Self-dual and non-self dual axially symmetric caloron solutions in $SU(2)$ Yang-Mills theory

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Abstract

New static regular axially symmetric solutions of $SU(2)$ Euclidean Yang-Mills theory are constructed numerically. They represent calorons having non-trivial Polyakov loop at spacial infinity. The solutions are labeled by two integers m, n. It is shown that besides known, charge one self-dual periodic instanton solution, there are other non-self dual solutions of the Yang-Mills equations naturally composed out of pseudoparticle constituents.

1 Introduction

The interplay between properties of self-dual BPS monopole solutions [1] and instantons [2, 3] caused a lot of attraction over last decade. It was shown that exact caloron solutions, i.e. the periodic instantons at finite temperature on $\mathbb{R}^3 \times S^1$, for which component A_0 approaches a constant at spacial infinity [5, 7], $A_0 \to 2\pi i\omega = 2\pi i\omega^a\sigma^a$, are composed out of Bogomol'nyi-Prasad-Sommerfeld (BPS) monopole constituents [8]. This periodic array of instantons corresponds to the non-trivial Polyakov loop (holonomy) around S^1 at spacial infinity. In the periodic gauge $A_\mu(\mathbf{r}, x_0 + T) = A_\mu(\mathbf{r}, x_0)$ the Polyakov loop operator is defined as

$$\mathcal{P}(\mathbf{r}) = \mathrm{Tr} \lim_{r\to\infty} P \exp\left(\int_0^T A_0(\mathbf{r}, x_0)dx_0\right) , \qquad (1)$$

where T is the period in the imaginary time direction, which is related with finite temperature Θ as $T = 1/k\Theta$, and P denotes the path ordering. Non-trivial value of $\mathcal{P}$ acts like a Higgs field in adjoint representation labeling the vacua, because under a gauge transformation $U(\mathbf{r})$ it transforms as

$$\mathcal{P}(\mathbf{r}) \to U(\mathbf{r})\mathcal{P}(\mathbf{r})U^{-1}(\mathbf{r}) \qquad (2)$$

Alternatively, one can formulate the model in $\mathbb{R}^4$ by fixing periodicity modulo gauge transformations. Indeed, the temporal component $A_0 = 2\pi i\omega^a\sigma^a$ can be gauged away by non-periodic gauge transformation $U(\mathbf{r}, x_0) = \exp\{2\pi i x_0\omega^a\sigma^a\}$ and then

$$A_\mu(\mathbf{r}, x_0 + T) = e^{2\pi i\omega^a\sigma^a} A_\mu(\mathbf{r}, x_0)e^{-2\pi i\omega^a\sigma^a} \qquad (3)$$

For the self-dual caloron solutions, considered in [8] (so called KvBLL calorons), the field strength vanishes at spatial infinity, or, equally, it is a pure gauge there, and the constituents are just BPS monopole-antimonopole pairs. The property of self-duality allows us to apply very powerful formalism of ADHM-Nahm construction [4] to obtain different exact multi-caloron configurations [8] and analyse properties of the BPS monopole constituents. In particular,

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it was shown that as the size of charge one $SU(2)$ caloron is getting larger than the period T, the caloron is splitting into constituents which represent the monopole-antimonopole pair configuration. The properties of these saddle point solutions in related $SU(2)$ Yang-Mills-Higgs (YMH) model were discussed first by Taubes [11], and different monopole-antimonopole systems were constructed numerically in [12, 13, 14], both in BPS limit and beyond.

However, besides the self-dual instantons, also solutions to the second order order Euler-Lagrange equations of the euclidean Yang-Mills (YM) theory are known [9]. Also recently the non-self dual instanton-anti-instanton pair static configuration was constructed [10], which represent a saddle point configurations, the deformation of the topologically trivial sector.

For the non-self dual instantons the action is finite but the field strength behaves different at spacial infinity and the action is not proportional to the Chern-Pontryagin topological charge.

In the present work we study the static axially symmetric $SU(2)$ YM caloron solutions on $R^3 \times S^1$ with non-trivial holonomy and find regular numerical solutions which are labeled by two integers (n, m as their counterparts in the YMH system, the monopole-antimonopole chains and the circular vortices [14]. Similar to the case of axially symmetric instantons discussed in [10], only $m = 1$ solutions are self dual, the calorons labeled by $m \geq 2$ however are non-self dual. The latter configurations are composed of constituents and corresponds to the monopole-antimonopole chains and/or vortex-like solutions.

In section II we present the action of the euclidean YM theory, the axially symmetric ansatz and the boundary conditions imposed to get regular solution. We will make a detailed numerical study of the solutions of the corresponding second order field equations. In section III we discuss the properties of the caloron solutions, in particular the dependence on the temperature.

2 Euclidean $SU(2)$ action and axially symmetric ansatz

We consider the usual $SU(2)$ YM action

$$S = \frac{1}{2}\int d^4x \mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right) = \frac{1}{4}\int d^4x \left(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu}\right)^2 \mp \frac{1}{2}\int d^4x \mathrm{Tr}\left(F_{\mu\nu}\widetilde{F}_{\mu\nu}\right) \tag{4}$$

in Euclidean space $R^3 \times S^1$ with one periodic dimension $x_0 \in [0, T]$ and in normalization where the gauge coupling $e^2 = 1$. Here $su(2)$ gauge potential is $A_\mu = A_\mu^a \tau^a/2$ and the field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. The topological charge is defined as

$$Q = \frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}\int d^4x \mathrm{Tr} F_{\mu\nu}F_{\rho\sigma} \tag{5}$$

and for the self-dual configurations $S = 8\pi^2 Q$.

To construct new regular caloron solutions of the corresponding *second order* field equations and investigate dependence of these solutions on the boudary conditions, we employ the known axially symmetric Ansatz for the gauge field

$$A_\mu dx^\mu = \left(\frac{K_1}{r}dr + (1-K_2)d\theta\right)\frac{\tau_\varphi^{(n)}}{2e} \quad - \quad n\sin\theta\left(K_3\frac{\tau_r^{(n,m)}}{2e} + (1-K_4)\frac{\tau_\theta^{(n,m)}}{2e}\right)d\varphi;$$

$$A_0 = A_0^a\frac{\tau^a}{2} \quad = \quad \left(K_5\frac{\tau_r^{(n,m)}}{2} + K_6\frac{\tau_\theta^{(n,m)}}{2}\right),$$

which was previously applied to the Yang-Mills-Higgs system [14]. The Ansatz is written in the basis of $su(2)$ matrices $\tau_r^{(n,m)}, \tau_\theta^{(n,m)}$ and $\tau_\varphi^{(n)}$ which are defined as the dot product of the Cartesian vector of Pauli matrices $\vec{\tau}$ and the spacial unit vectors

$$\begin{aligned}
\hat{e}_r^{(n,m)} &= (\sin(m\theta)\cos(n\varphi), \sin(m\theta)\sin(n\varphi), \cos(m\theta)) , \\
\hat{e}_\theta^{(n,m)} &= (\cos(m\theta)\cos(n\varphi), \cos(m\theta)\sin(n\varphi), -\sin(m\theta)) , \\
\hat{e}_\varphi^{(n)} &= (-\sin(n\varphi), \cos(n\varphi), 0) ,
\end{aligned} \tag{6}$$

respectively. The gauge field functions K_i, $i = 1, \dots, 6$ depend on the coordinates r and θ.

Recall that although the ansatz (6) is static, there is a time dependent gauge transformation which can eliminate the temporal component A_0, then the fields A_k will have a periodic time dependence modulo gauge transformation.

Substitution of the axially symmetric ansatz (6) into definition of the topological charge Q yields similar to [14, 10]

$$Q = \frac{n}{2}\left[1 - (-1)^m\right] ,$$

that is the configurations labeled by an even integer m correspond to the topologically trivial sector and represent saddle point solutions.

The number of the structure functions of the ansatz (6) evidently exceeds what one need to solve first order self-duality equations, in components there are only 3 equations on 6 functions and the system is overdetermined. Thus, a self-dual configuration corresponds to reduction of the ansatz (6). Actually the Harrington-Shepard solution [5] as well as KvBLL calorons [8] were constructed on the Corrigan-Fairlie-'t Hooft, or Jackiw-Nohl-Rebbi ansatz [6] $A_\mu = i\bar{\eta}_{\mu\nu}\partial_\nu \ln \phi$ and its generalizations.

To satisfy the condition of finitness of the total Euclidean action (4) we require that the field strength vanishes at the spatial boundary as $\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) \to O(r^{-4})$ as $r \to \infty$. In the regular gauge the value of the component of the gauge potential A_0 at spatial infinity approaches a constant, i.e.,

$$A_0 \to \frac{i\beta}{2}\tau_r^{(n,m)} \tag{7}$$

This corresponds to the holonomy operator (1)

$$\mathrm{Tr}\mathcal{P}(\mathbf{r}) = \mathrm{Tr}\exp\left(\frac{i\beta T}{2}\tau_r^{(n,m)}\right) = \mathrm{Tr}\, U \exp\left(\frac{i\beta T}{2}\tau_z\right) U^{-1} = \cos\frac{\beta T}{2} , \tag{8}$$

where $U \in SU(2)$ and $\beta \in [0; 2\pi/T]$. Using the classical scale invariance we can fix $\beta = 1$.

Let us consider deformations of the topologically trivial sector and the Harrington-Shepard caloron solution with trivial holonomy. Then the boundary conditions at infinity are

$$\begin{aligned}
\text{for even } m = 2k \ : \ A_0 &\longrightarrow \beta\hat{e}_r^{(n,m)} = \beta U\tau_z U^\dagger , \quad A_k \longrightarrow i\partial_k U U^\dagger , \\
\text{for odd } m = 2k+1 \ : \ A_0 &\longrightarrow \beta\hat{e}_r^{(n,m)} , \quad A_\mu \longrightarrow U A_{\mu\infty}^{(n,1)} U^\dagger + i\partial_\mu U U^\dagger ,
\end{aligned}$$

where $U = \exp\{-ik\theta\tau_\varphi^{(n)}\}$ and $A_{\mu\infty}^{(n,1)}$ is the self-dual charge n generalized Harrington-Shepard caloron solution [5]. We will not require, however, that the gauge field has to be self-dual, i.e., $F_{\mu\nu} \neq \pm\widetilde{F}_{\mu\nu}$, in general

In terms of the profile functions of the ansatz (6) these boundary conditions read:

$$\begin{aligned}
&K_1 \longrightarrow 0 , \quad K_2 \longrightarrow 1 - m , \quad K_3 \longrightarrow \frac{\cos\theta - \cos(m\theta)}{\sin\theta} \ (\text{for odd } m) , \\
&K_3 \longrightarrow \frac{1 - \cos(m\theta)}{\sin\theta} \ (\text{for even } m) , \quad K_4 \longrightarrow 1 - \frac{\sin(m\theta)}{\sin\theta} , \quad K_5 \longrightarrow 1 , \quad K_6 \longrightarrow 0 .
\end{aligned}$$

Regularity at the origin requires

$$\begin{aligned}
&K_1(0,\theta) = 0 , \quad K_2(0,\theta) = 1 , \quad K_3(0,\theta) = 0 , \quad K_4(0,\theta) = 1 , \\
&\sin(m\theta)K_5(0,\theta) + \cos(m\theta)K_6(0,\theta) = 0 \qquad \partial_r\left[\cos(m\theta)K_5(r,\theta) - \sin(m\theta)K_6(r,\theta)\right]\big|_{r=0} = 0.
\end{aligned}$$

Regularity on the z-axis, finally, requires

$$K_1 = K_3 = K_6 = 0 , \quad \partial_\theta K_2 = \partial_\theta K_4 = \partial_\theta K_5 = 0 ,$$

3 Numerical results

The regular caloron solutions with finite action density and proper asymptotic behavior can be constructed numerically by imposing these boundary conditions and solving the resulting system of 6 coupled non-linear partial differential equation of second order. As usually, to obtain regular solutions we have to fix the gauge condition as $\partial_r A_r + \partial_\theta A_\theta = 0$ (reduced Lorentz gauge), or $r\partial_r K_1 - \partial_\theta K_2 = 0$ [14, 15] and introduce the compact radial coordinate $x = r/(1+r) \in [0:1]$.The numerical calculations were performed with the software package FIDISOL based on the Newton-Raphson iterative procedure [16].

The simplest class of the solutions corresponds to the $m = 1$. It turns out that, similar to [10] these solutions are self-dual. we check this by numerical calculation of the integrated action density and direct substitution of the solutions into the first order self-duality equations. Furtermore, the $m = n = 1$ solution is spherically symmetric finite temperature solution [5] of unit topological charge. The $m = 1, n \geq 2$ solutions are axially symmetric and the action density distribution has a shape of a torus.

The $m \geq 2$ configurations satisfy only the second order Yang-Mills field equations and are not self-dual. Similar to their couterparts in YMH theory [14], the solutions with $n = 1, m = 2, 3, 4\dots$ represent chains of interpolating instantons and anti-instantons of unit charge. A general property of these solutions is that the corresponding action density posess m clear maxima on the axis of symmetry (see Fig 1). Thus, we can distinguis m individual consituents and identify these with non-self dual chain of periodic instantons. Also, the topological charge density posesses m local extrema on the z axis, whose locations coinsides with maxima of the action density. The positive and negative extrema alternate between the locations of individual constituents.

The same general behavior was observed for all other solutions of different types. Generally, increasing of the winding number n which is related with topological charge of each individual constituent pseudoparticle, yelds shift of the local extrema of the action density away from the symmetry axis. For example, for a configuration with $n = m = 3$ (triple charged instanton-anti-instanton-instanton system) we found three maxima on the $z\rho$ plane, which corresponds to the surface of triple torus with one maximum on x, y plane and two other, placed symmetrically above and below this plane (see Fig 2). The counterparts of these configurations in YMH theory are monopole-vortex rings systems [14]. Again, the radius of the tori and relative distance between its location decreases as $\Delta\rho_0 \sim 1/\beta$. The numerical results indicate that the intergated action of the $m \geq 2$ configurations for all non-zero values of temperature remains above the self-duality bound. Since the counterpats of these solutions in YMH theory corresponds to the sphaleron-like solutions, there is a reason to belive that such caloron solutions also are unstable and corresponds to the saddle points of the action functional. Note that the variation of the temperature does not lead to the chain-vortex bifurcations, which were observed in the YNH systems in external electromagnetic field [17] or in the limit of large scalar self-coupling [18].

4 Conclusions

To summarise, we have constructed axially symmetric caloron solutions of the d=4 Euclidean $SU(2)$ YM theory by numerical solution of the second order Yang-Mills equations. Similar to the monopole-antimonopole axially symmetric solutions of the YMH theory, the calorons are labelled by two winding numbers (n, m) and the topological charge of the configuration is $Q = \frac{n}{2}[1-(-1)^m]$. The action density of the configuration has non-trivial shape and local maxima of the action functional allows us to identify location of each individual constitutent. Besides configurations with $m = 1$, which are selfdual, the solutions do not saturate the self-duality bound.

For the chain solutions with $n = 1, 2$ there are instantons and anti-instantons, which are

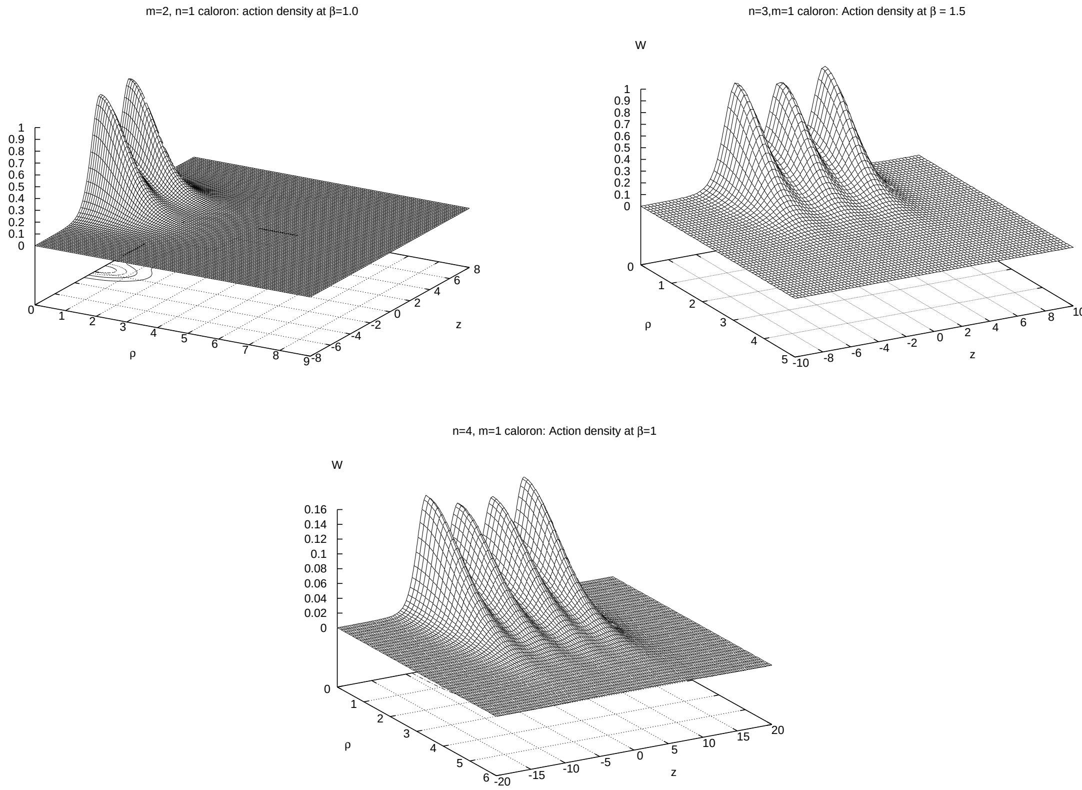

Figure 1: The action densities of the $m = 2, 3, 4$ and $n = 1$ instanton–anti-instanton chains are shown in coordinates z, ρ.

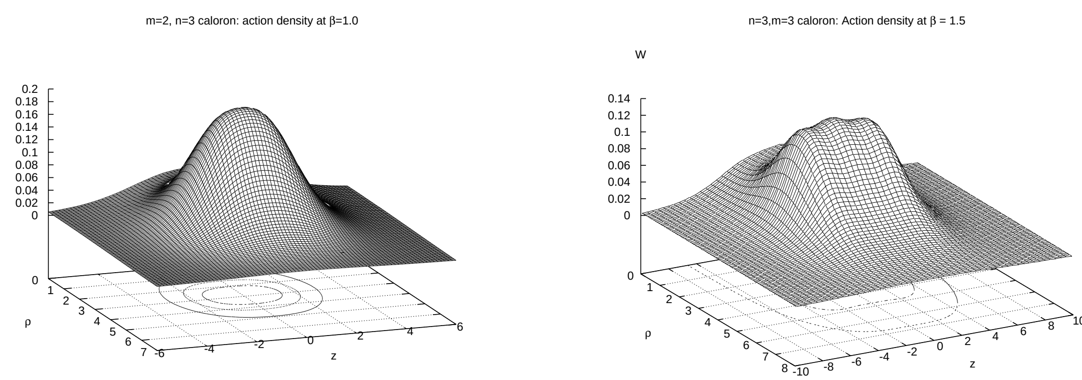

Figure 2: The action density distribution is plotted for the $n = 2$ and $n = 3$ calorons with winding number $m = 3$, respectively.

located on the axis of symmetry in alternating order. For configurations of higher topological charge the action density forms a torus-like shape.

The caloron solutions described here are restricted because the Ansatz (6) possesses the reflection Z_2-symmetry with respect to the xy plane. This is not the symmetry of the KvBLL solution which has only the $O(2)$ symmetry with respect to the rotation about the axis of symmetry [8]. To describe general non-self dual axially symmetric caloron solutions, also with non-trivial holonomy, one has to implement an extended Ansatz for the gauge field which includes complete set of 12 profile functions. and consider different set of boundary conditions. The results of the related calculations will be reported elsewhere.

Although both the configurations considered above as well as KvBLL calorons admit the constituent interpretation with lumps being accociated with monopoles, there is an important difference. The former caloron solutions, in a general case are defined along the same positive simple root, which corresponds to a given $SU(2)$ subgroup of $SU(N)$. For example, the configuration with winding numbers $n = 1, m = 2$ corresponds to the monopole-antimonopole pair solution described in [13, 14]. The monopole constituents of the $SU(N)$ KvBLL calorons [8] are defined in a different way, e.g., the $SU(2)$ caloron configuration describes monopole of positive charge embedded along positive simple root with the asymptotic $A_0 \to \beta$, and a Weyl-reflected antimonopole with asymptotic $A_0 \to 2\pi/T - \beta$.

It would be interesting to see how the non-self-dual calorons presented here can be relevant for QCD, in particular how the corresponding saddle point configurations may contribute to the process of the confinement-deconfinement phase transition.

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