1))d^3\vec{x}. \tag{3}$$

This model now has 3 parameters, the pion decay constant F_π, the pion mass m_π, and the Skyrme parameter a. These can be fitted to experimental values or be used as free parameters fitted to other measurable quantities. This is what Atkins, Napi and Witten [4][3] did in their papers where they quantised the zero modes (rigid body) of the Skyrme model and determined the values of these parameters by fitting the model to the experimental values of the proton, delta and pion masses. This has lead to the model agreeing with experiments to within 10% and 30% depending on the quantity being studied.

Although the Skyrme model is not a fundamental theory, it has the advantage that one can use it to compute various properties of nuclei including configurations with a relatively large baryon number. In Figure 1 we present the baryon density of several configuration. The last one, $B = 32$, is not the lowest energy configuration when $m = 0$ but is believed to be so when the value of the pion mass is large enough [5].

As shown by several authors [6][7][8], the zero mode quantisation of Atkins, Nappi and Witten is not fully correct because it assumes that the Skyrmion behaves like a rigid body under rotation and it does'nt. In their work, Battye, Krusch and Sutcliffe have redone the zero mode quantisation of the Skyrmion taking into account its deformation under rotation. Their result is that the model leads to the existence of a Delta resonance only if one takes for the pion mass a number that is much larger than the experimental value. As the Delta is only a resonance, this suggests that it makes more sense to fix the parameters of the Skyrme model using other experimental values.

This was done by R. Battye, N. Manton and P. Sutcliffe[10] who have used the mass of the spinless nuclei B=8 and B=12. The absence of spin implies that one does not need to quantise the zero mode, avoiding the deformation problem.

Another problem of the Skyrme model is that when the mass term vanishes or takes a small value, the classical solutions with the lowest energy have the shape of hollow shells. This is very different from what is seen from experiments. One way to fix this problem is to take for the pion mass a value much larger than the experimental value [5][9].

To avoid this problem, we propose to consider other mass terms and see if one can have non shell like solutions for large baryon numbers with a physically reasonable value of the pion mass.

The question of whether the modifications of the mass term we consider are really possible within QCD seems to be unresolved at present. However, in view of the importance of such modifications for the properties of multiskyrmions which model the properties of real nuclei, we decided to proceed further in the hope that the possible uncertainties can be solved or cured in the future.

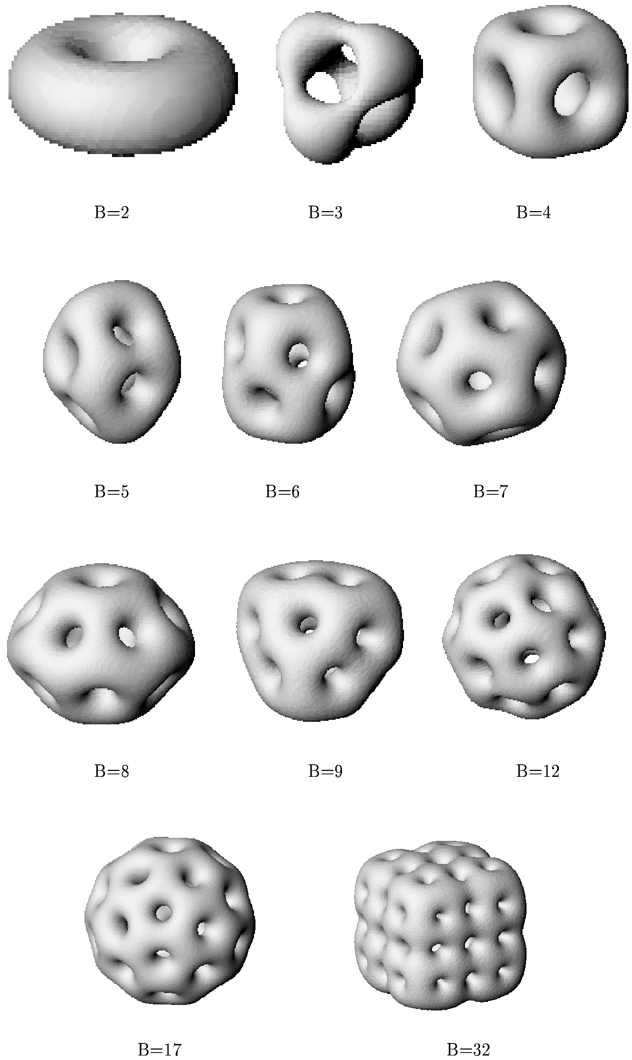


Figure 1: Baryon density for several Skyrmion configurations ($m = 0$). The presented B=32 is a saddle point configuration.

2 New Mass Term

The mass term appearing in (3) is not unique[11][12]. To see this, we consider the parametrisation of the Skyrme field in terms of the pion field $U = \sigma 1 + i\vec{\tau}.\vec{\pi}$. Then it is very easy to show that the unitary constraint $U^\dagger U = 1$ implies that $\sigma^2 + |\vec{\pi}|^2 = 1$ and that in the limit where $|\vec{\pi}|^2 << 1$,

$$V = m_\pi^2 Tr(1 - U) = m_\pi^2(1 - \sigma) = m_\pi^2|\vec{\pi}|^2 + O(|\vec{\pi}|^4),$$

showing that the parameter m_π appearing in (4) can indeed be interpreted as the pion mass.

However, the mass term (4) is not the only one that can be interpreted as a pion mass term. In particular the expression

$$V_p = \frac{m_\pi^2}{p^2} Tr(1 - U^p) \qquad (4)$$

has the same asymptotic value in the limit of low pion mass:

$$V_p = \frac{m_\pi^2}{p^2} Tr(1 - U^p) \tag{5}$$

$$= -m_\pi^2 \left[\vec{\pi}^2 - \frac{p^2}{12}\vec{\pi}^4 + \frac{p^4}{360}\vec{\pi}^6 - ... \right]. \tag{6}$$

In principle we can take any linear combination of V_p for different values of p, but in what follows, we will only consider a few fixed values of p.

Notice also that if one takes the limit $p \to 0$, the potential term does not vanish. This case will turn out to be particularly interesting.

Having introduced this new mass term, we must now find out what difference it makes to the solutions and their properties.

3 Rational Map Ansatz

The only way to compute static Skyrmion solutions is to solve the field equations numerically. This is rather difficult and time consuming, so we decided instead to start the study of the properties of the solutions of the new massive Skyrme model by using the so called rational map ansatz [13] which, in the massless case, provides a very good approximation to the static solutions.

The rational map ansatz involves the introduction of the spherical coordinates in $\mathbf{R^3}$, so that a point $\mathbf{x} \in \mathbf{R^3}$ is given by a pair (r, ξ), where $r = |\mathbf{x}|$ is the distance from the origin, and ξ is a Riemann sphere coordinate giving the point on the unit two-sphere which intersects the half-line through the origin and the point $\mathbf{x}$, *ie* $\xi = \tan(\frac{\theta}{2})e^{i\varphi}$, where θ and φ are the usual spherical coordinates on the unit sphere.

Then one observes [14] that a general $SU(2)$ matrix, U, can always be written in the form

$$U = \exp(if(2P - \mathcal{I})) \tag{7}$$

where f is real and P is a 2×2 hermitian projector *ie* $P = P^2 = P^\dagger$. The rational map ansatz assumes that f depends only on the radial coordinate, *ie* $f = f(r)$, and that the projector depends only on the angular coordinates,*ie* $P(\xi, \bar{\xi})$.

The projector is then taken in the form

$$P = \frac{\mathbf{f} \otimes \mathbf{f}^\dagger}{|\mathbf{f}|^2} \tag{8}$$

where $\mathbf{f}(\xi)$ is a 2-component vector and its entries are degree k polynomials in ξ. It is obvious from (8) that $\mathbf{f}$ is defined up to an overall factor and that one can thus write in full generality $\mathbf{f} = (1, R)^t$, where $R(\xi)$ is the ratio of $R = \frac{f_1}{f_2}$. It is easy to show that the winding number Q of the field configuration, *ie* the baryon number, is given by the higher degree of the polynomial f_1 and f_2.

For $B = 1$ this ansatz produces the exact one Skyrmion field solution. When $B > 1$ the ansatz (7) is not compatible with the full equation of motion, so the ansatz cannot produce any exact multi-Skyrmion configurations. However, as was shown in [13] and [9], it gives approximate field configurations which turn out to be very close to the numerically computed minimal energy states. To do this one selects a specific map $\mathbf{f}$ and puts it into the Skyrme energy functional (3). Performing the integration over the angular coordinates results into the following one-dimensional energy functional for $f(r)$

$$E = \frac{1}{3\pi} \int \left(r^2 f'^2 + 2B(f'^2 + 1)\sin^2 f + \mathcal{I}\frac{\sin^4 f}{r^2} \right) dr$$

$$+ \quad 2\frac{m^2}{p^2} r^2 \left[1 - \cos(pf)\right] \Big) dr, \tag{9}$$

where $\mathcal{I}$ denotes the integral

$$\mathcal{I} = \frac{1}{4\pi}\int\left(\frac{1+|\xi|^2}{1+|R|^2}\left|\frac{dR}{d\xi}\right|\right)^4 \frac{2i\ d\xi d\bar{\xi}}{(1+|\xi|^2)^2}\,. \tag{10}$$

The values of $\mathcal{I}$ have already been calculated; in what follows we take its values from [9]. The equation for the profile function f is then given by

$$\begin{aligned} \ddot{f}[r^2+2\sin^2 f] \quad &+ \quad 2r\dot{f} + 2(\dot{f}^2-1)B\,\sin f\cos f \;-\; I\,\frac{2\sin^3 f\cos f}{r^2} \\ &- \quad \frac{m^2}{p}r^2\,\sin(pf) \;=\; 0. \end{aligned} \tag{11}$$

If we take a close look at the potential density,

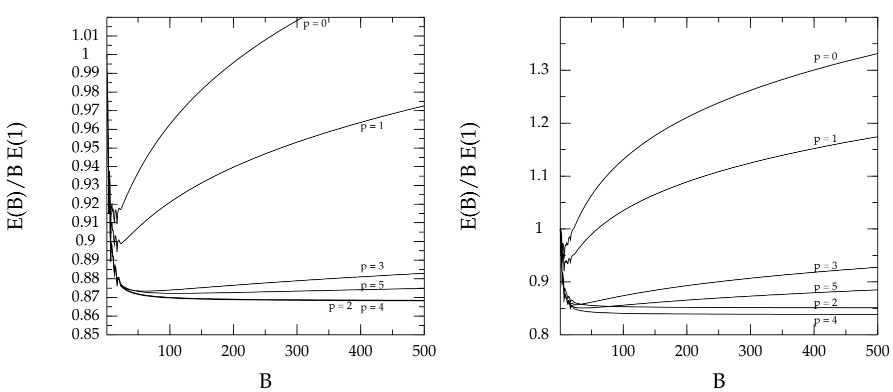
$$V(r) = \frac{m^2}{p^2}(1-\cos(pf(r))) \tag{12}$$

we notice that at the origin, where $f=\pi$, we have

$$\begin{aligned} p=0 \quad &: \quad V(0) = m^2\frac{\pi^2}{2} \\ p\,\text{odd} \quad &: \quad V(0) = 2m^2 \\ p\,\text{even} \quad &: \quad V(0) = 0 \end{aligned}$$

Thus we see that when p is null or odd, the energy density does not vanish at the origin, *i.e.* at the centre of the configuration. This means that shell like configurations are not favoured and that one can expect that for large enough values of the mass, solutions will have a non-shell like shapes.

To analyse this further, we have computed low energy configurations for different values of the mass and the parameter p using the rational map ansatz. The results are summarised in figure (2) in which we plot the normalized energy



m=0.361 m=1.3

Figure 2: Normalized energy (13) of multiskyrmion configurations for (a) $m = 0.362$, $E(1) = 1.274$. (b) $m = 1.300$, $E(1) = 1.486$

$$En = \frac{E(B)}{B\,E(1)} \tag{13}$$

as a function of the baryon number B, for several values of the mass [1] and for p from 0 to 5.

The normalized energy (13) is a dimensionless energy which describes the binding of the configuration by comparing it to that of the $B = 1$ solution. Note that when $En > 1$ the B multiskyrmion configuration has an energy larger than the energy of B single Skyrmions thus showing that this configuration is unstable. The jagged curve near the origin is caused by the value of $\mathcal{I}$ which varies rapidly when B is small. When $B > 22$, we have taken $\mathcal{I} = 1.28B^2$ (see [9]) and so the curves are smooth.

Looking at the figures we see that, as B becomes very large, the normalized energy converges to a finite value when p is even, but that it slowly diverges when p is odd. This was also observed by Battye and Sutcliffe [5] in the case of $p = 1$. Even for large values of the mass, the curve for $p = 3$ crosses the value of $En = 1$ but only for a very large value of the baryon number. On the other hand, the curve for $p = 0$, crosses the bound state threshold for a relatively small baryon number, indicating that in this case the solution is likely to have non shell like shape for a low baryon number even when the value of the mass if relatively small, *ie* physical.

We also see that, for a given parity of p, the energy at a given value of B decreases as we increase p by a multiple of 2, *ie* $En_p(B) > En_{p+2}(B)$.

In [12] we have shown that for odd values of p, and for $p = 0$, the binding energy has a maximum and for larger B the shell configurations are unstable because $E/B \overset{B>>}{\rightarrow} \sqrt{m}\,B^{1/8}$. For even p, $p \neq 0$, we have shown that the ratio E/B and the binding energy per baryon unit both approach constant values $\sim \sqrt{m}$ for large m.

We have also computed a few exact solution numerically and have found that the rational map ansatz works well when $m \neq 0$. So far we have tested this for $B = 4$ and $B = 2$ both for $p = 1$ and $p = 2$.

One property worth investigating for these low energy configurations is their ability to decay into two or more shells of smaller baryon charges.

To do this we have computed the derivative of the energy with respect to B and compared the obtained values with the energy per baryon of some small B configurations of low energy (typically $B = 2$, $B = 4$, $B = 7$ and $B = 17$). When the value of the derivative is larger than the energy per baryon of other configurations, it implies that the larger configuration can decay into two shells.

We summarize our observations in the table (1) where we have given the threshold value corresponding to several decay modes. The threshold values are the values of B above which the decay is always possible, but sometimes some configurations with lower values of B (smaller than 32) can decay in such a mode too (but only for very special values of B). The values given with a '$\geq$' refer to the values obtained by comparing the energy of the configuration directly (low B) instead of using the derivative of the energy.

4 Conclusions

We have shown that the Skyrme model can have several types of the so called mass terms for which the mass parameter can still be interpreted as the pion mass. We have studied a specific class of terms parametrised by an integer, which we called p.

The properties of the solutions depends very much on p. When p is odd or zero, shell like configurations are unstable for larger values of B and thus, the minimum energy solutions are likely to be non-shell like thus resembling more real nuclei.

[1] In our numerical calculations we take for pions $m = m_\pi = 0.36192$ and for kaons $m_{st} = 1.29996$. In what follows, in the text and in the captions, we refer to those values as $m = 0.362$ and 1.300 respectively.

m=0.362					
p	$B=1$	$B=2$	$B=4$	$B=7$	$B=17$
0	>95	>70	≥ 26	≥ 18	≥ 20
1	>380	>250	≥ 50	≥ 28	≥ 27
3	-	-	-	>410	>100
5	-	-	-	-	>390
m=1.300					
p	$B=1$	$B=2$	$B=4$	$B=7$	$B=17$
0	≥ 18	≥ 18	≥ 8	≥ 7	≥ 17
1	≥ 27	≥ 25	≥ 14	≥ 8	≥ 18
3	-	>480	>120	≥ 45	≥ 34
5	-	-	>350	>110	≥ 33

Table 1: Decay of low energy configurations into sub shells. Each column corresponds to a decay mode and the baryon charge given corresponds to the threshold from which the decay is always possible.

When p is even, shell like configurations are very stable for large Bs. These solutions are very much like those seen in the massless case.

A special mass term is obtained by taking the limit $p \to 0$. This results in configurations with interesting properties. In particular, the shell like configurations are in this case unstable for low values of the mass at low baryon number indicating that this variant of the Skyrme model is a good candidate to describe nuclei.

More investigations are needed to compute the exact solutions of the Skyrme model with the new mass terms and to study their properties.

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