Effective theories: aspects of renormalization

V. Vereshagin[a*](speaker), K. Semenov-Tian-Shansky[b†], A. Vereshagin[c‡]

[a] *St.-Petersburg State University*

St.Petersburg, Petrodvoretz, 198504, Russia

[b] *Université de Liège and St.-Petersburg State University*

Institut de Physique, B5a, Université de Liège au Sart Tilman, B4000 Liège 1, Belgium

[c] *Bergen University and St.-Petersburg State University*

Institute of Physics and Technology, Allegt.55, N5007 Bergen, Norway

Abstract

We discuss the basic principles that allow to construct meaningful perturbation series for S-matrix elements in canonically nonrenormalizable theories which do not contain massless particles with spin $J > 1/2$. Two new ones are suggested — uniformity and summability. It is shown that these principles lead to infinite system of bootstrap conditions which restrict the allowed values of the physical (observable) parameters appearing in the extended perturbation scheme built for a given localizable effective theory.

Below we explain the corner stones of our approach to the problem of S-matrix calculations in effective theories. Because of lack of time/space we restrict ourselves by purely qualitative discussion. The details can be found in our publications [1] – [5]. The papers [1] – [3] are devoted to preliminary analysis; [4] and [5] contain the systematic recital of axioms, statements and proofs with corresponding formulae.

First we would like to explain the main mathematical idea that underlies the logic scheme of our reasonings. As well known, the *formal series* that appear, i.g., in the process of iterative resolving various (differential, integral, functional) equations, as a rule, do not present the true solution. The reason is that to acquire meaning the series must converge at least in certain sufficiently small domain of the space of variables (and whatever eventual parameters). Hence, to get the true solution one needs to consider the iterative series inside its convergence area (if it exists) and then perform the analytic continuation to the domain of interest. Here it is pertinent to recall that the analytic continuation of a series is nothing but certain kind of its reordering (resummation).

The written above suggests the following set of postulates which might be useful in solving the problems that appear in quantum field theories conventionally called as nonrenormalizable (see, e.g., [6]). First, as usual, the convergence of Dyson series for S-matrix should be understood as the *weak convergence*. Second, this series should be taken as just a formal (iterative) solution to the relevant Schrödinger equation. Third, it looks reasonable to only consider the special class of theories — those producing the polynomially bounded (in momentum space) physical (renormalized) amplitudes at every step of loop expansion. This restriction is explained by the necessity to fulfil the causality requirement (see, e.g., [7]) in perturbative calculations.

Surely, this set is not complete. In what follows we discuss two additional general principles and several requirements which allow us to derive certain algebraic conditions for the physical parameters of conventionally nonrenormalizable theories. Note that we only consider the case of four-dimensional space-time.

*e-mail: vvv@av2467.spb.edu

†e-mail: cyr-stsh@mail.ru

‡e-mail: alexand@fi.uib.no

Two notes are in order. First, in nonrenormalizable theory there are no limitations on the possible structure of required counterterms except those originating from the eventual linear symmetry. For simplicity, below we imply that the only symmetry of a theory is Lorentz invariance. The presence of any internal symmetry would not create a problem. Second, in what follows we only discuss theories which do not contain massless particles with spin $J > 1/2$.

Now we need to define what is *effective theory*. We say that *the field theory is effective if the quantum interaction Hamiltonian in the interaction picture contains all the monomials consistent with a given algebraic (linear) symmetry*. This is Weinberg's definition [8] rectified in [4]. ¿From the written above it follows that every nonrenormalizable theory with the full set of counterterms included in the Hamiltonian is, by definition, effective theory. Clearly, the effective theories are as renormalizable as ordinary renormalizable ones, the only difference being that, in contrast to renormalizable theories, they require formulating an infinite number of renormalization prescriptions (RP's). Because of this reason theories of this kind are not considered in traditional textbooks on renormalization theory.

It should be stressed that effective theories, by the very definition, present intrinsically quantum constructions. The popular now functional approach cannot be used in this case. As well known, the general problem of canonical quantization of classical theories with higher powers (and orders) of time derivatives still remains unsolved. That is why we rely upon alternative scheme of constructing the quantum theory first suggested by S. Weinberg (the detailed description and corresponding references can be found in [9]). This scheme can be illustrated as follows.

Asymptotic states (including vacuum)

$\Downarrow$

Creation (annihilation) operators $a^+(p)$, $(a^-(p))$ depending on momenta; their commutation relations

$\Downarrow$

Free causal fields (local operators) $\phi(x)$

$\Downarrow$

Interaction Hamiltonian H_{int}: Lorentz invariant sum of local monomials constructed from free fields and their derivatives of arbitrary high orders and powers

$\Downarrow$

Formal Dyson series for S-matrix elements (here T_W stands for Wick's T-product)

$$S_{fi} = \langle f|T_W exp\left\{-i\int H_{int}dx\right\}|i\rangle\,.$$

¿From the shown above schematic construction it is clearly seen that Weinberg's approach is only adjusted for S-matrix calculations. The off-shell extension of matrix elements would require attracting special suggestions. For this reason we call an effective theory constructed on the basis of this scheme as *effective scattering theory* (EST). In our work we only consider theories of this very type.

As mentioned above, one of the most difficult problems connected with renormalization of effective theories (which, as already noted, are renormalizable by the very construction) is the necessity to formulate an infinite number of renormalization prescriptions needed to fix the finite parts of counterterms. This looks impracticable until one finds a regularity effectively reducing the number of independent prescriptions. It seems quite natural to look for the mathematical expression of such a regularity in terms of *independent* parameters appearing in a theory. Therefore one needs to point out the full set of those parameters. Note, that even in the case of ordinary renormalizable theories the number of independent parameters not always coincides with the number of coupling constants in the Hamiltonian — the Yukawa model presents an example of such a situation. In the case of EST this problem requires a special study just because

the number of Hamiltonian parameters is actually infinite and only certain their combinations appear in S-matrix elements.

Here it is pertinent to recall the difference between *essential* and *redundant* parameters (see [9]). Those combinations of the Hamiltonian parameters which only contribute to Green functions but do not appear in expressions for S-matrix elements are called redundant. All the other independent combinations are called essential. This definition is not quite rigorous but we hope it is quite sufficient for qualitative understanding. For example, the gauge fixing parameter in Standard Model is redundant as well as the wave function renormalization constant. In contrast, the physical (or, the same, observable) mass of electron is the essential parameter.

Clearly, in the case of EST the set of conditions relating theoretical formulae to experimentally measurable quantities (in fact, this is nothing but the set of RP's) only contains the essential parameters. Therefore, in looks quite natural to begin the search of regularity, which might reduce the number of required renormalization prescriptions in EST, from the detailed study of those combinations of Hamiltonian parameters which appear in S-matrix graphs of various loop orders. This problem is completely solved now. The results are described in [4], where it is shown that an arbitrary S-matrix graph only depends on those combinations of Hamiltonian parameters (including the counterterm coupling constants) which provide the Lorentz invariant parametrization of all N-particle vertices when all their lines are considered on mass shell $p_i^2 = m_i^2$. Here N is (arbitrary) number of legs and p_i — the four-momentum of the i-th leg. In [4] these combinations were called as *resultant parameters* and the corresponding vertices — as *effective resultant vertices*. To use the effective resultant vertex as an element of Feynman rules, one has to make use of the above-mentioned Lorentz invariant on-shell form and consider it as a function of $4N$ independent variables $p_{i\mu}$ (components of four-momenta). Another element of Feynman rules — propagator — should be taken in the "minimal" form: with the relevant spin sum in numerator. Each one of the resultant parameters must be supplied with the *level index* l (the same for all parameters appearing in the vertex under consideration because this index relates to the vertex as a whole) needed to preserve the correct loop counting. When this is done one has to only consider the graphs without self-closed lines (bubbles), the corresponding loop order of a given graph being just the number of loops plus the sum of level indices of all its vertices. The special convenience of this parametrization is explained by the following reasons (see [4]). First, the resultant parameters of different levels are independent. Second, the set of resultant parameters (of various levels) is *full* and *closed*. This means that to calculate the S-matrix elements of a given loop order L one only needs to fix the resultant parameters of the levels $l \leq L$, the lower level parameters remaining unchanged when one calculates the graphs of higher loop orders. The important thing is that, as shown in [5], all singularities of L-loop graphs turn out to be fixed by the resultant parameters of lower levels $l < L$.

The resultant parametrization provides a solution to the problem of isolating the redundant parameters — those which the S-matrix does not depend upon. The next problem of EST is to point out the guiding principle which could fix the order of summation of an infinite number of graphs contributing to S-matrix elements with a given (fixed!) number of loops. The infiniteness of this number follows from the fact that the Hamiltonian contains an infinite number of monomials constructed from the same set of field operators and differing from one another by the total number of derivatives. To solve this problem we attract the *summability requirement* [5]. It is formulated as follows: *in every sufficiently small domain of the complex space of kinematical variables there must exist an appropriate order of summation of the formal series of contributions coming from the graphs with a given number of loops, such that the reorganized functional series converges. Altogether, these series must define a unique analytic function with only those singularities which are peculiar to individual graphs.* This requirement is analogous to the maximal analyticity principle used in the analytic theory of S-matrix. For this reason sometimes we call it as *analyticity principle*. This principle is widely used in modern theory of divergent series (see, i.g., [10]). Note that it does not fix the true summation order

itself, it only states that this order does exist. The thing is that it is very difficult, if ever possible, to point out any concrete order just because the latter depends on the domain in question. Instead, it is quite possible to impose general limitations on the allowed type of analytic functions which result from such a summation. One of limitations of this kind is provided by the above-mentioned condition of polynomial boundedness (in momenta) of S-matrix elements. Surely, it restricts the class of allowed amplitudes but this restriction is too general. To proceed further we need somewhat more constructive condition.

For this reason we attract one more requirement called in [5] as *asymptotic uniformity* (or, simply uniformity) *condition*. It reads as follows. *The degree of the bounding polynomial which specifies the asymptotics of a given loop order amplitude must be equal to that specifying the asymptotics of the full (experimental) amplitude of the process under consideration.* Surely, this latter degree may depend on the type of the process in question as well as on the values of the variables kept fixed. We refer to experimental asymptotics just because we would like to deal with physically reasonable schemes. In contrast, the requirement of equality of bounding polynomial degrees for different loop order amplitudes mirrors our wish to construct the perturbation series in a way that allows one to avoid unnecessary mutual contractions between different terms of loop series and, hence, to perform a comparison of the finite order calculations with experimental data.

It should be stressed that the requirements of uniformity and summability are nothing but independent subsidiary conditions fixing the type of perturbation scheme which we would like to work with. Surely, there is no guarantee that on this way one can construct the most general expressions for S-matrix elements in effective theory. Nevertheless, there is a hope to construct at least meaningful ones presented by the Dyson's type perturbation series only containing the well-defined items. Note that both these requirements are trivial in the case of conventional renormalizable theory.

Now it remains to explain the concept of *localizability* introduced in [5]. As noted above, the primary object in Weinberg's scheme of constructing the EST is the space of asymptotic states. This space may only contain the particle states which are stable with respect to the type of interaction in question. In such a space there is no room for resonance states because, by the very definition, resonances cannot survive outside the interaction area — their lifetime is finite in contrast to that of true stable particles. This circumstance creates a serious problem in the case when Weinberg's scheme is used for constructing the EST. To explain this problem let us consider the model example.

Suppose that there is only one kind of stable particles — light pseudoscalar "pions" (without isospin). In this case the interaction Hamiltonian contains an infinite set of four-particle vertices which can be always presented as follows:

$$\mathcal{H}_4 = \sum_{i,j,k=0}^{\infty} G_{ijk}\,\pi(D^{\mu_i}D^{\nu_j}\pi)(D_{\nu_j}D^{\xi_k}\pi)(D_{\xi_k}D_{\mu_i}D^{\nu_j}\pi)\ , \quad D_{\mu_i} \equiv \partial_{\mu_1}\ldots\partial_{\mu_i}\ .$$

Hence the tree level four particle amplitude $A(s,t,u)$ takes a form of (formal) power series

$$A(t,\nu_t) = \sum_{i,j=0}^{\infty} C_{ij}\ t^i\ {\nu_t}^j$$

in arbitrary pair of independent kinematical variables (here we took t and $\nu_t \equiv (s-u)$ as such a pair). The coefficients C_{ij} in this series are the functions of G_{ijk} and, possibly, of pion mass μ. Let us now consider this series at, say, $t = 0$:

$$A(0,\nu_t) = \sum_{j=0}^{\infty} C_{0j}\ {\nu_t}^j.$$

There are three possibilities. First: it may happen that this series diverges everywhere in complex-ν_t plane. Second: it converges everywhere. Third: there is a circle (with raduis R) such that for $|\nu_t| < R$ this series converges while at $|\nu_t| > R$ it diverges.

The first situation is not interesting just because in this case the tree level amplitude does not exist and we have nothing to discuss. In what follows it is tacitly implied that the set of coupling constants G_{ijk} is constructed in such a way that this situation takes no place.

The second possibility also is not interesting. This is because in this case the tree level amplitude would contradict the polynomial boundedness condition and, hence, the causality principle would happen broken. To avoid this trouble one has to require that $C_{0j} = 0$ for $j > J = 0$. This is quite possible but not interesting because it would correspond to a conventional renormalizable theory ϕ^4. Alternatively, the conjecture $J \geq 1$ leads to inner contradiction because it corresponds to nonrenormalizable theory and, hence, the counterterms inevitably result in appearing an infinite power series.

At last, the third possibility deserves more attentive consideration. First of all we would like to stress that we do not discuss the exceptional case when the circle $|\nu_t| = R$ presents the natural analyticity boundary (barrier). In what follows we tacitly imply that this is not the case. Conversely, we imply that the series under consideration may be analytically continued beyond this boundary and this continuation results in certain analytical function well defined in the complex-ν_t plane except possible isolated singular points. The concept of localizability introduced in [5] specifies the concrete class of effective scattering theories which we only work with. It is formulated as follows: *the effective theory is called localizable if, first, its formal series for all the tree level amplitudes converge at certain small domains in relevant spaces of kinematical variables and, second, in those domains they can be reproduced by well-defined in wider domains tree level series of the modified perturbation scheme (of Dyson's type) based on the new (formal!) "interaction Hamiltonian" that contains auxiliary fields with spins J_i and "masses" M_i which do not belong to the spectrum of one-particle asymptotic states*. The auxiliary fields may be interpreted as those describing particles (resonances) which are unstable with respect to decays into the true asymptotic states of initial theory. The term *mass* in this case should be understood as just a convention. The auxiliary Hamiltonian in [5] is called as *extended Hamiltonian* and the corresponding perturbation series — as *extended perturbation scheme*. Note that the free Hamiltonian remains the same as that of initial EST.

The essence of the given above definition can be shortly explained as follows. The tree level amplitudes produced by the extended perturbation scheme (based on auxiliary Hamiltonian) must realize an analytic continuation of the tree level amplitudes obtained in the initial perturbation scheme to a wider domain. To ensure this requirement all the singularities of the initial tree level series must be reproducible as separate items of the new series following from the extended perturbation scheme. Thus the extended perturbation scheme presents just an auxiliary construction only needed to properly define the tree level series of the initial localizable effective theory in a wider domain. Once the original tree level amplitudes are well defined in the whole space of relevant variables, the calculation of loops becomes just a matter of machinery.

It should be stressed that our definition does not imply any extension of the initial space of asymptotic states — it remains unchanged and only contains those states which correspond to true stable particles. The normalization condition remains the same as that in the initial effective theory and the unit operator $\hat{\mathbf{1}}$ in Fock space takes a form of infinite sum over the states only containing stable particles:

$$\hat{\mathbf{1}} = \sum_{n=0}^{\infty} |n\ (\text{stable!})\rangle\langle n\ (\text{stable!})|.$$

There is no need in defining the action of auxiliary "field operators" on the vectors of this space because we never need to know the result. We only need to know the explicit expressions for

corresponding propagators defined in accordance with cluster decomposition principle and the Lorentz symmetry requirements. These expressions follow directly from Weinberg's scheme.

Now we can explain few more technical points that render the described above general reasonings into a practically useful calculational scheme which provides one with an infinite set of explicit equations connecting among themselves the numerical values of physical observables. For simplicity, below we discuss as an example the model case when the only stable particles in a theory are pions (isospin $I_\pi = 1$, mass μ) and kaons (isospin $I_K = 1/2$, mass m). Let us consider the amplitude of elastic pion-kaon scattering $\pi_a(k_1) + K_\alpha(p_1) \to \pi_b(k_2) + K_\beta(p_2)$ (here $a, b = 1, 2, 3$ and $\alpha, \beta = 1, 2$ stand for isotopic indices) as a function of scalar variables $s = (k_1 + p_1)^2$, $t = (k_1 - k_2)^2$, $u = (k_1 - p_2)^2$ ($s + t + u = 2\mu^2 + 2m^2$). This amplitude can be presented as follows

$$M^{ba}_{\beta\alpha} = \delta_{ba}\delta_{\beta\alpha}\, A + i\,\varepsilon_{bac}\sigma_{\beta\alpha}\, B\,,$$

where A and B stand for invariant amplitudes depending on $s\,, t\,, u$.

To make use of the discussed above general principles we need to specify the degrees of bounding polynomials that restrict the asymptotic behavior of invariant amplitudes. For this we attract the experimental information. As known from experiment, at small values of the momentum transfer the high energy behavior of hadron scattering amplitudes is described by Regge asymptotics. Therefore, the known values of intercepts can be used to specify the degrees of bounding polynomials in three mutually intersecting hyperlayers $B_x\{\nu_x \in \mathbf{C};\ x \in \mathbf{R},\ x \sim 0\}$, where the variable $x = (s, t, u)$ (hyperlayer thickness) stands for the momentum transfer in corresponding channel while the energy-like variables ν_x are defined as follows: $\nu_s = (u - t)$, $\nu_t = (s - u)$, $\nu_u = (t - s)$. Having those values, we can write down the Cauchy formula which provides the value of relevant combination $f(\nu_x, \mathbf{x})$ of amplitudes A and B at every point in corresponding hyperlayer B_x:

$$f(\nu_x, \mathbf{x}) = \sum_{n=0}^{N_x} \frac{1}{n!} f^{(n)}(0, \mathbf{x}) {\nu_x}^n + \frac{z^{N_x+1}}{2\pi i} \sum_{k=1}^{+\infty} \oint\limits_{C_k} \frac{f(\xi, \mathbf{x})}{\xi^{N_x+1}(\xi - \nu_x)}\, d\xi\ .$$

Here N_x stands for the relevant bounding polynomial degree and $x = (s, t, u)$; contours C_k surround possible cuts and isolated singular points. It is implied that singular points (in which the cuts start) are enumerated in order of increasing modulo. In those cases when $N_x < 0$ (so-called superconvergence) the first sum should be omitted and in the second one the value $N_x = -1$ should be taken. This relation provides the uniformly converging functional series for the N_x-bounded function $f(\nu_x, \mathbf{x})$ in B_x. In the case under consideration we can write down two series of this kind in every one of three hyperlayers B_x.

Let us now turn to extended perturbation scheme. In accordance with the (tacitly implied) property of localizability we can introduce the auxiliary resonance fields and construct the extended Hamiltonian, no limitations on allowed values of spin and mass of resonances being imposed except those commonly accepted (finite number of resonances with the same mass value, absence of boson resonances with isospin $I > 1$, etc.). The corresponding Feynman rules allow us to construct the formal series for tree level amplitudes which, by condition, are summable. The summability and polynomial boundedness requirements tell us that these series must result in polynomially bounded amplitudes A and B, the degrees of corresponding bounding polynomials being fixed by the known values of Regge intercepts. Thus we can write down the relevant Cauchy formulae in every one of three hyperlayers B_x. The localizability principle tells us that the only singularities of the resulting functions are simple poles provided by the graphs with particle exchanges. Hence all the integrals can be done explicitly and the corresponding infinite sums result in uniformly converging series of Mittag-Leffler type (poles plus correcting polynomials, if needed). In the hyperlayers which correspond to $N_x \geq 0$ those series, along with triple couplings and resonance masses, also contain $(N_x + 1)$ unknown functions $f^{(n)}(0, \mathbf{x})$. In accordance with the summability principle two series which present the

same analytic function in two different hyperlayers, say in B_s and B_t, must coincide identically in the mutual intersection domain $D_u \equiv B_t \cap B_s$. This equality provides a source of an infinite number of *tree level bootstrap conditions* for the parameters of resonance spectrum (masses and triple coupling constants). Indeed, at every point in D_u (say, at $s = t = 0$) two series must be equal as well as all their derivatives. Clearly, this also relates to the domains D_t and D_u.

All written above concerning the tree level bootstrap constraints remains true with respect to those constraints of arbitrary level. In [5] it is shown that a part of tree level constraints (called as the first kind bootstrap conditions) allow one to express the unknown functions $f^{(n)}(0, \mathbf{x})$ in terms of resonance masses and triple coupling constants. The remaining constraints (called as the second kind bootstrap conditions) present an infinite number of self-consistency conditions for the *physical* (measurable) parameters of localizable EST. The reason is that the second kind bootstrap conditions restrict the allowed values of renormalization prescriptions. Besides, it is shown that the bootstrap conditions of higher levels $l \geq 1$ only contain the second kind constraints. Moreover, it is shown that in theories with the amplitudes which asymptotic behavior is governed by known Regge intercepts, the system of independent renormalization prescriptions only contains those fixing the counterterm vertices with $n \leq 3$ lines, while other prescriptions are determined by self-consistency requirements. The prescriptions for $n \leq 3$ cannot be taken arbitrary: an infinite number of second kind bootstrap conditions should be respected.

The mathematical problem of self-consistency of the infinite system of second kind bootstrap conditions still awaits a solution. That is why, to demonstrate fruitfulness of the discussed above general principles we have analyzed (see [3]) the well known model based on Eyler's B-function (more precisely, on the Lovelace amplitude). It is shown that the corresponding bootstrap constrains are nothing but an infinite system of identities for Pochhammer symbols, the most part of which hardly could be found in the literature. To check the physical reasonability of these principles we obtained and compared with known data several constraints for the parameters of resonances in $\pi\pi$, πK, πN and KN elastic scattering processes. In all these cases the results turned out to be rather promising. They will be published elsewhere (a part of them has been discussed already in our talks at International workshops, the list of references can be found in [5]).

It remains to say that the material discussed above relates to so-called *minimal* renormalization prescriptions — those needed to fix the on-shell amplitudes of arbitrary loop order. The difference between minimal and nonminimal prescriptions is explained in detail in [4]; in [5] we only discuss the minimal RP's. Clearly, to calculate the loop graphs one needs to formulate the set of nonminimal prescriptions. At this point the general principles considered in our talk also play an important role. This problem will be analyzed in a separate publication.

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