Two-loop vertex integrals in general kinematics

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Abstract

New method of calculation of master integrals using differential equations and asymptotical expansion is presented. This method leads to the results exact in space-time dimension D having the form of the convergent power series. As an application of this method, we consider the two–loop master integrals of the vertex topology in general kinematics.

1 Introduction

Multiloop calculations become conventional nowadays. This is partly because of the high precision of modern high-energy experiments. These calculations, in comparison with one-loop calculations, are technically much more complicated. Systematic approach comprises the tensor reduction and integration-by-parts (IBP) reduction [1, 2], which effectively reduce the integral corresponding to any multi-loop diagram, to the linear combination of the master integrals (MIs). The coefficients in this linear combination often have poles of some finite order at $\epsilon = 0$ (space-time dimension is $D = 4 - 2\epsilon$). Therefore, it is necessary to know the expansion of the master integrals in ϵ, at least, a few first terms of this expansion. Successful method of calculating this expansion term-by-term is based on the application of the differential equations method [3, 4, 5, 6, 7]. The results for two-loop diagrams are usually expressed in terms of (generalized) harmonic polylogarithms [6, 7] (see, however, Ref. [5]). Due to rapid increase of calculational complexity when increasing the order of expansion, the calculations using this method are typically done up to $O(\epsilon)$ terms. However, it may be not sufficient for the calculation of some diagrams. Therefore, it is highly desirable to calculate master integrals exactly in D dimensions. One of the most powerful methods of such calculation is the application of Mellin–Barnes transformation [8, 9]. However, for the case of several external invariants and/or many internal lines this method becomes hard to apply.

In this report, we present a method of obtaining the asymptotic expansion (AE) in inverse powers of some large scale s of multi-loop master integrals exact in space–time dimension D. Moreover, this expansion has a finite radius of convergence which can be determined from ODE theory, thus being a power series representation accessible for numerical calculations in definite kinematic region. In particular, we consider the two-loop vertex master integrals in general kinematics. This report is mostly based on our paper [10], however, the results concerning non-planar two-loop vertex master integral are presented for the first time.

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2 Differential equations for master integrals

Let us consider loop integral

$$\int \frac{d^D l_1\, d^D l_2 \dots}{D_1^{\alpha_1} D_2^{\alpha_2} \dots} t^{\mu\nu\dots}, \tag{1}$$

where D_k are quadratic in l_i polynomials and $t^{\mu\nu\dots}$ is a tensor polynomial with respect to l_i. The first step is to express it as a linear combination of different tensors, constructed of external momenta and $g_{\mu\nu}$ with the coefficients being scalar integrals. After this step, the calculation of the tensor integral is reduced to that of scalar integrals. The next step is to use IBP identities and express the scalar integrals via some finite number of master integrals. IBP identities are based on the fact that the integral of the total derivative is zero in dimensional regularization:

$$0 = \int d^D l_1\, d^D l_2 \dots \frac{\partial}{\partial l_i^\mu} q^\mu \left[D_1^{\alpha_1} D_2^{\alpha_2} \dots\right]^{-1} \tag{2}$$

If we calculate the derivative explicitly, we obtain recurrence relation for each choice of $\partial/\partial l_i^\mu$, q^μ, and α_k, [1]. Note, that some of these relations (obtained for $q = l_i$) contain the space-time dimension D. The infinite system of the recurrence relations is usually solved starting from the simplest relation and continuing until the required integrals are expressed as a linear combination of the master integrals [2]. Coefficients of this linear combination are functions of the external invariants and D, often having poles at some integer D, including $D = 4$.

Master integrals depend on the external invariants. Derivatives with respect to the external invariants can be expressed via the derivatives with respect to the external momenta. E.g., for two external momenta $J = J(p^2, q^2, s = (p+q)^2)$.

$$\frac{\partial J}{\partial s} = \frac{(q\cdot p)p - p^2 q}{2((q\cdot p)^2 - q^2p^2)} \cdot \frac{\partial J}{\partial p} \tag{3}$$

Differentiating master integrals with respect to the external momenta explicitly and then using IBP identities, we can express the right-hand side via the master integrals of the same topology and its subtopologies [4]. Note, that differential equations may give nontrivial information for the integrals depending on two or more invariants. If the given topology contains only one master integral, then we obtain differential equation for this master integral

$$\frac{\partial}{\partial s} J = f(s)J + h(s)\,, \tag{4}$$

where $h(s)$ depends on simpler integrals and is assumed to be known. The solution of this equation is

$$J(s) = J_h(s)\left[\int_{s_0}^{s} J_h^{-1}(s')h(s')ds'\right] + J_h(s)J_h^{-1}(s_0)J(s_0)\,, \tag{5}$$

where $J_h(s) = \exp[\int f(s)ds]$ is the solution of the corresponding homogeneous equation. The choice of the point s_0, where the boundary conditions are posed, is very important for further transformations. While for massive propagator-type integrals (and in general, for two-scale integrals) there are, in fact, two natural possibilities, $s_0 = 0$ or $s_0 = \infty$, for the integrals with three and more invariants there are more options. In Ref. [7] the boundary conditions were fixed in the point where the denominator in Eq. (3) vanishes. In our approach we are interested in the asymptotical expansion for large s and assume that it is already known for $J_h^{-1}(s)h(s)$. Then it is convenient to let s_0 tend to infinity:

$$J(s) = J_h(s)\left(\left[\int_{s_0}^{s} J_h^{-1}(s')h(s')ds'\right] + J_h^{-1}(s_0)J(s_0)\right)\Bigg|_{s_0\to\infty}\,. \tag{6}$$

For MIs considered in the present paper, the correct choice of the ϵ sign allows to calculate the limit $s_0 \to \infty$ separately in the first and the second terms in Eq. (6). In this case, we can

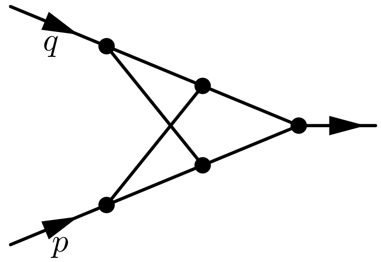


Figure 1: Two distinct topologies of the two-loop vertex integrals

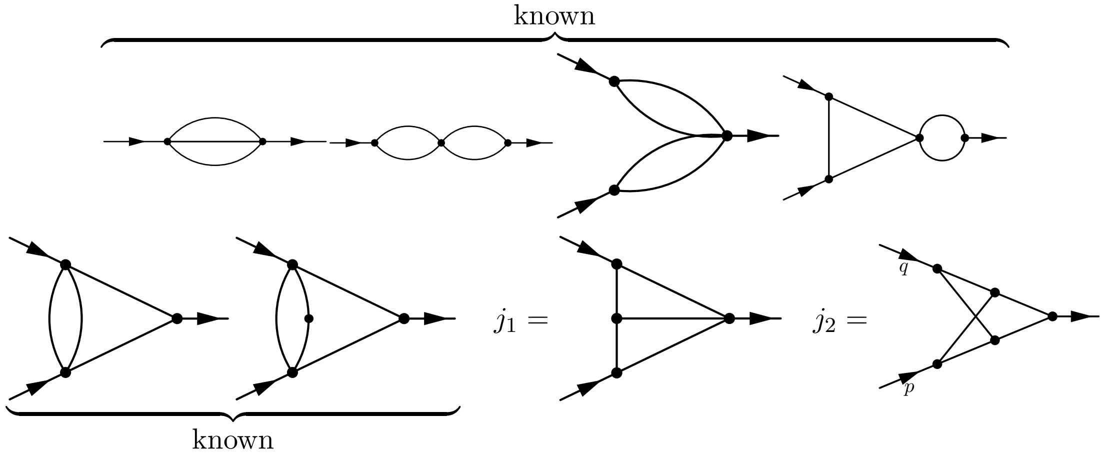


Figure 2: Master integrals for two-loop vertex integrals

integrate term-by-term the AE of the integrand in Eq. (6). At this stage, the problem of finding of the whole AE of J is reduced to the calculation of its main asymptotics. In the more general case, when the limits can not be taken separately, the problem is reduced to the calculation of few first terms of the asymptotics of J.

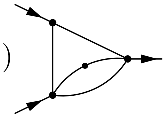
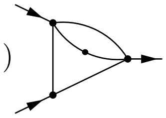
3 Two-loop vertex integrals

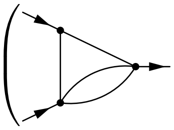
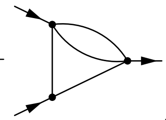
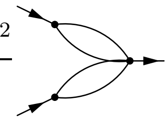
In this Section we apply the differential equation technique for the calculation of the two-loop massless vertex integrals with all external legs off-shell. There are two distinct biggest topologies depicted in Fig. 1. Using the IBP reduction procedure, we find the master integrals depicted in Fig. 2.

The two most complex master integrals with 6 and 5 propagators are not known in D dimensions. Let us consider the master integral j_1 with 5 propagators. Differentiating with respect to s, we obtain the following differential equation

$$\frac{\partial}{\partial s} j_1 = \frac{(3-D)y}{2(y^2 - p^2 q^2)} j_1 + \frac{\tilde{h}_1(s, p^2, q^2)}{2(y^2 - p^2 q^2)}\,, \tag{7}$$

where $y = p \cdot q$, and

$$\tilde{h}_!(s,p^2,q^2) = -\,(y+p^2)$$  $$-\,(y+q^2)$$ 

$$+\,\frac{(3-D)(3D-10)}{D-4}\Bigg($$  $$+$$  $$\Bigg) + \frac{2(3-D)^2}{D-4}$$  $,$

(8)

The solution of the homogeneous part of Eq. (7) is

$$j_{1h}(s,p^2,q^2) = (-2y)^{-1+2\epsilon}\left(1-\frac{p^2q^2}{y^2}\right)^{-1/2+\epsilon} \overset{s\to\infty}{\longrightarrow} (-s)^{-1+2\epsilon}\,. \tag{9}$$

The choice $\epsilon > 0$ allows us to take the limit separately in the first and the second terms in Eq. (6). Indeed, the leading asymptotics of $[y^2 - p^2q^2]^{-1}\tilde{h}(s,p^2,q^2)$ is $s^{-2+\epsilon}$. Thus, the integrand in Eq. (6) behaves as $s^{-1-\epsilon}$ and the integral converges when its lower limit is replaced by ∞. Therefore, we can confine ourselves to calculation of the main asymptotics of j_1 that determines the limit

$$\left(j_{1h}^{-1}(s_0,p^2,q^2)j_1(s_0,p^2,q^2)\right)\big|_{s_0\to\infty}\,. \tag{10}$$

We calculate this asymptotics with the help of Feynman parametrization

$$j_1(s,p^2,q^2) = \frac{\Gamma[1+2\epsilon]}{(4\pi)^D}\int_0^\infty \frac{\prod_{i=1}^5 dx_i\,\delta(1-x_{12345})(x_{13}x_{24}+x_5x_{1234})^{-1+3\epsilon}}{(-sx_3x_4x_5 - p^2x_3(x_5x_{12}+x_1x_{24}) - q^2x_4(x_5x_{12}+x_2x_{13}))^{1+2\epsilon}}\,. \tag{11}$$

where $x_{i_1,i_2,\ldots} = x_{i_1} + x_{i_2} + \cdots$. The region, providing the main asymptotics, is

$$0 \leqslant x_3 \sim x_4 \ll x_1 \sim x_2 \sim x_5 \leqslant 1\,. \tag{12}$$

The result of integration is

$$j_1(s,p^2,q^2) \overset{s\to\infty}{\longrightarrow} \frac{\pi^2}{(4\pi)^D}\frac{\Gamma[2\epsilon]}{\epsilon\sin(\pi\epsilon)\sin(2\pi\epsilon)}\frac{(-s)^{-1+2\epsilon}}{(-q^2)^{2\epsilon}(-p^2)^{2\epsilon}}\,. \tag{13}$$

We use the following expansion

$$[y^2-p^2q^2]^{-1}j_{1h}^{-1}(s,p^2,q^2) = (-s)^{-1-2\epsilon}\frac{2\Gamma[\epsilon]}{\Gamma[2\epsilon]}\sum_{k,l=0}^{\infty}\Gamma\left[\begin{matrix}k+l+1+2\epsilon,\,k+l+1+\epsilon\\ k+1+\epsilon,\,l+1+\epsilon\end{matrix}\right]\frac{(p^2/s)^k(q^2/s)^l}{k!\;l!}\,, \tag{14}$$

where $\Gamma\left[\begin{smallmatrix}x_1,x_2,\ldots\\ y_1,y_2,\ldots\end{smallmatrix}\right] = \frac{\Gamma[x_1,x_2,\ldots]}{\Gamma[y_1,y_2,\ldots]} = \frac{\Gamma[x_1]\Gamma[x_2]\cdots}{\Gamma[y_1]\Gamma[y_2]\cdots}$. Substituting (14) in Eq. (6) and integrating over s, we obtain

$$
\begin{aligned}
j_1 = {}& (-2y)^{-1+2\epsilon}\left(1-\frac{p^2q^2}{y^2}\right)^{-1/2+\epsilon}(4\pi)^{-D} \\
&\times\left\{\Gamma[2\epsilon]\frac{\pi^2}{\epsilon\sin(\pi\epsilon)\sin(2\pi\epsilon)}(-q^2)^{-2\epsilon}(-p^2)^{-2\epsilon}\right. \\
&+4\Gamma[2\epsilon]\pi^2\cot^2(\pi\epsilon)(-q^2)^{-\epsilon}(-p^2)^{-\epsilon}(-s)^{-2\epsilon}\sum_{k,l=0}^{\infty}\frac{(p^2/s)^k(q^2/s)^l}{k!\ l!}\Gamma\left[\begin{matrix}l+k+2\epsilon, l+k+1+\epsilon\\ 1+\epsilon, l+1+\epsilon, k+1+\epsilon\end{matrix}\right] \\
&+\epsilon^2\Gamma[-\epsilon]^3\Gamma[\epsilon]\sum_{k,l=0}^{\infty}\frac{(p^2/s)^k(q^2/s)^l}{k!\ l!} \\
&\times\left\{-(-s)^{-4\epsilon}\Gamma\left[\begin{matrix}k+l+4\epsilon, k+l+1+2\epsilon\\ 1-3\epsilon, l+1+4\epsilon, 1+k+\epsilon\end{matrix}\right]\frac{1}{2\epsilon}{}_3F_2\left[\left.\begin{matrix}-k, 3\epsilon, 2\epsilon\\ 1+l+4\epsilon, 1+2\epsilon\end{matrix}\right|1\right]\right. \\
&-(-s)^{-2\epsilon}(-q^2)^{-2\epsilon}\Gamma\left[\begin{matrix}l+k+1-2\epsilon, k+l+2\epsilon\\ 1-3\epsilon, l+1-2\epsilon, 1+k+\epsilon\end{matrix}\right]\frac{1}{2\epsilon}{}_3F_2\left[\left.\begin{matrix}-k, 3\epsilon, 2\epsilon\\ -l-k+2\epsilon, 1+2\epsilon\end{matrix}\right|1\right] \\
&+(-s)^{-3\epsilon}(-p^2)^{-\epsilon}\Gamma\left[\begin{matrix}2\epsilon, \epsilon, k+l+3\epsilon, k+l+1+\epsilon\\ 1-3\epsilon, 3\epsilon, 1+l+2\epsilon, 1+k+\epsilon\end{matrix}\right] \\
&\left.\left.+(-s)^{-\epsilon}(-p^2)^{-\epsilon}(-q^2)^{-2\epsilon}\Gamma\left[\begin{matrix}2\epsilon, k+l+1-\epsilon, k+l+\epsilon\\ 1-\epsilon, 1+l-2\epsilon, 1+k+\epsilon\end{matrix}\right]+\begin{pmatrix}p\leftrightarrow q\\ k\leftrightarrow l\end{pmatrix}\right\}\right\}.
\end{aligned}
\tag{15}
$$

Here, the replacement $\binom{p\leftrightarrow q}{k\leftrightarrow l}$ applies to all the terms in inner braces. Symbol δ_c imposes the constraint, c, on the limits of multiple sums so that, e.g.,

$$
\sum_{n,m}\delta_{m+n=0}f(m,n)\equiv\sum_{\substack{n,m\\ m+n=0}}f(m,n)\,.
$$

Note, that all hypergeometric functions in Eq. (15) are reduced to the polynomials in ϵ. As known, j_1 is finite at $D=4$. It is easy to check that ϵ-poles in individual terms cancel and the finite part reproduce the AE of the well-known result of [11].

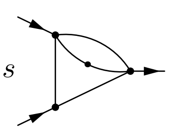
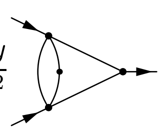
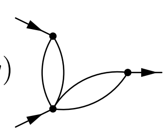
Let us consider the homogeneous system of the differential equations in s for all MIs appearing in the subtopologies of j_1. The point $s=\infty$ is a regular singular point of this system. Then, from the theory of ODE, it follows that the above expansion has a finite radius of convergence, determined by the closest singularity of the coefficients of this system. In our case, it is located in the point $y^2=p^2q^2$. Thus, the convergence region of Eq. (15) is determined by the condition

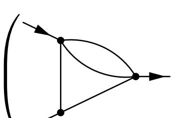
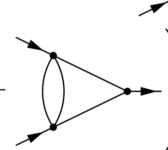
$$
|s|>|\sqrt{-p^2}\pm\sqrt{-q^2}|^2\,. \tag{16}
$$

In the kinematical region where p^2 (or q^2) is large enough the expansion (15) is not valid anymore. To obtain the expansion of the integral j_1 in inverse powers of p^2, valid for large enough p^2, we consider the differential equation for j_1 with respect to p^2:

$$
\frac{\partial j_1(s,p^2,q^2)}{\partial p^2}=\left(\frac{D-4}{p^2}-\frac{(3-D)(y+q^2)}{2[y^2-p^2q^2]}\right)j_1(s,p^2,q^2)+\frac{1}{2[y^2-p^2q^2]}\tilde{h}_1'(s,p^2,q^2)\,, \tag{17}
$$

where

$\tilde{h}_1'(s,p^2,q^2)=s$  $+\dfrac{sy}{p^2}$  $-2\dfrac{(D-3)^2}{p^2(D-4)}(p^2+y)$ 

$+\dfrac{(3D-10)(D-3)}{p^2(D-4)}(p^2+y)\Big($  $+$  $\Big)$. (18)

The homogeneous solution of Eq. (18) is

$$j'_{1h}(s,p^2,q^2) = (-p^2)^{-2\epsilon}(2y)^{-1+2\epsilon}\left(1-\frac{p^2q^2}{y^2}\right)^{-1/2+\epsilon} \overset{p^2\to\infty}{\longrightarrow} (-p^2)^{-1}\,. \tag{19}$$

Similar to the previous case, the choice $\epsilon > 0$ allows us to take the limit separately in the first and the second terms in Eq. (6). The calculation of the main asymptotics of $j_1(s,p^2,q^2)$ at $p^2 \to \infty$ is slightly more difficult than that $s \to \infty$. Using the parametric representation we have

$$j_1(s,p^2,q^2) = \frac{\Gamma[1+2\epsilon]}{(4\pi)^D}\int_0^\infty \frac{\prod_{i=1}^5 dx_i\,\delta(1-\sum x_i)(x_{13}x_{24}+x_5x_{1234})^{-1+3\epsilon}}{(-sx_3x_4x_5 - p^2x_3(x_5x_{12}+x_1x_{24}) - q^2x_4(x_5x_{12}+x_2x_{13}))^{1+2\epsilon}}\,. \tag{20}$$

As it is well-known, the sum in the argument of the δ-function in Eq. (20) can run over arbitrary subset of x_i. We choose it as $\sum x_i = x_{1235}$. The region, providing the main asymptotics, is

$$0 \leqslant x_3 \ll x_1 \sim x_2 \sim x_4 \sim x_5 \lesssim 1\,. \tag{21}$$

After the integration over x_3 we obtain

$$j_1(s,p^2,q^2) \overset{p^2\to\infty}{\longrightarrow} \frac{\Gamma[2\epsilon]}{(4\pi)^D}\frac{(-q^2)^{-2\epsilon}}{-p^2}\int_0^\infty \frac{dx_1dx_2dx_4dx_5\delta(1-x_{125})\beta^{-2\epsilon}x_4^{-2\epsilon}}{(\beta+x_4x_1)(\beta+x_4x_{15})^{1-3\epsilon}}\,, \tag{22}$$

where $\beta = x_1x_2+x_2x_5+x_5x_1$. Now we use the Mellin–Barnes representation to separate β and x_4x_{15} in denominator and take the integrals over x_i. The final integration over s can be done by applying the Barnes second lemma (see, e.g., [12]) and results in

$$j_1(s,p^2,q^2) \overset{p^2\to\infty}{\longrightarrow} \frac{1}{(4\pi)^D}\frac{(-q^2)^{-2\epsilon}}{-p^2}\frac{\Gamma[1-\epsilon]^3\Gamma[2\epsilon]}{\epsilon(1-2\epsilon)^2\Gamma[1-3\epsilon]}\,{}_3F_2\left[{1,1,1-\epsilon \atop 2-2\epsilon,2-2\epsilon}\Big|1\right] \tag{23}$$

The final result for the AE of $j_1(s,p^2,q^2)$ in the region of large p^2 reads

$$\begin{aligned} j_1 =& \frac{(2y)^{-1+2\epsilon}(-p^2)^{-2\epsilon}}{(4\pi)^D}\left(1-\frac{p^2q^2}{y^2}\right)^{-1/2+\epsilon}\Bigg\{(-q^2)^{-2\epsilon}\frac{\Gamma[1-\epsilon]^3\Gamma[2\epsilon]}{\epsilon(1-2\epsilon)^2\Gamma[1-3\epsilon]}\,{}_3\mathrm{F}_2\left[{1,\;1,\;1-\epsilon \atop 2-2\epsilon,2-2\epsilon}\Big|1\right] \\ &+\frac{\Gamma[\epsilon]}{\Gamma[2\epsilon]}\sum_{k,l=0}^{\infty}\left(\frac{s}{p^2}\right)^k\left(\frac{q^2}{p^2}\right)^l\Bigg[(-q^2)^{-\epsilon}(-p^2)^{-\epsilon}\frac{\Gamma[1-\epsilon]^4\Gamma[\epsilon]^2}{2\epsilon\Gamma[1-2\epsilon]^2}\,B_{k,l} \\ &+\frac{\Gamma[1-\epsilon]^2}{2\epsilon\Gamma[1-3\epsilon,1-2\epsilon]}\Big((-q^2)^{-\epsilon}(-p^2)^{-\epsilon}C^{(1)}_{k,l} + (-s)^{-2\epsilon}(-q^2)^{-\epsilon}(-p^2)^{\epsilon}C^{(2)}_{k,l} \\ &+(-s)^{-2\epsilon}(-q^2)^{-2\epsilon}(-p^2)^{2\epsilon}C^{(3)}_{k,l} + (-p^2)^{-2\epsilon}(C^{(4)}_{k,l}+C^{(5)}_{k,l}) \\ &+(-s)^{-2\epsilon}(C^{(6)}_{k,l}+C^{(7)}_{k,l}) + (-q^2)^{-2\epsilon}C^{(8)}_{k,l}\Big)\Bigg]\Bigg\}\,, \end{aligned} \tag{24}$$

$$A_{k,l,m,n} = \frac{(-1)^{n+m}}{n!m!(k-m)!(l-n)!}\,\Gamma\left[{k+l-m-n+1+2\epsilon,\,k+l-m-n+1+\epsilon \atop k-m+1+\epsilon,\;l-n+1+\epsilon}\right]\,, \tag{25}$$

$$B_{k,l} = \frac{1}{k+l+\epsilon}(A_{k,l,1,0}\delta_{k>0} - A_{k,l,0,1}\delta_{l>0} - A_{k,l,0,0})\,, \tag{26}$$

$$C^{(1)}_{k,l} = \frac{\pi^2}{\sin[2\pi\epsilon]\sin[\pi\epsilon]}\frac{\Gamma[2\epsilon]}{\Gamma[3\epsilon]}\frac{2}{k!l!}\,\Gamma\left[{k+l+3\epsilon,\,k+l+\epsilon \atop k+2\epsilon,\,l+1+\epsilon}\right]\,, \tag{27}$$

$$C^{(2)}_{k,l} = \frac{\pi\Gamma[2\epsilon,1-3\epsilon]}{\sin[2\pi\epsilon]}\,\frac{2k}{k!l!}\,\Gamma\left[{k+l+\epsilon,\,k+l-\epsilon \atop k+1-2\epsilon,\,l+1+\epsilon}\right]\,, \tag{28}$$

$$C^{(3)}_{k,l} = \frac{\pi\sin[\pi\epsilon]}{\sin^2[2\pi\epsilon]}\Gamma[2\epsilon, 1-3\epsilon]\frac{-2}{k!l!}\, k\,\Gamma\begin{bmatrix} k+l-\epsilon, k+l-2\epsilon \\ k+1-2\epsilon, l+1-2\epsilon \end{bmatrix}, \tag{29}$$

$$C^{(4)}_{k,l} = \frac{\pi}{\sin[2\pi\epsilon]}\Gamma\begin{bmatrix} k+l+1+\epsilon, k+l+2\epsilon \\ k+1+\epsilon, l+1+\epsilon \end{bmatrix}\frac{\Gamma[-\epsilon]}{k!l!}\, {}_3\mathrm{F}_2\left[\begin{matrix} 1, -l, 1-\epsilon \\ 1+k+\epsilon, 1+2\epsilon \end{matrix}\Big|1\right], \tag{30}$$

$$C^{(5)}_{k,l} = \frac{\pi^2}{\sin^2[2\pi\epsilon]}\frac{\Gamma[2\epsilon]}{\Gamma[3\epsilon]}\frac{2}{k!l!}\,\Gamma\begin{bmatrix} k+l+3\epsilon, k+l+2\epsilon \\ k+2\epsilon, l+1+2\epsilon \end{bmatrix}, \tag{31}$$

$$C^{(6)}_{k,l} = \frac{\pi\Gamma[-\epsilon]}{\sin[2\pi\epsilon]}\frac{k}{k!l!}\,\Gamma\begin{bmatrix} k+l+2\epsilon, k+l \\ k+1, l+1+\epsilon \end{bmatrix} {}_3\mathrm{F}_2\left[\begin{matrix} -l, 2\epsilon, 1-\epsilon \\ k+1, 1+2\epsilon \end{matrix}\Big|1\right]\delta_{l+k>0}\,, \tag{32}$$

$$C^{(7)}_{k,l} = \sum_{m=1}^{k}\sum_{n=0}^{l}\frac{A_{k,l,m,n}}{k+l}(3m\epsilon - n(1-\epsilon) - 2\epsilon(1+2\epsilon))\, m\,\Gamma[m+n, m+n-1-\epsilon, 2\epsilon-m, -2\epsilon-n]\,. \tag{33}$$

$$\begin{aligned} C^{(8)}_{k,l} = \sum_{m=0}^{k}\sum_{n=0}^{l}\frac{A_{k,l,m,n}}{k+l}\Big[&(n(1-\epsilon)-3m\epsilon)\,\Gamma[m+n, m+n-1-\epsilon, 1-2\epsilon-m, 2\epsilon-n]\delta_{m+n>1} \\ &+ \Gamma[1-2\epsilon, -\epsilon, 2\epsilon](m/2 - n\,\epsilon/(1-2\epsilon))\delta_{m+n=1}\Big]\,. \end{aligned} \tag{34}$$

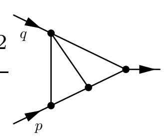
Again, the convergence region is determined by the distance to the closest singular point of differential equation:

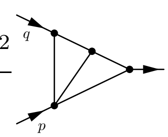
$$|-p^2| > |\sqrt{-s} \pm \sqrt{-q^2}|^2\,. \tag{35}$$

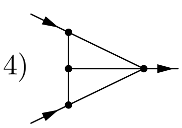
Similar to the previous case, at $D=4$ we reproduce from Eq. (24) the AE of the well-known result of Ref. [11].

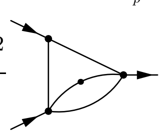
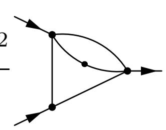
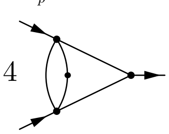
Let us consider now the most complicated master integral j_2 with 6 denominators (non-planar vertex integral), which is symmetrical with respect to permutations of s, p^2, and q^2. The differential equation for this integral has the form

$$\frac{\partial}{\partial s}j_2 = \frac{(D-6)y}{2(y^2-p^2q^2)}j_2 + \frac{\tilde{h}_2(s,p^2,q^2)}{2(y^2-p^2q^2)}\,, \quad (y = p\cdot q) \tag{36}$$

$\tilde{h}_2(s,p^2,q^2) = 2(D-4)\dfrac{y+q^2}{s}$ 

$+ 2(D-4)\dfrac{y+p^2}{s}$ 

$- 2(D-4)$ 

$+ 4\dfrac{y+q^2}{s}$  $+ 4\dfrac{y+p^2}{s}$  $- 4$ 

(37)

Solution of the corresponding homogeneous equation is

$$j_{2h}(s,p^2,q^2) = (-2y)^{D-6}\left(1-\frac{p^2q^2}{y^2}\right)^{\frac{D-6}{2}} \stackrel{s\to\infty}{\longrightarrow} (-s)^{D-6}\,.$$

We can fix the boundary conditions by calculating the leading asymptotics of J for $\epsilon<0$. Using the parametric representation we can represent the master integral J in the form

$$\begin{aligned} j_2(s,p^2,q^2) = &-\frac{\Gamma[2+2\epsilon]}{(4\pi)^D}\times \\ \times \int_0^\infty &\frac{\prod_{i=1}^6 dx_i\,\delta(1-x_{1234})(x_{14}x_{23}+x_{1234}x_{56})^{3\epsilon}}{(-s(y_{245}+y_{136}+y_{56}x_{1234}) - p^2(y_{125}+y_{346}+y_{23}x_{1456}) - q^2(y_{126}+y_{345}+y_{14}x_{2356}))^{2+2\epsilon}}\,, \end{aligned} \tag{38}$$

where $y_{i_1 i_2 \ldots} = x_{i_1} x_{i_2} \cdots$ The main asymptotics is provided by the region

$$x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_5 \sim x_6 \sim 1\,. \tag{39}$$

We have

$$j_2(s,p^2,q^2)(-s)^{2+2\epsilon} \overset{s\to\infty}{\longrightarrow} -\frac{\Gamma[2+2\epsilon]}{(4\pi)^D} \int\limits_0^\infty \frac{\prod_{i=1}^6 dx_i\, \delta(1-x_{1234})(x_{14}x_{23}+x_{56})^{3\epsilon}}{(y_{245}+y_{136}+y_{56})^{2+2\epsilon}} \tag{40}$$

Using the double Mellin–Barnes transform to separate y_{245}, y_{136}, and y_{56} in the denominator and making a change of variables $x_2 = a\eta$, $x_3 = a\bar\eta$, $x_1 = \bar a \xi$, $x_4 = \bar a \bar\xi$, $x_5 = b\bar z$, $x_6 = bz$ ($\bar z = 1-z$ etc.), we can take the integrals over b, a, z, η, and ξ, subsequently. The result is

$$j_2(s,p^2,q^2)(-s)^{2+2\epsilon} \overset{s\to\infty}{\longrightarrow} -(4\pi)^{-D} \frac{\Gamma[-\epsilon]^2}{\Gamma[-2\epsilon,-3\epsilon]} \times$$
$$\times \int\limits_{-i\infty-3\epsilon/2}^{i\infty-3\epsilon/2} \frac{d\omega d\phi}{(2\pi i)^2} \Gamma[1-\omega, 1-\phi, 2\epsilon+\omega+\phi, \epsilon+\omega+\phi, -2\epsilon-\omega, -2\epsilon-\phi] \left(\frac{\Gamma[\omega,\phi]}{\Gamma[\omega+\phi]}\right)^2 \tag{41}$$

It is not difficult to obtain its ϵ expansion up to ϵ^0

$$j_2(s,p^2,q^2) \overset{s\to\infty}{\longrightarrow} \frac{(-s)^{-2-2\epsilon}}{(4\pi)^D} \times$$
$$\times \left(-\frac{1}{\epsilon^4} + \frac{2\gamma}{\epsilon^3} + \frac{\pi^2 - 2\gamma^2}{\epsilon^2} + \frac{4\gamma^3 + 83\zeta(3) - 6\gamma\pi^2}{3\epsilon} - \frac{166}{3}\gamma\zeta(3) - \frac{2}{3}\gamma^4 + 2\gamma^2\pi^2 + \frac{59}{120}\pi^4 + O(\epsilon)\right) \tag{42}$$

Note, that the asymptotical expansion for all integrals entering $\tilde h_2$ is already known. Thus, we have all ingredients needed to obtain the asymptotical expansion of the master integral j_2. To save space, we do not present the explicit form of this expansion. As in the above cases, it has the form of the power series in p^2/s and q^2/s. The coefficients in this series are some finite sums which we did not succeed to rewrite in compact form. The convergence region is determined by the condition (16). We have checked that for $D = 4$ our result for j_2 reproduces the well-known result of Ref.[11].

4 Conclusion

In this report we present the approach to the calculation of the dimensionally regularized multiloop integrals. This approach is based on the differential equations method and asymptotical expansion. As a demonstration of our method, we have calculated the asymptotical expansions of the master integrals depicted in Fig.2 when one of the invariants is large. Moreover, we have shown that these asymptotical expansions have a finite convergence radius determined by Eq.(16), thus being the power series accessible for the numerical calculations. It worth noting that master integral j_1 was known only in $O(\epsilon)$ order so far [7] and this expansion was much more lengthier than our exact representation. Our results can be easily expanded up to any order of ϵ. We have checked that in the limit $\epsilon \to 0$ our results coincides with the well-known result of Ref. [11]. The result for special kinematics when one or two external momenta are on mass shell ($p^2 = 0$ and/or $q^2 = 0$) can be readily obtain from Eqs. (15)-(24).

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