# Avoiding cosmological chaos with $S$-branes

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**Abstract**

In 4 dimensions there are cosmological models which exhibit chaotic BKL oscillations as one approaches the initial singularity *i.e.* the Kasner parameters "oscillate" between different values as $\tau \to +0$. Similar BKL oscillations have been found in higher dimensional, string inspired cosmological models, which have form fields and scalar dilaton fields, and it has been conjectured that this behavior is generic in such higher dimensional cosmologies. In this essay we give an example where this chaotic behavior is avoided. The system is 5 dimensional spacetime with 3-form fields; the metric is diagonal and the form field describes a "maximal" set of *composite*, electric $S$-branes. When the composite $S$-branes are replaced by non-composite $S$-branes chaotic oscillation occur again. This leads us to propose the conjecture that the occurrence or non-occurrence of chaotic oscillating behavior for diagonal metrics depends on the character of the $S$-brane – composite or non-composite.

In [1] Belinskii, Lifshitz and Khalatnikov investigated the dynamics of 4-dimensional Bianchi-IX cosmological solutions. They found that near the initial singularity these solutions took a Kasner form but with the Kasner parameters performing a never-ending chaotic oscillations as one approached the singularity. The asymptotic metric was

$$ds^2_{as} = -d\tau^2 + \sum_{i=1}^{n} \tau^{2\alpha^i} g^{(i)}_{m_i n_i}(x_i) dx_i^{m_i} dx_i^{n_i} \ , \tag{1}$$

where $g^{(i)}$ are Einstein metrics for the spatial coordinates, having dimension given by $d_i$. The Kasner parameters, $\alpha^i$, satisfy the following two conditions:

$$\sum_{i=1}^{n} d_i \alpha^i = 1 \quad , \quad \sum_{i=1}^{n} d_i (\alpha_i)^2 = 1 \tag{2}$$

The oscillatory behavior arose in that the Kasner parameters were time-dependent, $\alpha^i(\tau)$, and took constant values for some time interval $\tau \in [\tau_{k+1}, \tau_k)$ $k = 0, 1, 2....$ but took different values in neighboring intervals. A graphical way of picturing this behavior was given by Chitre in [2] in terms of "billiard" description. In this "billiard" picture the anisotropy part of scale factors was described by a point that was "moving freely" in the Lobachevsky plane, $H_2$, until it

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encountered a "hard wall" corresponding to a constraint (inequality) on the Kasner parameters due to gravity or various matter sources. At the wall there is a "reflection" after which the parameters, $\alpha^i$ take new values and "move freely" until they encounter another "hard wall". If the Lobachevsky plane is completely enclosed this oscillation or switching of the parameters persists indefinitely.

In [3, 4, 5, 6, 7, 8] these results from 4D were extended to higher dimensions with block-diagonal metrics and various fluid or matter sources using the billiard approach on generalized Lobachevsky spaces. Additionally in [7] mathematical conditions were given under which one would get asymptotic Kasner versus BKL type oscillations of the metric. These conditions were formulated in terms of brane $U^I$ co-vectors

$$U^I(\alpha) \equiv U_i^I \alpha^i \equiv \sum_{i \in I} d_i \alpha^i, \tag{3}$$

The brane is labeled by index $I$ where $I = \{i_1, \ldots, i_k\}$, $i_1 < \ldots < i_k$, describes the location of brane, *i.e.* the brane is "attached" to the sub-manifold $M_s = M_{i_1} \times \ldots \times M_{i_k}$. In [7] (3) also contained terms associated with scalar fields. These are omitted since in what follows we do not consider a scalar field. The scalar products of the $U^I$ co-vectors were defined using the inverse ($G^{ij} = \delta_{ij} + \frac{1}{2-D}$) of the minisuperspace metric [9] [10].

$$(U^I, U^I) = G^{ij} U_i^I U_j^I = d_I \left(1 + \frac{d_I}{2-D}\right). \tag{4}$$

In the above $d_I$ is the dimension of the brane world-volume manifold. Asymptotic Kasner type behavior occurred if there existed a Kasner set $\alpha$ such that for all branes with $(U^I, U^I) > 0$

$$U^I(\alpha) > 0. \tag{5}$$

If, on the other hand, one had that for any Kasner set $\alpha$ there existed $I$ such that $(U^I, U^I) > 0$ and

$$U^I(\alpha) \leq 0, \tag{6}$$

then BKL oscillating behavior occurred near the singularity.

In [11] it was conjectured that this never-ending oscillating behavior is generic to a certain class of string theory inspired cosmologies (see [12] for a review and [13] for recent work on simple models with 2-forms). The system considered in [11] was $D$ dimensional gravity coupled to a scalar field, and a $p$-form field, whose action is

$$S_g = \int d^D x \sqrt{|g|} \left( R(g) - g^{MN} \partial_M \varphi \partial_N \varphi - \sum_p \frac{1}{p!} e^{2\lambda_p \varphi} (F^p)^2 \right) \tag{7}$$

where $g$ is the multidimensional metric, $\varphi$, is a scalar, dilaton field, and $F^p = dA^{p-1}$ is a $p$-form field. The action (7) is the one used for investigating $S$-branes [10, 14, 15, 16].

In this essay we present an example, involving higher dimensional gravity and a *composite* form field, where the BKL oscillating behavior is avoided. When the form field is taken to be *non-composite* BKL oscillations are reinstated.

The multidimensional system considered here is similar to that of [17] but without the scalar field. The action is given by

$$S = \int d^D x \sqrt{|g|} \left[ R[g] - \frac{1}{q!} F_{[q]}^2 \right], \tag{8}$$

The metric is diagonal and is taken to have the form

$$ds^2 = -\mathrm{e}^{2\gamma(t)} dt^2 + \sum_{i=1}^{n} \mathrm{e}^{2\phi^i(t)} (dx^i)^2. \tag{9}$$

The $(q = p + 2)$-form, $F$, is taken in the electric composite $Sp$-brane ansatz of the form

$$F = \sum_{I \in \Omega_e} d\Phi^I \wedge \tau(I) \tag{10}$$

where $\Phi^I$ is a scalar potential. This ansatz is composite since it is a sum of terms rather than a single term as in the non-composite case. In (10) $\Omega_e$ is the set of multi-indices having the form $\{i_0, i_1, ..., i_p\}$ with all $i_k$ being among the numbers $1, ..., n$. Also in (10) we have introduced a volume form of rank $p+1$, $\tau(I) \equiv dx^{i_0} \wedge \ldots \wedge dx^{i_p}$ with $I = \{i_0, ..., i_p\}$ corresponding to the brane submanifold with the coordinates $x^{i_0}, \ldots, x^{i_p}$. The ansatz functions $\gamma(t), \phi^i(t)$, and the $q$-forms are all assumed to depend only on $t$.

The $q = (p+2)$-form took the form [17]

$$F_{[q]} = \frac{1}{(p+1)!} Q_{i_0 \ldots i_p} \exp(2\phi^{i_0} + \cdots + 2\phi^{i_p}) dt \wedge dx^{i_0} \wedge \cdots \wedge dx^{i_p}. \tag{11}$$

$Q_{i_0 \ldots i_p}$ are components of antisymmetric form coinciding with constant charge densities $Q_I$ where $I \in \Omega_e$ and $i_0 < \cdots < i_p$. Since the Ricci tensor is diagonal one finds

$$F_{iM_2 \ldots M_q} F_j^{M_2 \ldots M_q} \propto T_{ij} = 0, \qquad i \neq j. \tag{12}$$

where $T_{ij}$ is the stress-energy tensor. This leads to constraints among the charge densities, $Q_{i_0 \ldots i_p}$. The functional dependence on $t$ can be absorbed by defining $\bar{Q}_{i_0 \ldots i_p} = Q_{i_0 \ldots i_p} \exp(2\phi^{i_0} + \cdots + 2\phi^{i_p})$. With this definition the constraints in (12) can be written as [17]

$$C_i^j \equiv \sum_{i_1, \ldots, i_p = 1}^{n} \bar{Q}_{i i_1 \ldots i_p} \bar{Q}^{j i_1 \ldots i_p} = 0, \tag{13}$$

$i \neq j$, $i, j = 1, \ldots, n$. These $\bar{Q}_{i_0 \ldots i_p}$ can be viewed as "running" charge densities with the functional dependence coming from $\phi^{i_k}(t)$.

In [17] it was shown that for spacetimes with $D = 4m + 1$ and form fields with $p = 2m - 1$ $(m = 1, 2, 3, ...)$ these constraints could be satisfied if the non-running charge densities were self-dual or anti-self-dual (*i.e.* $Q_{i_0 \ldots i_p} = \pm \frac{1}{(p+1)!} \varepsilon_{i_0 \ldots i_p j_0 \ldots j_p} Q^{j_0 \ldots j_p} = \pm (*Q)_{i_0 \ldots i_p}$ – where $\varepsilon_{i_0 \ldots i_p j_0 \ldots j_p}$ is the completely anti-symmetric symbol) and if all the scale factors were the same $\phi^i(t) = \phi(t)$

From [17] [18] if the above conditions are satisfied then the equations of motion for the action (8) have the following solution

$$\phi(t) = \frac{2}{n} f(t) , \qquad f(t) = -\ln \left[ (t - t_0) |Q^2 K|^{1/2} \right], \tag{14}$$

$$ds^2 = -e^{2n\phi(t)} dt^2 + e^{2\phi(t)} \sum_{i=1}^{n} (dx^i)^2, \tag{15}$$

$$F = e^{2f(t)} dt \wedge Q, \qquad Q = \frac{1}{(p+1)!} Q_{i_0 \ldots i_p} dx^{i_0} \wedge \cdots \wedge dx^{i_p}, \tag{16}$$

where $K = -\frac{n}{4(n-1)}$ and $Q^2 = \sum_{I \in \Omega_e} Q_I^2$. In the above the ansatz function $\gamma(t) = \sum_{i=1}^n \phi^i(t) = n\phi(t)$ because all $\phi^i(t)$ are the same. The charge density form $Q$ is of rank $n/2 = 2m$ and is self-dual or anti-self-dual. This solution with composite form field does not have oscillating Kasner like behavior as one approaches the singularity at $t \to +\infty$ (or $\tau \to +0$). It should be stressed that the solution presented above is a general one when all charge densities $Q_I$ are non-zero.

We now specialize to the case $D = 5$ and $p = 1$ (*i.e.* the form fields are 3-forms) and show explicitly that one does not have BKL oscillating behavior. For these choices – $D = 5$, $n = 4$,

$p = 1$ – the solution in (14) (15) (16) becomes

$$ds^2 = -d\tau^2 + \tau \sqrt{\left| \frac{Q^2}{3} \right|} \sum_{i=1}^{4} (dx^i)^2 \quad , \quad F_{[3]} = \frac{1}{(2)!} d\tau \wedge Q_{ij} dx^i \wedge dx^j , \tag{17}$$

where we have used $d\tau^2 = e^{2n\phi} dt^2 \rightarrow d\tau = -e^{n\phi} dt$ to write the results in terms of $\tau$. Also in (17) $I = \{i, j\}$, $i < j$; $i, j = 1, 2, 3, 4$, and all charges – $Q_{12}, Q_{13}, Q_{14}, Q_{23}, Q_{24}, Q_{34}$ – are non-zero. As can be seen from (17) this solution has neither an oscillating nor even a Kasner-type behavior (*i.e.* writing the metric in Kasner form, (1), the Kasner powers do not satisfy (2) near the singularity. From the billiard representation point of view the solution in (17) corresponds to a static point in the billiard.

Omitting the constraints (13) leads to a model with six 3-forms having the action

$$S_{5br} = \int_M d^5 z \sqrt{|g|} \left[ R[g] - \frac{1}{3!} \sum_I (F^I_{[3]})^2 \right] , \tag{18}$$

instead of (8) and a non-composite ansatz

$$F^I_{[3]} = d\Phi^I \wedge \tau(I), \tag{19}$$

instead of the composite one (10).

Using the results of [5, 7] one can show that these non-composite solutions with non-zero charge densities, $Q_I$, of the 1-branes (strings) give never-ending oscillating behavior near the singularity as $\tau \rightarrow +0$. The walls of the "billiard" come from the fact that the scalar products of all $U$-vectors obey

$$(U^I, U^I) \equiv G^{ij} U^I_i U^I_j = \frac{2}{3} > 0, \tag{20}$$

for $I = \{i, j\}$, $i < j$, [5, 7]. In (20) we have used (4) with $d_I = 2$.

One will get never-ending oscillating behavior for the non-composite model since for any Kasner set, $\alpha = (\alpha^i)$, there exists a brane $U$-vector which satisfies

$$U^I(\alpha) = \alpha^i + \alpha^j \leq 0, \tag{21}$$

for $I = \{i, j\}$, $i < j$. Using the results [5, 7, 18] we find $\alpha = (0, 0, 0, 1), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ and 6 other sets obtained by permutations. These are points on the Kasner sphere $S^2$ coinciding with the eight vertices of the "deformed" cube. For these point the equality in (21) holds, and from (20) we see that the conditions in (6 are satisfied giving BKL oscillating behavior near the singularity.

One can also use the billiard representation to show that this example gives never-ending oscillating behavior. The billiard representation gives a 3-dimensional billiard "table" with six walls that form a deformed cube. From the relationships in [7] the explicit form of this billiard belongs to the 3-dimensional Lobachevsky space ("ball") $H^3 = D^3 = \{\vec{z} \in \mathbf{R}^3 : |\vec{z}| < 1\}$ and the walls of the billiard are given by the six inequalities: $|\vec{z} - \vec{v}_k| > \sqrt{2}$. where $k = 1, ..., 6$ where $\vec{v}_1 = -\vec{v}_6 = (\sqrt{3}, 0, 0)$, $\vec{v}_2 = -\vec{v}_5 = (0, \sqrt{3}, 0)$, and $\vec{v}_3 = -\vec{v}_4 = (0, 0, \sqrt{3})$. The enclosure of this billiard is not compact, but according to the "illumination" theorem of [5, 7] the billiard has a finite volume, thus indicating BKL oscillations. This can be seen in terms of the "illumination" of the Kasner sphere by point sources of "light". The sources of light are located at the points $\vec{v}_k$, $k = 1, ..., 6$, and completely illuminate the unit sphere $S^2 = \{\vec{z} \in \mathbf{R}^3 : |\vec{z}| = 1\}$.

Thus, in this essay we have presented an example showing that the never-ending BKL oscillating behavior near the singularity with diagonal metric can be broken when *non-composite* branes are replaced by *composite* ones.

In general some of the walls of the billiard could be "destroyed" due to some of the charge densities being zero. This was the case in the 4D example of [18]. "Destroying" some of the billiard walls would lead to asymptotic Kasner-like, non-oscillatory behavior. In the example presented here however all the walls survive since all the charge densities, $Q_{ij}$ $(i < j)$, are non-zero and obey (anti)-self-duality relations. In this case the constraints select only those solutions that have simple isotropic behavior (with non-Kasner and non-oscillating behavior near the singularity).

These examples suggest a conjecture that an analogous effect may take place for 11-dimensional supergravity and 10-dimensional models of superstring origin. In other words, one may consider the following hypothesis: for certain supergravity models (say, for $D = 11$, or $D = 10$ $IIA$ supergravities) there are no composite $S$-brane solutions with diagonal metrics that have a BKL oscillating behavior near the singularity.

The main conclusion of this essay is that the never-ending oscillating behavior near the singularity of solutions in certain cosmological models with form fields can be broken or avoided. This avoidance can be attributed to the form of the ansatz considered here: a diagonal metric, and composite electric $Sp$-brane ansatz for the form fields. This result comes from the constraints on the brane charge densities which are due to the diagonality of the metric and the compositeness of the brane system. This shows that BKL like oscillating behavior may not be generic to such higher dimensional cosmologies.

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