Fermionic and bosonic determinants in QCD at finite temperature

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Abstract

We compute the functional determinant for the fluctuations around the most general self-dual configuration with unit topological charge for 4D $SU(2)$ Yang-Mills with one compactified direction. This configuration is called "instanton with non-trivial holonomy" or "KvBLL caloron" [1, 2]. It is generalization of the usual instantons for the case of non-zero temperature. We extend earlier results of [3] onto arbitrary values of paramiters.

1 Introduction

Since pioneered work of Callan, Dashen and Gross [4], where it was proposed to approximate the QCD path-integral by a superposition of a dilute set of instantons, there was significant success done by many authors which lead to the instanton liquid model [5, 6, 7]. This model has many lattice and phenomenological confirmations. An instanton-like lumpy structure has been observed in lattice studies using techniques like cooling [8, 9, 10, 11, 12, 13]. It explains successfully the chiral symmetry breaking [14], describes hadronic correlators and details of hadronic structure [15, 16, 17] and solves the $U(1)_A$ problem [18].

However, the standard instanton liquid model could not describe confinement [19]. In [3] it was shown analytically that consideration of the more general solutions with non-trivial holonomy (KvBLL calorons) [1, 20, 2] leads to the existence of two phases with phase transition temperature $T_c \simeq 1.1\Lambda$. In [21] diluted non-interacting gaze of the KvBLL calorons was studied in details. This approach is also motivated by lattice observations [22, 23, 24, 25, 26, 27, 28, 29, 30].

The KvBLL caloron [1, 2] is a generalization of the BPST instantons [31] and Harrington-Shepard Instantone with trivial-holonomy [32]. It is a self-dual gauge field configuration periodical in one Euclidean time direction with period $1/T$, where T- temperature. It is characterize by an additional gauge invariant - holonomy or eigenvalues of the Wilson line that goes along the time direction. Recently the higher charge calorons were obtained [33]. The fascinating feature of the KvBLL caloron is that it consists of two BPS dyons for $SU(2)$ gauge group (see fig.1).

In general, to take into account the effect of quantum fluctuations around a classical solution one expands the Euclidean action as follows [34]

$$Z_{\rm cl} = e^{-S_{\rm cl}} \int d(\text{collective coordinates}) \cdot \text{Jacobian} \cdot \text{Det}'^{-1/2}(W_{\mu\nu}) \cdot \text{Det}(-D_\mu^2), \qquad (1)$$

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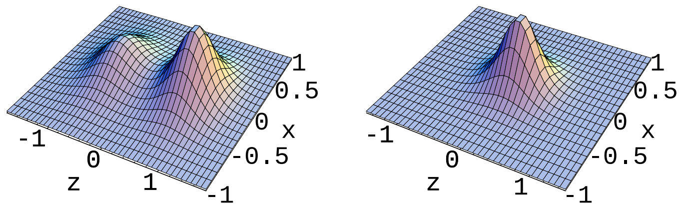

Figure 1: The action density of the KvBLL caloron as function of z, x at fixed $t = y = 0$. At large separations r_{12} the caloron is a superposition of two BPS dyon solutions (left, $r_{12} = 1.5/T$). At small separations they merge (right, $r_{12} = 0.6/T$).

which is the single-pseudoparticle contribution to the partition function. Here D_μ is a covariant derivative in adjoint representation, $\mathrm{Det}'(W_{\mu\nu})$ denotes the non-zero mode determinant of the quadratic form of the Euclidean action, parametrized by the collective coordinates of the classical solution. For self-dual field one can show [35] that $\mathrm{Det}'(W_{\mu\nu}) = \mathrm{Det}^4(-D_\mu^2)$. Thus the determinant $\mathrm{Det}(-D_\mu^2)$ determines the weight of the quasi-particle or the probability with which it occurs in the partition function of the theory. The quantum determinant for the case of zero temperature was computed by 't Hooft [18] in 70's and it still plays an important role for phenomenological and theoretical studies of strong interaction physics. The finite temperature generalization was made by Gross, Pisarski and Yaffe [36]. They found the weight of the instanton with trivial holonomy i.e. with unit value of the Wilson loop going along periodic euclidean time direction. More recently exact analytical expressions were computed for the determinants in the fundamental and adjoint representations of the $SU(2)$ gauge group and *arbitrary* holonomy in [3, 37]. Unfortunately, these expressions are extremely cumbersome and occupy significant part of a hard disk space. Nevertheless, in [37] the results of numerical evaluation were presented.

In this article we argue the existence of a simple relation between determinants in adjoint and fundamental representations. Concretely, if the determinant in fundamental representation is written in the form [37] (we take $T = 1$ and restore exact T dependence in the last section only)

$$\begin{aligned} \log \mathrm{Det}(-\nabla^2)\big|_{T=1} &= \log \mathrm{Det}(-\nabla^2)\big|_{T=0} + A(\mathrm{v}, r_{12}) \\ &+ V\left[P\left(\frac{2\pi - \mathrm{v}}{2}\right) - \frac{\pi^2}{12}\right] + \frac{\pi r_{12}}{2} P''\left(\frac{2\pi - \mathrm{v}}{2}\right), \end{aligned} \tag{2}$$

where $P(\mathrm{v}) = \frac{\mathrm{v}^2(2\pi - \mathrm{v})^2}{12\pi^2}$ - a perturbative potential, v - a quantity connected with holonomy (when $\mathrm{v} = 0$ and $\mathrm{v} = 2\pi$ the holonomy is trivial, see [3, 37] for notations), and r_{12} is a distance between constituent dyons or $r_{12} = \pi\rho^2 T$, where ρ is an instanton size. The determinant in adjoint representation is simply

$$\begin{aligned} \log \mathrm{Det}(-D^2)\big|_{T=1} &= \log \mathrm{Det}(-D^2)\big|_{T=0} + 16A(\mathrm{v}, r_{12}) + \log\left(1 + \frac{r_{12}\mathrm{v}\overline{\mathrm{v}}}{2\pi}\right) \\ &+ VP(\mathrm{v}) + 2\pi r_{12} P''(\mathrm{v}) \end{aligned} \tag{3}$$

(we denote $\overline{\mathrm{v}} = 2\pi - \mathrm{v}$). This connections is in spirit of the one found by Gross, Pisarski and Yaffe. This relation provides an independent check of the results of [3]. In particular the large r_{12} asymptotic, found there, can be easily rederived on the base of this relation. All analytical and numerical results of [37] extend automatically on the isospin-1 case. The function $A(\mathrm{v}, r_{12})$ is know with a good accuracy from [37].

The way we argue the relation (4) is the following: using an exact expressions for the determinants [3, 37] we calculate analytically expansion in powers of $1/r_{12}$ (see Appendix A), and check the relation up to the $1/r_{12}^{10}$. Then we check the relation numerically for a several values of r_{12} and v with a precision 10^{-5}. This calculation involves 3-fold integration of an expression of several Mb size and by itself is rather nontrivial and is possible due to the numerical and analytical power of Mathematica.

In section II we review old results related to KvBLL caloron important for the derivation. In section III we derive the result basing on the $1/r_{12}$ expansion and in section IV we calculate the quantum weight of the KvBLL caloron and make more accurate the estimation for the temperature of the phase transition made in [3].

2 Old results

Before proceeding to argue the relation (4) between determinants in different representations let us first remind results concerning determinants in the background of KvBLL caloron.

2.1 Zero temperature

When the size of the KvBLL caloron ρ or distance between constituent BPS dyons $r_{12} = \pi\rho^2 T$ is small compare to $1/T$ the caloron reduces to the usual BPST instanton. For the BPST instanton the 't Hooft [18] results for isospin-1/2 and isospin-1 determinants are

$$\log \mathrm{Det}(-\nabla^2)\big|_{T=0} = \frac{1}{6}\log\mu\rho + \alpha(1/2)\ , \qquad \alpha(1/2) = \frac{\gamma_E}{6} - \frac{17}{72} + \frac{\log\pi}{6} - \frac{\zeta'(2)}{\pi^2} \tag{4}$$

$$\log \mathrm{Det}(-D^2)\big|_{T=0} = \frac{2}{3}\log\mu\rho + \alpha(1), \qquad \alpha(1) = \frac{2\gamma_E}{3} - \frac{16}{9} + \frac{\log 2}{3} + \frac{2\log(2\pi)}{3} - \frac{4\zeta'(2)}{\pi^2}\ , \tag{5}$$

where μ is a Pauli-Villars regulator.

2.2 Nonzero temperature, trivial holonomy

The determinant in the case of the trivial holonomy was calculated by Gross, Pisarski and Yaffe [36]. Then the holonomy becomes trivial the caloron becomes spherical symmetric. Consequently, the resulting expressions are much more simpler. Nevertheless it have not been shown analytically that the isospin-1 and isospin-$\frac{1}{2}$ are related even for this more simple case.

For the isospin-$\frac{1}{2}$ the result reads

$$\log\det(-\nabla^2)\big|_{T=1} = \log\det(-\nabla^2)\big|_{T=0} + A(r_{12}) - \frac{\pi r_{12}}{6} \tag{6}$$

where $r_{12} = \pi\rho^2$ can be interpreted as a distance between dyons when the holonomy become nontrivial (we take $T = 1$). As it was verified numerically the isospin-1 determinant can be written in the form

$$\log\det(-D^2) = \log\det(-D^2)\big|_{T=0} + 16A(r_{12}) + \frac{4\pi r_{12}}{3}\ , \tag{7}$$

where $A(r_{12})$ has the following asymptotics

$$A(r_{12}) = -\frac{\pi r_{12}}{36} + \mathcal{O}\left(r_{12}^{3/2}\right) = \frac{1}{18} - \frac{\gamma_E}{6} - \frac{\pi^2}{216} - \frac{\log(r_{12}/\pi)}{12} + \mathcal{O}\left(\frac{1}{r_{12}}\right) \tag{8}$$

2.3 Non-trivial holonomy, isospin-$1/2$

The task of calculating the determinant in the background of the caloron with nontrivial holonomy is more complicated because the field configuration has not spherical symmetry and has additional parameter v, that is connected with the value of the holonomy (when $\mathrm{v} = 0, 2\pi$ the holonomy becomes trivial). In [37] an expression for the isospin-1/2 was found for all distances r_{12} and holonomies $0 \leq \mathrm{v} \leq 2\pi$

$$\log \mathrm{Det}(-\nabla^2) = \log \mathrm{Det}(-\nabla^2)\big|_{T=0} + A(\mathrm{v}, r_{12}) + \left[P\left(\frac{\overline{\mathrm{v}}}{2}\right) - \frac{\pi^2}{12}\right] V + P''\left(\frac{\overline{\mathrm{v}}}{2}\right)\frac{\pi r_{12}}{2} \tag{9}$$

where function $A(\mathrm{v}, r_{12})$ is fitted by (26) and has the following large r_{12} asymptotic (in Appendix A we give more terms in eq.(27))

$$\begin{aligned} A(\mathrm{v}, r_{12}) &= \frac{\log(2\pi)}{6} - \frac{\mathrm{v}\log\mathrm{v}}{12\pi} - \frac{\overline{\mathrm{v}}\log\overline{\mathrm{v}}}{12\pi} + \frac{1}{18} - \frac{\gamma}{6} - \frac{\pi^2}{216} - \frac{\log(r_{12}/\pi)}{12} \\ &- \frac{1}{12 r_{12}\pi}\left(\log(\mathrm{v}\overline{\mathrm{v}}r_{12}^2/\pi^2) - \frac{23\pi^2}{72} + 2\gamma + \frac{37}{6}\right) + \mathcal{O}\left(\frac{1}{r_{12}^2}\right) \end{aligned} \tag{10}$$

and for small r_{12} it is

$$A(\mathrm{v}, r_{12}) = \frac{(3\mathrm{v}\overline{\mathrm{v}} - 2\pi^2) r_{12}}{72\pi} + \mathcal{O}\left(r_{12}^{3/2}\right) \tag{11}$$

Note that (9) is a generalization of (6), that satisfies all asymptotics. Thus, eq.(9) generalizes eq.(4) to arbitrary values of holonomy.

2.4 Non-trivial holonomy, isospin-1

Isospin-1 or Ghost determinant plays an important role since it determines the statistical weight of the configuration. In [3] the large r_{12} purely analytic expression for asymptotic was found

$$\begin{aligned} \log \mathrm{Det}(-D^2) &= V\,P(\mathrm{v}) + \frac{2}{3}\log\mu + \frac{3\pi - 4\mathrm{v}}{3\pi}\log\mathrm{v} + \frac{3\pi - 4\overline{\mathrm{v}}}{3\pi}\log\overline{\mathrm{v}} + \frac{5}{3}\log(2\pi) + 2\pi P''(\mathrm{v}) r_{12} \\ &+ \frac{1}{r_{12}}\left[\frac{1}{\mathrm{v}} + \frac{1}{\overline{\mathrm{v}}} + \frac{23\pi}{54} - \frac{8\gamma_E}{3\pi} - \frac{74}{9\pi} - \frac{4}{3\pi}\log\left(\frac{\mathrm{v}\overline{\mathrm{v}}\,r_{12}^2}{\pi^2}\right)\right] + c_1 + \mathcal{O}\left(\frac{1}{r_{12}^2}\right) \end{aligned} \tag{12}$$

where

$$c_1 = \log 2 + \frac{5}{3}\log\pi - \frac{8}{9} - 2\,\gamma_E - \frac{2\,\pi^2}{27} - \frac{4\,\zeta'(2)}{\pi^2}\,. \tag{13}$$

The most nontrivial is a constant c_1. It can be easily rederived independently, using result for isospin-1/2 (10) and (4).

3 Derivation of the relation

In this section we derive the relation (4) from the comparison of the large r_{12} asymptotics. As we shell show this relation is right at least up to the 10^{th} order in $1/r_{12}$. The method of the calculation is taken from [3].

The derivative of the determinant with respect to a parameter is

$$\frac{\partial \log\det(-D^2)}{\partial r_{12}} \equiv -\int \mathrm{Tr}\left(\partial_{r_{12}} A_\mu J_\mu\right) \tag{14}$$

Where J_μ is a vacuum current related to Green function of the covariant Laplas operator in the background of the KvBLL caloron. One of the results of [3] and [37] is an expression for the

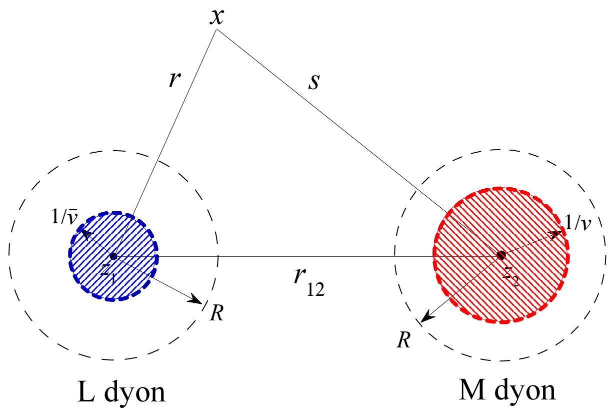

Figure 2: Three regions of integration for well separated dyons.

vacuum current J_μ that is a rational function of r, s, $R = e^{r\overline{\mathrm{v}}}$, $S = e^{s\mathrm{v}}$, $E_0 = e^{2i\pi x_0}$ and v, where r, s are distances from the BPS dyons (see. fig.2), $1/\mathrm{v}$ and $1/\overline{\mathrm{v}}$ are their core sizes.

The main point in the expansion is to divide space into tree domains: two balls of radius R, such that $1/\mathrm{v}, 1/\overline{\mathrm{v}} \ll R \ll r_{12}$, surrounding the centers of the constituent BPS dyons, and all the rest space. Then we expand expression in the core regions in powers of $1/r_{12}$ near each core and integrate it over core domain. The expression outside cores has an exponential precision and the only source of the $1/r_{12}$ terms here is the nontrivial domain of integration.

The vacuum current of the isospin-1 can be naturally divided into tree pieces $J_\mu = J^{\mathrm{r}}_\mu + J^{\mathrm{s}}_\mu + J^{\mathrm{m}}_\mu$ (see [3] for notations). Let us denote by $\frac{\partial \log \det^{\mathrm{r,s,m}}_{\mathrm{core}}}{\partial r_{12}}$ the contributions to the $\frac{\partial \log \det(-D^2)}{\partial r_{12}}$ that comes from $J^{\mathrm{r,s,m}}_\mu$ i.e.

$$\frac{\partial \log \det^{\mathrm{r,s,m}}_{\mathrm{core}}}{\partial r_{12}} \equiv -\int_{\mathrm{core}} \mathrm{Tr}\left(\partial_{\mathrm{v}} A_\mu J^{\mathrm{r,s,m}}_\mu\right) \tag{15}$$

where integration is over two ball of the radius R. The total $\frac{\partial \log \det(-D^2)}{\partial r_{12}}$ is a sum of these three contributions and a contribution that comes from the integration over the rest space $\frac{\partial \log \det(-D^2)_{\mathrm{far}}}{\partial r_{12}}$. In Appendix A the expansion of these contributions in powers of $1/r_{12}$ is given. One can easily see that in all order the following equalities hold

$$\frac{\partial \log \mathrm{Det}^{\mathrm{s}}_{\mathrm{cores}}}{\partial r_{12}} = 4\frac{\partial A}{\partial r_{12}} - \frac{2\log\frac{r_{12}}{R}}{3r_{12}^2\pi} + \frac{1}{3r_{12}} + \frac{\pi}{8r_{12}^2} - \frac{23}{18r_{12}^2\pi} + (R^n \text{ terms}) \tag{16}$$

$$\frac{\partial \log \mathrm{Det}^{\mathrm{r+m}}_{\mathrm{cores}}}{\partial r_{12}} = 12\frac{\partial A}{\partial r_{12}} - \frac{2\log\frac{r_{12}}{R}}{r_{12}^2\pi} + \frac{1}{r_{12}} + \frac{3\pi}{8r_{12}^2} + \frac{5}{6r_{12}^2\pi} - \frac{2\pi}{r_{12}(r_{12}\mathrm{v}\overline{\mathrm{v}} + 2\pi)} + (R^n \text{ terms}) \tag{17}$$

here we do not write R^n terms as they all get cancel with the similar terms in the contribution of the far from dyons region.

$$\frac{\partial \log \mathrm{Det}^{\mathrm{r+m+s}}_{\mathrm{far}}}{\partial r_{12}} = \frac{8\log\frac{r_{12}}{R}}{3r_{12}^2\pi} + \frac{8-9\pi^2}{18\pi r_{12}^2} + 2\pi P''(\mathrm{v}) + (R^n \text{ terms}) \tag{18}$$

adding up contributions from 'far' and 'core' regions we have

$$\frac{\partial \log \mathrm{Det}(-D^2)}{\partial r_{12}} = \partial_{r_{12}}\left(16A + \frac{1}{3}\log r_{12} + \log\left(2\pi + r_{12}\mathrm{v}\overline{\mathrm{v}}\right) + 2\pi P''(\mathrm{v})\right) \tag{19}$$

integrating it to the small values of r_{12}, where KvBLL caloron reduces to the ordinary BPST instanton and the determinant is known, we can write

$$\log \mathrm{Det}(-D^2) = \log \mathrm{Det}(-D^2)\big|_{T=0} + 16A(\mathrm{v}, r_{12}) + \log\left(1 + \frac{r_{12}\mathrm{v}\overline{\mathrm{v}}}{2\pi}\right) + 2\pi P''(\mathrm{v}) r_{12} + VP(\mathrm{v}) \tag{20}$$

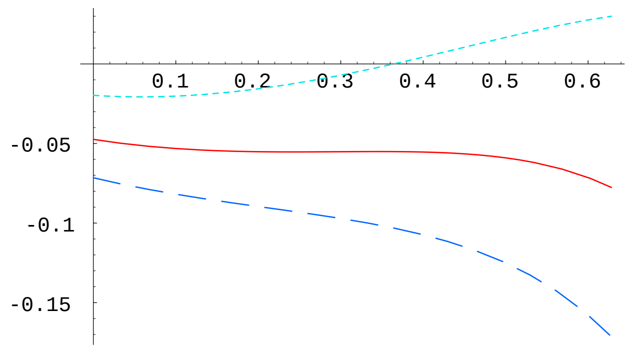

Figure 3: Free energy of the caloron gas in units of T^3V at $T = 1.5\Lambda$ (dotted), $T = 1.325\Lambda$ (solid) and $T = 1.25\Lambda$ (dashed) as function of the asymptotic value of A_4 in units of T.

We claim that this answer is exact. It gives right large r_{12} asymptotic (13) and consistent with trivial holonomy results (6), (7). Moreover, we tested it numerically for a several values of r_{12} and v with a precision of order 10^{-5}. We consider this as a serious prove of the relation (20).

4 Quantum weight

In this section we renew the main result of [3] quantum weight of the KvBLL caloron. The concept of the quantum weight is discussed in details, for example, in [3]. For a self-dual configuration it reads

$$\mathcal{Z} = \int \prod_{i=1}^{p} d\xi_i e^{-S_{cl}} \left(\frac{\mu}{g\sqrt{2\pi}}\right)^p J\,\mathrm{Det}^{-1}(-D^2) \tag{21}$$

where ξ_i are coordinates on the moduli space of the configuration, g is a gauge coupling, and J is a measure on the moduli space. It can be expressed it terms of metric on the moduli space

$$J = \sqrt{\det g_{ij}} \tag{22}$$

in [1, 3] it was found that

$$J = 8(2\pi)^8 \rho^3 \left(1 + \frac{r_{12}}{2\pi} \mathrm{v}\overline{\mathrm{v}}\right) \tag{23}$$

the total number of zero modes is 8. The associated collective coordinates are z_μ - position of KvBLL caloron center, one gauge orientation and two angles, that determine the orientation is space, combined into $\mathcal{O}$ and instanton size ρ. One can parameterize the module space by two 3D coordinates of dyons and two color orientations of the dyons. It turns our that the determinant does not depend on the color orientations.

$$\int \prod_{i=1}^{8} d\xi_i\; J = \int d^4z\; d^4\mathcal{O}\; d\rho\; \rho^3 \left(1 + \frac{r_{12}}{2\pi} \mathrm{v}\overline{\mathrm{v}}\right) 16\; (2\pi)^{10} = \int d^3z_1\; d^3z_2\; \left(1 + \frac{r_{12}}{2\pi} \mathrm{v}\overline{\mathrm{v}}\right) \frac{1}{r_{12}} 16\; (2\pi)^7 \tag{24}$$

combining this with (21) and (4) we come to

$$\begin{aligned}\mathcal{Z}_{\mathrm{KvBLL}} &= \int d^3z_1\, d^3z_2\, T^6\, C_A \left(\frac{8\pi^2}{g^2}\right)^4 \left(\frac{\Lambda e^{\gamma_E}}{4\pi T}\right)^{\frac{22}{3}} \left(\frac{1}{T r_{12}}\right)^{\frac{4}{3}} \\ &\times\; \exp\left[-V\,P(\mathrm{v}) - 16A(\mathrm{v}, r_{12}) - 2\pi\, r_{12}\, P''(\mathrm{v})\right],\end{aligned}$$

$$C_A = 2^8\pi^2 \exp\left(\frac{16}{9} - 8\gamma_E + \frac{4\zeta'(2)}{\pi^2} + \frac{2}{3}\log 2\right) \tag{25}$$

Surprisingly, the moduli space measure exactly cancels with the third term in expression (20) for $\mathrm{Det}(-D^2)$.

$A(r_{12}, \mathrm{v})$ was fitted in [37] by

$$A(\mathrm{v}, r_{12}) \simeq -\frac{1}{12}\log\left(1+\frac{\pi r_{12}T}{3}\right) - \frac{r_{12}\alpha}{216\pi(1+r_{12}T)} + \frac{0.00302\, r_{12}(\alpha + 9\mathrm{v}\overline{\mathrm{v}}/T)}{2.0488 + r_{12}^2T^2} \tag{26}$$

where $\alpha = 18\mathrm{v}\log\frac{\mathrm{v}}{T} + 18\overline{\mathrm{v}}\log\frac{\overline{\mathrm{v}}}{T} - 216.611T$. This expression has a maximum absolute error 5×10^{-3}.

4.1 Estimation of the T_c

Here we make slightly more accurate the crude estimation of the free energy of ensemble of KvBLL calorons without taking into account an interaction made in [3]. We do not repeat the details here and just give the result.

To obtain a phase transition one has to consider a gas of the calorons. The density of the calorons increases when the temperature becomes smaller. At some critical temperature T_c the density becomes sufficient to override the perturbativ potential $P(\mathrm{v})$ and nontrivial values of holonomy becomes preferable (see fig.3). Our new estimation for T_c is 1.3Λ which is only 20% bigger then in [3].

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A Series expansion with respect to $1/r_{12}$

In this appendix we give results of the expansion in powers of $1/r_{12}$. This expressions are used to obtain eq.(16) and eq.(17).

$$\begin{aligned}\frac{\partial A(\mathrm{v},r_{12})}{\partial r_{12}} &= -\frac{1}{24r_{12}} + \left[\frac{25}{144\pi}+\frac{\gamma_E}{12\pi}-\frac{23\pi}{1728}+\frac{\log(\mathrm{v}r_{12}/\pi)}{12\pi}\right]\frac{1}{r_{12}^2} + \frac{1}{12\pi r_{12}^3\mathrm{v}} - \frac{1}{24\pi r_{12}^4\mathrm{v}^2}\\
&+\left[\frac{1}{36\pi}-\frac{\pi^3}{2160}\right]\frac{1}{r_{12}^5\mathrm{v}^3} + \left[-\frac{1}{48\pi}+\frac{\pi^3}{576}\right]\frac{1}{r_{12}^6\mathrm{v}^4} + \left[\frac{1}{60\pi}-\frac{73\pi^3}{10800}-\frac{\pi^5}{2835}\right]\frac{1}{r_{12}^7\mathrm{v}^5}\\
&+\left[-\frac{1}{72\pi}+\frac{343\pi^3}{12960}+\frac{11\pi^5}{3888}\right]\frac{1}{r_{12}^8\mathrm{v}^6} + \left[\frac{1}{84\pi}-\frac{769\pi^3}{8640}-\frac{2285\pi^5}{127008}-\frac{\pi^7}{2016}\right]\frac{1}{r_{12}^9\mathrm{v}^7}\\
&+\left[-\frac{1}{96\pi}+\frac{1169\pi^3}{4608}+\frac{15025\pi^5}{145152}+\frac{1111\pi^7}{172800}\right]\frac{1}{r_{12}^{10}\mathrm{v}^8} + (\mathrm{v}\leftrightarrow\overline{\mathrm{v}}) + \mathcal{O}\left(\frac{1}{r_{12}^{11}}\right)\end{aligned} \tag{27}$$

We divide $\frac{\partial\log\det^{\mathrm{r}}_{\mathrm{core}}}{\partial r_{12}}$ into two parts $\frac{\partial\log\det^{\mathrm{r1}}_{\mathrm{core}}}{\partial r_{12}}$ and $\frac{\partial\log\det^{\mathrm{r2}}_{\mathrm{core}}}{\partial r_{12}}$

$$\begin{aligned}\frac{\partial\log\det^{\mathrm{r1}}_{\mathrm{core}}}{\partial r_{12}} &= \left[\frac{3}{8}-\frac{\pi^2}{9}\right]\frac{1}{r_{12}^2\mathrm{v}} + \left[-\frac{3}{8}+\frac{\pi^2}{9}-\frac{11\pi^4}{1890}\right]\frac{1}{r_{12}^3\mathrm{v}^2} + \left[\frac{2873}{6720}-\frac{7\pi^2}{144}+\frac{437\pi^4}{75600}+\frac{2\pi^6}{2205}\right]\frac{1}{r_{12}^4\mathrm{v}^3}\\
&+\left[-\frac{33}{70}-\frac{19\pi^2}{144}+\frac{10693\pi^4}{302400}-\frac{547\pi^6}{52920}\right]\frac{1}{r_{12}^5\mathrm{v}^4} + \left[\frac{215}{448}+\frac{149\pi^2}{288}-\frac{1651\pi^4}{21600}+\frac{1321\pi^6}{26460}\right]\frac{1}{r_{12}^6\mathrm{v}^5}\\
&+\left[-\frac{353\pi^2}{288}+\frac{38939\pi^4}{138240}+\frac{8023081\pi^6}{121927680}+\frac{143\pi^8}{32256}\right]\frac{1}{r_{12}^7\mathrm{v}^6}\\
&+\left[\frac{32687}{71680}+\frac{77\pi^2}{32}+\frac{412879\pi^4}{460800}+\frac{386711\pi^6}{564480}+\frac{7049\pi^8}{108000}\right]\frac{1}{r_{12}^8\mathrm{v}^7}\\
&+\left[-\frac{153973}{337920}-\frac{407\pi^2}{96}-\frac{4711609\pi^4}{1382400}-\frac{368419\pi^6}{145152}-\frac{2376127\pi^8}{4536000}-\frac{8461\pi^{10}}{1372140}\right]\frac{1}{r_{12}^9\mathrm{v}^8}\\
&+\left[\frac{77537}{168960}+\frac{111\pi^2}{16}+\frac{50991947\pi^4}{4838400}+\frac{630755\pi^6}{72576}+\frac{14417821\pi^8}{4536000}+\frac{27623\pi^{10}}{177870}\right]\frac{1}{r_{12}^{10}\mathrm{v}^9} + (\mathrm{v}\leftrightarrow\overline{\mathrm{v}}) + \mathcal{O}\left(\frac{1}{r_{12}^{11}}\right)\end{aligned} \tag{28}$$

$$\frac{\partial \log \det^{\rm r2}_{\rm core}}{\partial r_{12}} = \left[\frac{5}{2\pi}+\frac{\gamma_E}{\pi}+\frac{\pi}{36}+\frac{\log({\rm v}R/\pi)}{\pi}\right]\frac{1}{r_{12}^2}+\frac{1}{\pi}\frac{1}{r_{12}^3{\rm v}}-\frac{1}{2\pi}\frac{1}{r_{12}^4{\rm v}^2}+\left[\frac{1}{3\pi}-\frac{\pi^3}{180}\right]\frac{1}{r_{12}^5{\rm v}^3} \tag{29}$$
$$+\left[-\frac{1}{4\pi}+\frac{\pi^3}{48}\right]\frac{1}{r_{12}^6{\rm v}^4}+\left[\frac{1}{5\pi}-\frac{73\pi^3}{900}-\frac{4\pi^5}{945}\right]\frac{1}{r_{12}^7{\rm v}^5}+\left[-\frac{1}{6\pi}+\frac{343\pi^3}{1080}+\frac{11\pi^5}{324}\right]\frac{1}{r_{12}^8{\rm v}^6}$$
$$+\left[\frac{1}{7\pi}-\frac{769\pi^3}{720}-\frac{2285\pi^5}{10584}-\frac{\pi^7}{168}\right]\frac{1}{r_{12}^9{\rm v}^7}+\left[-\frac{1}{8\pi}+\frac{1169\pi^3}{384}+\frac{15025\pi^5}{12096}+\frac{1111\pi^7}{14400}\right]\frac{1}{r_{12}^{10}{\rm v}^8}$$
$$+({\rm v}\leftrightarrow\overline{\rm v})+\mathcal{O}\left(\frac{1}{r_{12}^{11}}\right)$$

$$\frac{\partial \log \det^{\rm s}_{\rm core}}{\partial r_{12}} = \left[\frac{1}{18\pi}+\frac{\gamma_E}{3\pi}+\frac{\pi}{108}+\frac{\log({\rm v}R/\pi)}{3\pi}\right]\frac{1}{r_{12}^2}+\frac{1}{3\pi r_{12}^3{\rm v}}-\frac{1}{6\pi r_{12}^4{\rm v}^2}+\left[\frac{1}{9\pi}-\frac{\pi^3}{540}\right]\frac{1}{r_{12}^5{\rm v}^3} \tag{30}$$
$$+\left[-\frac{1}{12\pi}+\frac{\pi^3}{144}\right]\frac{1}{r_{12}^6{\rm v}^4}+\left[\frac{1}{15\pi}-\frac{73\pi^3}{2700}-\frac{4\pi^5}{2835}\right]\frac{1}{r_{12}^7{\rm v}^5}+\left[-\frac{1}{18\pi}+\frac{343\pi^3}{3240}+\frac{11\pi^5}{972}\right]\frac{1}{r_{12}^8{\rm v}^6}$$
$$+\left[\frac{1}{21\pi}-\frac{769\pi^3}{2160}-\frac{2285\pi^5}{31752}-\frac{\pi^7}{504}\right]\frac{1}{r_{12}^9{\rm v}^7}+\left[-\frac{1}{24\pi}+\frac{1169\pi^3}{1152}+\frac{15025\pi^5}{36288}+\frac{1111\pi^7}{43200}\right]\frac{1}{r_{12}^{10}{\rm v}^8}$$
$$+({\rm v}\leftrightarrow\overline{\rm v})+\mathcal{O}\left(\frac{1}{r_{12}^{11}}\right)$$

Analogously, we divide $\frac{\partial \log \det^{\rm m}_{\rm core}}{\partial r_{12}}$ into two parts $\frac{\partial \log \det^{\rm m1}_{\rm core}}{\partial r_{12}}$ and $\frac{\partial \log \det^{\rm m2}_{\rm core}}{\partial r_{12}}$. It is very convenient to extract the factor $\left(1+\frac{r_{12}{\rm v}\overline{\rm v}}{2\pi}\right)$ from the denominator of this contribution before making an expansion

$$-\left(1+\frac{r_{12}{\rm v}\overline{\rm v}}{2\pi}\right)\frac{\partial \log \det^{\rm m1}_{\rm core}}{\partial r_{12}} = \left[\frac{11}{8}-\frac{\pi^2}{9}\right]\frac{1}{r_{12}}-\frac{11\pi^4}{1890}\frac{1}{r_{12}^2{\rm v}}+\left[\frac{353}{6720}+\frac{\pi^2}{16}-\frac{\pi^4}{25200}+\frac{2\pi^6}{2205}\right]\frac{1}{r_{12}^3{\rm v}^2} \tag{31}$$
$$+\left[-\frac{59}{1344}-\frac{13\pi^2}{72}+\frac{4147\pi^4}{100800}-\frac{499\pi^6}{52920}\right]\frac{1}{r_{12}^4{\rm v}^3}+\left[\frac{19}{2240}+\frac{37\,\pi^2}{96}-\frac{12421\,\pi^4}{302400}+\frac{419\,\pi^6}{10584}\right]\frac{1}{r_{12}^5{\rm v}^4}$$
$$+\left[\frac{-17\,\pi^2}{24}+\frac{115529\,\pi^4}{230400}+\frac{121033\,\pi^6}{2488320}+\frac{143\,\pi^8}{32256}\right]\frac{1}{r_{12}^6{\rm v}^5}$$
$$+\left[\frac{85\,\pi^2}{72}-\frac{25373531\,\pi^4}{11059200}-\frac{82760843\,\pi^6}{487710720}-\frac{6627049\,\pi^8}{96768000}\right]\frac{1}{r_{12}^7{\rm v}^6}$$
$$+\left[\frac{43}{118272}-\frac{11\,\pi^2}{6}-\frac{868243\,\pi^4}{345600}-\frac{4707133\,\pi^6}{2540160}-\frac{2080069\,\pi^8}{4536000}-\frac{8461\,\pi^{10}}{1372140}\right]\frac{1}{r_{12}^8{\rm v}^7}$$
$$+\left[\frac{367}{112640}+\frac{259\,\pi^2}{96}+\frac{2555653\,\pi^4}{358400}+\frac{297697\,\pi^6}{48384}+\frac{95569\,\pi^8}{36000}+\frac{286483\,\pi^{10}}{1920996}\right]\frac{1}{r_{12}^9{\rm v}^8}+({\rm v}\leftrightarrow\overline{\rm v})+\mathcal{O}\left(\frac{1}{r_{12}^{10}}\right)$$

$$-\left(1+\frac{r_{12}{\rm v}\overline{\rm v}}{2\pi}\right)\frac{\partial \log \det^{\rm m2}_{\rm core}}{\partial r_{12}} = \left[\frac{\pi}{18}-\frac{11}{16\pi}\right]\frac{\rm v}{r_{12}}+\left[\frac{3}{16\pi}-\frac{\pi}{18}+\frac{11\pi^3}{3780}\right]\frac{1}{r_{12}^2} \tag{32}$$
$$+\left[-\frac{2873}{13440\pi}+\frac{7\pi}{288}-\frac{437\pi^3}{151200}-\frac{\pi^5}{2205}\right]\frac{1}{r_{12}^3{\rm v}}+\left[\frac{33}{140\pi}+\frac{19\pi}{288}-\frac{10693\pi^3}{604800}+\frac{547\pi^5}{105840}\right]\frac{1}{r_{12}^4{\rm v}^2}$$
$$+\left[-\frac{215}{896\,\pi}-\frac{149\,\pi}{576}+\frac{1651\,\pi^3}{43200}-\frac{1321\,\pi^5}{52920}\right]\frac{1}{r_{12}^5{\rm v}^3}+\left[\frac{353\,\pi}{576}-\frac{38939\,\pi^3}{276480}-\frac{8023081\,\pi^5}{243855360}-\frac{143\,\pi^7}{64512}\right]\frac{1}{r_{12}^6{\rm v}^4}$$
$$+\left[-\frac{77\,\pi}{64}+\frac{9496217\,\pi^3}{7372800}+\frac{4253821\,\pi^5}{36126720}+\frac{1008007\,\pi^7}{27648000}\right]\frac{1}{r_{12}^7{\rm v}^5}$$
$$+\left[\frac{153973}{675840\,\pi}+\frac{407\,\pi}{192}+\frac{4711609\,\pi^3}{2764800}+\frac{368419\,\pi^5}{290304}+\frac{2376127\,\pi^7}{9072000}+\frac{8461\,\pi^9}{2744280}\right]\frac{1}{r_{12}^8{\rm v}^5}$$
$$+\left[-\frac{77537}{337920\,\pi}-\frac{111\,\pi}{32}-\frac{50991947\,\pi^3}{9676800}-\frac{630755\,\pi^5}{145152}-\frac{14417821\,\pi^7}{9072000}-\frac{27623\,\pi^9}{355740}\right]\frac{1}{r_{12}^9{\rm v}^7}+({\rm v}\leftrightarrow\overline{\rm v})+\mathcal{O}\left(\frac{1}{r_{12}^{10}}\right)$$

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