# Mass spectrum, actons and cosmological landscape

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**Abstract**

It is suggested that the properties of the mass spectrum of elementary particles could be related with cosmology. Solutions of the Klein-Gordon equation on the Friedmann type manifold with the finite action are constructed. These solutions (actons) have a discrete mass spectrum. We suggest that such solutions could select a universe from the string cosmological landscape. In particular the solutions with the finite action on de Sitter space are investigated.

## 1 Introduction

Understanding of the mass spectrum of elementary particles is an outstanding problem for physics. Why the elementary particles have their observed pattern of masses? Note that there is no answer to even a simpler question why the mass spectrum is discrete.

The mass in quantum field theory [1] is considered as an arbitrary parameter but in nature there is only a discrete set of masses of elementary particles.

We investigate the following proposal. We suggest that the mass parameter in the Klein-Gordon equation should be such that there exists a corresponding classical solution with the finite action. In other words we consider an eigenvalue problem for the Klein-Gordon equation.

There are not such nontrivial solutions in the Minkowski space. However we will show that the square integrable solutions of the Klein-Gordon equation on an important class of manifolds do exist. Moreover such solutions have the finite action (we call them *actons*) and they exist only for some discrete values of the mass, i.e. we obtain quantization of masses.

A finite mass spectrum was first obtained in [3] for de Sitter space, solutions on manifolds are considered in [4].

The requirement of the finiteness of the action in the Lorentz signature is natural, for example, in the case when we study the wave function of the Universe in real time in the semiclassical approximation [5] where it is just $\exp(iS)$, $S$ being the action. It is known that there are solutions of some nonlinear equations with finite action (instantons) but they exist only in the Euclidean time.

In the string landscape picture [6, 7, 8] our observed universe is viewed within a multiverse which contains every possible type of vacuum and the law of physics are determined by the anthropic principle. We suggest that the principle of the finiteness of the action proposed in this work can be used to restrict the multiplicity of universes.

The actons are classical solutions. They define masses of elementary particles. Then we have to quantize the system in this background as we do with solitons and instantons.

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The values of masses of scalar particles obtained this way so far are not very realistic but the mechanism of how to get the discrete mass spectrum seems does work.

The idea that the boundedness of the mass spectrum might be related with de Sitter geometry in the momentum space is considered in [9]. A symmetry which exploits the feature that de Sitter and Anti de Sitter space are related by analytic continuation is considered in [10].

Let $M$ be an $(n+1)$-dimensional manifold with a Lorentz metric $g_{\mu\nu}$, $\mu,\nu = 0,1,...,n$. Consider the Klein-Gordon equation [2] on $M$ for the real valued function $f$

$$\Box f + \lambda f = 0. \tag{1}$$

Here

$$\Box f = \nabla_\mu \nabla^\mu f = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu f),$$

$g$ is the determinant of $(g_{\mu\nu})$ and the real parameter $\lambda$ corresponds to the mass square.

We are interested in deriving the values of $\lambda$ for which there exist classical solutions $f \in C^2(M)$, satisfying the condition

$$\int_M f^2 \sqrt{|g|} dx < \infty \tag{2}$$

The condition (2) was first considered in [3] for solutions of the Klein-Gordon equation on de Sitter space. Let us note that the condition (2) includes the integration not only over the spatial variables as it is done usually for the quantum Klein-Gordon field [2] but also over the time-like variable.

The paper is composed as follows. In the next section square integrable solutions of the Friedmann type manifolds are constructed. Then solutions on de Sitter space and on the Friedmann space are considered.

## 2 Solutions on the Friedmann type manifolds

Let us consider a manifold $M = I \times N^n$ with a metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) dl^2. \tag{3}$$

Here $I$ is an interval on the real axis, $I \subset \mathbb{R}$, $a(t)$ is a smooth positive function on $I$, $N^n$ a Riemannian manifold and

$$dl^2 = h_{ij}(y) dy^i dy^j, \quad i,j = 1,...,n \tag{4}$$

is a Riemannian metric on $N^n$. Such manifolds $(M, g_{\mu\nu})$ will be called the Friedmann type manifolds.

Eq. (1) for the metric (3)takes the form

$$\ddot{f} + \frac{n}{a}\dot{a}\dot{f} - \frac{1}{a^2}\Delta_h f + \lambda f = 0 \tag{5}$$

where $\Delta_h$ is the Laplace-Beltrami operator fot the metric $h_{ij}$,

$$\Delta_h f = \frac{1}{\sqrt{h}} \partial_i (\sqrt{h} h^{ij} \partial_j f) \tag{6}$$

and the condition (2) reads

$$\int_M f^2 \sqrt{|g|} dx = \int_{I \times N^n} f^2 |a|^n \sqrt{h} dt dy < \infty \tag{7}$$

Let $q \geq 0$ be the eigenvalue of the operator $-\Delta_h$ on $N^n$ and $\Phi = \Phi(y)$ is the corresponding eigenfunction:

$$-\Delta_h \Phi = q\Phi, \tag{8}$$

$$\int_{N^n} \Phi^2 \sqrt{h} dy < \infty \tag{9}$$

We set

$$f = B(t) a(t)^{-\frac{n}{2}} \Phi(y). \tag{10}$$

Then from (5),(8) we obtain the Sturm-Liouville (Schrodinger) equation

$$\ddot{B} + [\lambda - v(t)] B = 0 \tag{11}$$

where

$$v(t) = \frac{n}{2}\frac{\ddot{a}}{a} + \frac{n}{2}(\frac{n}{2} - 1)\frac{\dot{a}^2}{a^2} - \frac{q}{a^2} \tag{12}$$

We look for solutions $B(t)$ of Eq. (11) in $L^2(I)$ since for functions of the form (10) the condition (7) takes the form

$$\int_I B^2 dt < \infty.$$

Consider the case $I = \mathbb{R}$.

*Let $M = \mathbb{R} \times N^n$ be the Friedman type manifold with the metric of the form (3),(4) such that there exists a solution $\Phi$ of Eq. (8) on $N^n$ which is not identically vanishing with an eigenvalue $q \geq 0$. Let the smooth positive function $a(t)$ on $\mathbb{R}$ is such that $v(t)$ (12) satisfies the condition*

$$v(t) \to \infty \quad if \quad |t| \to \infty. \tag{13}$$

*Then for given $q, \Phi$ (8),(9), the problem (1),(2) has an infinite family of solutions (actons) $f_j = B_j(t) a(t)^{-\frac{n}{2}} \Phi(y)$ with eigenvalues $\lambda_j,\ j = 1, 2, ...$ and moreover $\lambda_j \to \infty$ when $j \to \infty$.*

The statement is evident from quantum mechanics. It follows from a well known result by Weyl and Titchmarsh (see for example [11], Sect.5.12-5.13) that under condition (13) the Sturm-Liouville problem (11) has a discrete spectrum in $L^2(\mathbb{R})$.

Example. Let us take

$$a(t) = C \exp(\alpha t^{2k}),\ C > 0,\ \alpha > 0,\ k > 1. \tag{14}$$

Then

$$v(t) = \frac{n}{2}[\alpha 2k(2k-1)t^{2k-2} + (\frac{n}{2} - 1)(\alpha 2k)^2 t^{4k-2}] - qC^{-2}\exp(-\alpha t^{2k})$$

In this case the condition (13) is satisfied. One has a discrete spectrum $\lambda_1, \lambda_2, ...$ and $\lambda_j \to \infty$ when $j \to \infty$.

## 3 Solutions on de Sitter space

For de Sitter space one has: $M = \mathbb{R} \times S^3$,

$$ds^2 = dt^2 - \cosh^2 t \cdot h_{ij}(y) dy^i dy^j, \tag{15}$$

where $h_{ij}$ is the standard metric on the 3-dimensional sphere $S^3$. The eigenvalues of the operator $-\Delta_h$ on the 3-sphere are equal to $q = j(j+2),\ j = 0, 1, 2, ...$ and

$$v(t) = \frac{9}{4} - [\frac{3}{4} + j(j+2)]\frac{1}{\cosh^2 t} \tag{16}$$

We set

$$\alpha = \frac{3}{4} + j(j+2),\ \nu^2 = \frac{9}{4} - \lambda \tag{17}$$

Then Eq. (11) takes the form

$$\ddot{B} + [\frac{\alpha}{\cosh^2 t} - \nu^2]B = 0 \tag{18}$$

Theory of Eq. (18) is well known [11, 13] and it was used in [3] to construct square integrable solution of the Klein-Gordon equation on de sitter space. Spectrum for positive values of $\nu^2$ is discrete and for negative is continuous. We consider the first case, $\nu^2 > 0$.

Eq. (18) for $\alpha > 0$ has a solution in $L^2(\mathbb{R})$ iff

$$0 < \nu = \frac{1}{2}(\sqrt{1+4\alpha} - 1) - n, \ n = 0, 1, 2, ... \tag{19}$$

In our case, due to (17),

$$0 < \nu = j + \frac{1}{2} - n, \quad j, n = 0, 1, 2, ...$$

There is a family of square integrable solutions of Eq. (11) with eigenvalues $\lambda$ of the form

$$\lambda_{jn} = \frac{9}{4} - (j + \frac{1}{2} - n)^2, \quad (j, n = 0, 1, 2, ..., j + \frac{1}{2} - n > 0) \tag{20}$$

If $\lambda_{jn} \geq 0$ then we should have

$$0 < j + \frac{1}{2} - n \leq \frac{3}{2}, \quad j, n = 0, 1, 2, ...$$

and therefore either $j = n$ and $\lambda_{jn} = 2$, or $j = n + 1$ and $\lambda_{jn} = 0$.

In the case $j = n, \ \nu = 1/2$ for any $j = 0, 1, 2, ...$ Eq. (18) has a solution (acton) in $L^2(\mathbb{R})$ of the form

$$B_j(t) = \frac{1}{(\cosh t)^{1/2}} \sum_{s=0}^{j} \frac{(-j)_s (j+2)_s}{(3/2)_s s!} \frac{1}{(e^{2t}+1)^s}, \tag{21}$$

$$(k)_0 = 1, \ (k)_s = k(k+1)...(k+s-1).$$

In the case $j = n + 1, \ \nu = 3/2$ for any $j = 1, 2, ...$ Eq. (18) has a solution in $L^2(\mathbb{R})$ of the form

$$B_j(t) = \frac{1}{(\cosh t)^{3/2}} \sum_{s=0}^{j-1} \frac{(1-j)_s (j+3)_s}{(5/2)_s s!} \frac{1}{(e^{2t}+1)^s} \tag{22}$$

We have obtained that if the eigenvalues $\lambda = \lambda_{jn} \geq 0$ then either $\lambda = 0$, or $\lambda = 2$.

# 4 Solutions on the Friedmann space

**1.** In the inflation cosmology the following form of the Friedmann-de Sitter metric is often used:

$$ds^2 = dt^2 - e^{2Ht} \cdot h_{ij}(y) dy^i dy^j, \tag{23}$$

$h_{ij}$ is a Riemannian metric on a compact 3-dimensional manifold, $0 < t < \infty$ and $0 < H$ is Hubble‘s constant. In this case the function $v(t)$ (12) is

$$v(t) = \frac{9}{4}H^2 - qe^{-2Ht} \tag{24}$$

Eq. (11) on the semi-axis with boundary conditions $B(0) = B(\infty) = 0$ has an eigenvalue in this case. If the parameter $t$ is interpreted as the radius in spherical coordinates then we get the known model of deuteron (see, for example [13]). The solution has the form

$$B(t) = J_\nu(ce^{-Ht}),$$

where

$$c = \frac{\sqrt{q}}{H} > 0, \quad \nu = \frac{\sqrt{9H^2 - 4\lambda}}{2H} > 0,$$

and $J_\nu$ is the Bessel function. The eigenvalue $\lambda$ is derived from the relation $J_\nu(c) = 0$.

**2.** The Friedmann metric has the form

$$ds^2 = dt^2 - a^2(t)h_{ij}(y)dy^i dy^j \tag{25}$$

where $h_{ij}$ is a Riemannian metric on the manifold of positive, negative or flat curvature. The function $a(t)$ is derived from the Einstein-Friedmann equations

$$3\dot{a}^2/a^2 = 8\pi\rho - 3k/a^2, \quad 3\ddot{a}/a = -4\pi(\rho + 3p), \tag{26}$$

where $k = 1, -1, 0$ for the manifolds for manifolds of the positive, negative, or flat curvature respectively. The pressure $p$ and the mass density $\rho$ are related by an equation of state $p = p(\rho)$. In particular, for massless thermal radiation ($p = \rho/3$) in a 3-dimensional torus ($k = 0$) one has

$$a(t) = c\sqrt{t}, \quad c > 0, \quad 0 < t < \infty. \tag{27}$$

In this case

$$v(t) = -\frac{3}{16t^2} - \frac{q}{c^2 t}, \quad q > 0 \tag{28}$$

and the Sturm-Liouville equation (11) has a discrete spectrum for negative $\lambda$ :

$$\lambda_n = -\frac{4q^2}{c^4(4n+1)^2}, \quad n = 1, 2, ... \tag{29}$$

Indeed, if we denote $\lambda = -\nu^2$, $\nu > 0$ and define a new function $\varphi(x)$ by

$$B(t) = e^{-\nu t} t^{1/4} \varphi(2\nu t)$$

then from the Sturm-Liouville equation

$$\ddot{B}(t) + [\lambda + \frac{3}{16t^2} + \frac{q}{c^2 t}]B(t) = 0$$

we obtain that the function $\varphi(x)$ satisfies the equation for the degenerate hypergeometric function

$$x\varphi''(x) + (\frac{1}{2} - x)\varphi'(x) - (\frac{1}{4} - \frac{q}{2\nu c^2})\varphi(x) = 0.$$

It is known that the last equation has solutions with the required behavior at infinity only if

$$\frac{1}{4} - \frac{q}{2\nu c^2} = -n, \quad n = 1, 2, ...$$

which leads to (28).

For a compact 3-dimensional manifold of negative curvature ($k = -1$) the function $a(t)$ has the form

$$a(t) = \sqrt{t^2 - c^2}, \quad 0 < c < t < \infty \tag{30}$$

and the corresponding Sturm-Liouville problem also has a discrete spectrum for negative $\lambda$ .

## 5 Discussions and Conclusions

**1**. An action for the Klein-Gordon equation (1) has the form

$$S = \frac{1}{2}\int_M [(\nabla f, \nabla f) - \lambda f^2]\sqrt{|g|}dx \tag{31}$$

where

$$(\nabla f, \nabla f) = g^{\mu\nu}\partial_\mu f \partial_\nu f.$$

On solutions of the form (10) on the Friedmann type manifolds the action takes the form

$$S = \frac{1}{2}\int_{I\times N^n} [(\dot{B} - \frac{n}{2}\frac{\dot{a}}{a}B)^2\Phi^2 - a^{-2}B^2 h^{ij}\partial_i\Phi\partial_j\Phi - \lambda B^2\Phi^2]dt\sqrt{h}dy \tag{32}$$

On the solutions on de Sitter space of the form (21), (22) the integral (32) is convergent, i.e. the action is finite (moreover, $S = 0$).

**2.** Let us make the substitution into Eq. (5)

$$f = u(y,t)a(t)^{-\frac{n}{2}}$$

Then we obtain the following equation

$$\ddot{u} - a(t)^{-2}\Delta_h u + [\lambda - w(t)]u = 0 \tag{33}$$

where

$$w(t) = \frac{n}{2}\frac{\ddot{a}}{a} + \frac{n}{2}(\frac{n}{2} - 1)\frac{\dot{a}^2}{a^2}$$

We look for solutions satisfying the condition

$$\int_{\mathbb{R}\times N^n} u(y,t)^2 dt\sqrt{h}dy < \infty$$

There exists a well developed spectral theory for elliptic differential operators (see, for example [11, 12]. There is a spectral theory of the Liouville operator in ergodic theory of dynamical systems [14]. It would be interesting to develop a spectral theory for hyperbolic equations.

Consider for example on the Schwartz space $S(\mathbb{R}^2)$ of the functions $u = u(x,t)$ on the plane the hyperbolic differential operator of the form

$$Au = \frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u + \phi(x,t)u \tag{34}$$

where the smooth real valued function $\phi(x,t)$ admits a power bound on the variables $x, t$ at infinity. The operator $A$ admits a self-adjoint extension in $L^2(\mathbb{R}^2)$. In the particular simple case when the function $\phi$ has the form $\phi = x^2 - t^2$, the operator is the difference of the Schrodinger operators for two harmonic oscillators (this case was disscussed by Ginzburg and Markov). Hence it has in $L^2(\mathbb{R}^2)$ a complete system of eigenfunctions

$$u_{jn} = H_j(t)H_n(x)\exp\{-\frac{1}{2}(t^2 + x^2)\}, \tag{35}$$

$$Au_{jn} = \lambda_{jn}u_{jn}, \quad j, n = 0, 1, 2, ...,$$

with the corresponding eigenvalues $\lambda_{jn} = 2(n - j)$. Here $H_j$ are the Hermite polynomials.

The action

$$S = \frac{1}{2}\int_{\mathbb{R}^2}(\dot{u}^2 - u_x^2 - \phi u^2 + \lambda u^2)dtdx$$

is finite on solutions (35).

**3.** Note that together with (1) the so called equation with conformal coupling is considered:

$$\Box f + \xi R f + \lambda f = 0. \tag{36}$$

Here $R$ is the scalar curvature of the manifold $M$ and $\xi = (n-1)/4n$. For de Sitter space $R = 12,\ \xi = 1/6$. In this case Eq. (36) $\Box f + (2+\lambda) f = 0$ has square integrable solutions if $2 + \lambda = \lambda_{jn}$ (20) and in particular if $\lambda = 0$.

To conclude, in this note the square integrable solutions with the finite action (actons) of the Klein-Gordon equation for scalar field on manifolds have been considered. It would be interesting to study square integrable solutions on more general manifolds and also equations for fields with higher spins.

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