Low-energy effective action in non-anticommutative charged hypermultiplet model

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Abstract

We study the quantum aspects of a charged hypermultiplet in deformed $\mathcal{N} = (1,1)$ superspace with singlet non-anticommutative deformation of supersymmetry. We prove the renormalizability of this model and compute the leading contributions to the low-energy effective action. In particular, the holomorphic part of the effective action is shown to be a star-generalization of the standard holomorphic potential, while the antiholomorphic part is not. The component structure of these contributions to the effective action is also discussed.

1 Introduction

The interest in $\mathcal{N} = (1/2, 0)$ non-anticommutative deformations of supersymmetry originated with the papers [1], where these deformations were derived from superstring theory on a constant graviphoton background. As a rule, such deformations break supersymmetry only in the chiral sector of superspace, which is possible in Euclidean superspace. The key feature of non-anticommutative deformations on the quantum level is the preservation of renormalizability, which was established for $\mathcal{N} = (1/2, 0)$ Wess-Zumino [2] and super Yang-Mills (SYM) [3] models. This result is very non-trivial since the non-anticommutative deformations involve a parameter with negative mass dimension which plays the role of a new coupling constant. Since such theories appear to be renormalizable, their quantum dynamics should be explored. Indeed, the low-energy effective action of such models was considered in [4], where the corrections due to the non-anticommutative deformation were calculated. These results provide a promising new method of partial supersymmetry breaking which preserves renormalizability.

These surprises of $\mathcal{N} = (1/2, 0)$ theories motivated analogous investigations of deformed *extended* supersymmetric theories. There are several types of non-anticommutative deformations of the extended supersymmetry. The simplest one depends on a single scalar parameter I which appears in the anticommutator of the chiral $\mathcal{N} = (1,1)$ Grassmann coordinates,

$$\{\theta^\alpha_i, \theta^\beta_j\}_\star = 2I\varepsilon^{\alpha\beta}\varepsilon_{ij}. \tag{1}$$

Such a deformation was introduced in [5] and was named a chiral singlet deformation; the corresponding field theories are referred to as $\mathcal{N} = (1,0)$ non-anticommutative. Although more general types of deformations of extended supersymmetry have been considered in [6, 7], here we shall restrict ourselves to the singlet deformation (1), since this case is most elaborated now on the classical level [8]-[11] and its stringy origins have been established [8].

The quantum aspects of $\mathcal{N} = (1,0)$ non-anticommutative theories are more involved. In [8, 9, 10] it was shown that the $\mathcal{N} = (1,0)$ SYM and hypermultiplet models acquire a number of new classical interaction terms even in the Abelian case. In principle, such terms can spoil

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renormalizability. In our recent paper [12] we addressed this problem in two such $\mathcal{N}=(1,0)$ theories, namely the Abelian SYM model and the *neutral* hypermultiplet interacting with an Abelian gauge superfield. By computing all divergent contributions to the effective action we proved that both these models remain renormalizable. Note that these theories are deformations of free ones, and thus all interaction terms vanish in the undeformed limit $I \to 0$. A physically more important example is the *charged* hypermultiplet model, which – already prior to the deformation – features the interaction of a hypermultiplet with a background Abelian vector superfield. In particular, the low-energy effective action of this theory is goverend by the so-called holomorphic potential, which plays a significant role in the Seiberg-Witten theory [13]. Therefore, in the present work we study the low-energy effective action and renormalizability for the $\mathcal{N}=(1,0)$ non-anticommutative charged hypermultiplet model.

Theories with extended supersymmetry are most naturally described within the harmonic superspace approach [14, 15]. Hence, we will consider the $\mathcal{N}=(1,0)$ non-anticommutative charged hypermultiplet in the harmonic superspace that was studied on the classical level in [9, 10]. We are interested in the non-anticommutative deformation of the holomorphic effective action that was discussed in [16] using the harmonic superspace approach. Here we generalize the results of these works to the non-anticommutatively deformed hypermultiplet theory and compute the leading contributions to the effective action.

One of our main results is the proof of one-loop renormalizability of the deformed charged hypermultiplet system. It supports the idea that the non-anticommutative deformations in general do not spoil the renormalizability of supersymmetric theories. Next, we find the leading contributions to the (anti)holomorphic effective action including non-anticommutative corrections. We observe that the holomorphic and antiholomorphic pieces are deformed differently: the holomorphic piece is nothing but the star-generalization of the standard holomorphic potential while the antiholomorphic piece is not. We study also the component structure of the deformed effective action and derive the corrections to the standard terms in the (anti)holomorphic potential for the bosonic component fields.

The present contribution is based essentially on our recently published results [17].

2 Non-anticommutative charged hypermultiplet model

In the undeformed case the classical action of the charged hypermultiplet interacting with the external Abelian vector superfield is given by

$$S = \int d\zeta du\, \breve{q}^{+}(D^{++} + V^{++})q^{+}. \tag{2}$$

Here q^+ and its conjugate $\breve{q}^+$ are complex analytic superfields which describe the hypermultiplet, while V^{++} is a real analytic superfield which corresponds to the vector multiplet. The integration in (2) is performed over $\mathcal{N}=2$ analytic superspace with the measure $d\zeta du$. In the present work we consider the non-anticommutative generalization of this model in the case of chiral singlet deformation of $\mathcal{N}=(1,1)$ supersymmetry. Such a deformation was introduced in [5] and the corresponding field models were studied in [8]-[12]. It is effectively taken into account by the star product operator

$$\star = \exp\left[-I\varepsilon^{\alpha\beta}\varepsilon_{ij}\overleftarrow{Q}^{i}_{\alpha}\overrightarrow{Q}^{j}_{\beta}\right], \tag{3}$$

which should be placed everywhere instead of usual product of superfields in the classical actions. The constant I here is a parameter of non-anticommutativity, Q^i_α are the supercharges. In particular, the non-anticommutative generalization of the action (2) is given by [9]

$$S = \int d\zeta du\, \breve{q}^{+} \star \nabla^{++} \star q^{+}, \tag{4}$$

where we use the notations

$$\nabla^{++} = D^{++} + V^{++}, \qquad \nabla^{--} = D^{--} + V^{--}. \tag{5}$$

This action (4) is invariant under the following gauge transformations

$$\delta\breve{q}^{+} = -\breve{q}^{+} \star \lambda, \qquad \delta q^{+} = \lambda \star q^{+}, \qquad \delta V^{++} = -D^{++}\lambda - [V^{++}, \lambda]_\star. \tag{6}$$

In the deformed case the strength superfields W, $\bar{W}$ are defined by the standard equations

$$\bar{W} = -\frac{1}{4} D^{+\alpha} D^{+}_{\alpha} V^{--}, \qquad W = -\frac{1}{4} \bar{D}^{+}_{\dot\alpha} \bar{D}^{+\dot\alpha} V^{--}. \tag{7}$$

However, the superfield V^{--} is now given by a series

$$V^{--}(z,u) = \sum_{n=1}^{\infty} (-1)^n \int du_1 \dots du_n \frac{V^{++}(z,u_1) \star V^{++}(z,u_2) \star \dots \star V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\dots(u_n^+u^+)}, \tag{8}$$

which solves the star-deformed zero-curvature equation [8]

$$D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}]_\star = 0. \tag{9}$$

Let us introduce the "bridge" superfield $\Omega(z,u)$ as a general $\mathcal{N} = (1,1)$ superfield which relates the covariant harmonic derivatives $\nabla^{\pm\pm}$ with the plain ones $D^{\pm\pm}$:

$$\nabla^{++} = e^{\Omega}_\star \star D^{++} e^{-\Omega}_\star, \qquad \nabla^{--} = e^{\Omega}_\star \star D^{--} e^{-\Omega}_\star. \tag{10}$$

The bridge superfield was originally introduced in [14] for the undeformed $\mathcal{N} = 2$ SYM theory as an operator relating the $\mathcal{N} = 2$ superfields in the τ- and λ-frames. Using the bridge superfield Ω one can alternatively rewrite the equation (8) in the following two equivalent forms

$$V^{--}(z,u) = \int du' \frac{e_\star^{\Omega(z,u)} \star e_\star^{-\Omega(z,u')} \star V^{++}(z,u')}{(u^+u'^+)^2} = \int du' \frac{V^{++}(z,u') \star e_\star^{\Omega(z,u')} \star e_\star^{-\Omega(z,u)}}{(u^+u'^+)^2}. \tag{11}$$

The expression (11) can be checked directly to satisfy the zero-curvature condition (9).

It is well known [15] that the *free* propagator in the hypermultiplet model (2) is given by the following expression [1]

$$G_0^{(1,1)}(1|2) = -\frac{1}{\Box}(D_1^+)^4(D_2^+)^4 \frac{\delta^{12}(z_1 - z_2)}{(u_1^+u_2^+)^3} \tag{12}$$

which solves the equation $D^{++}G_0^{(1,1)}(1|2) = \delta_A^{(3,1)}(1|2)$, where $\delta_A^{(3,1)}(1|2)$ is the analytic delta-function. Let us define now the *full* propagator in the model (4) as a distribution satisfying the equation

$$\nabla^{++} \star G^{(1,1)}(1|2) = \delta_A^{(3,1)}(1|2). \tag{13}$$

The solution of (13) can formally be written as

$$G^{(1,1)}(1|2) = -\frac{1}{\hat\Box_\star} \star (D_1^+)^4(D_2^+)^4 \left\{ e_\star^{\Omega(1)} \star e_\star^{-\Omega(2)} \star \frac{\delta^{12}(z_1 - z_2)}{(u_1^+u_2^+)^3} \right\}, \tag{14}$$

where $\hat\Box_\star$ is a covariant box operator

$$\hat\Box_\star = -\frac{1}{2}(D^+)^4 \nabla^{--} \star \nabla^{--}. \tag{15}$$

[1] Note that we write here (and further) the box operator $\Box$ assuming that it is nothing but the Laplacian operator rather than a d'Alambertian one since we deal with the Euclidian space.

Clearly, it moves an analytic superfield to another analytic one. The operator (15), acting on the analytic superfield, can be represented in the form

$$
\begin{aligned}
\hat{\square}_\star \;\; = \;\; & \nabla^m \star \nabla_m - \frac{1}{2}(\nabla^{+\alpha} \star W) \star \nabla^-_\alpha - \frac{1}{2}(\bar\nabla^+_{\dot\alpha} \star \bar W) \star \bar\nabla^{-\dot\alpha} + \frac{1}{4}(\nabla^{+\alpha} \star \nabla^+_\alpha \star W) \star \nabla^{--} \\
& - \frac{1}{8}[\nabla^{+\alpha}, \nabla^-_\alpha]_\star \star W - \frac{1}{2}\{W, \bar W\}_\star . \qquad (16)
\end{aligned}
$$

Here $\nabla^\pm_\alpha = D^\pm_\alpha + V^\pm_\alpha$, $\bar\nabla^\pm_{\dot\alpha} = \bar D^\pm_{\dot\alpha} + \bar V^\pm_{\dot\alpha}$ are covariant spinor derivatives. Note that the expression (16) has a similar form as in the undeformed theory [18] with the simple star-modification of multiplication of superfields. This result is not surprising since eq. (16) is derived from (15) only by using the (anti)commutation relations between spinor derivatives which have the same form as in the undeformed theory.

3 Computation of low-energy effective action

In general, the effective action in the hypermultiplet model (4) is a functional of external strength superfields W, $\bar W$ and, possibly, the prepotential V^{++}. The part of the effective action which depends only on the superfields without spatial and covariant spinor derivatives can be written as

$$
\Gamma = \int d^4x d^4\theta \, \mathcal{F}_\star(W) + \int d^4x d^8\theta \, \mathcal{H}_\star(V^{++}, V^{--}, W, \bar W), \qquad (17)
$$

where $\mathcal{F}_\star$ and $\mathcal{H}_\star$ are some functions. Note that the antiholomorphic piece is prohibited in the non-anticommutative case. Indeed, the strength superfields W, $\bar W$ are not (anti)chiral, but *covariantly* (anti)chiral

$$
D^+_\alpha \bar W = \nabla^-_\alpha \star \bar W = 0, \qquad \bar D^+_{\dot\alpha} W = \bar\nabla^-_{\dot\alpha} \star W = 0. \qquad (18)
$$

Therefore the expression $\int d^4x d^4\bar\theta \bar{\mathcal{F}}_\star(\bar W)$ depends on θ variables

$$
D^-_\alpha \int d^4x d^4\bar\theta \, \bar{\mathcal{F}}_\star(\bar W) = \int d^4x d^4\bar\theta \, [\bar{\mathcal{F}}_\star(\bar W), V^-_\alpha]_\star \neq 0. \qquad (19)
$$

The rhs of (19) does not vanish since the star-product is not cyclic under $d^4x d^4\bar\theta$ integration. Therefore, the possible terms in the low-energy effective action which correspond to the anti-holomorphic potential in the limit $I \to 0$ is included into the function $\mathcal{H}_\star(V^{++}, V^{--}, W, \bar W)$ integrated in full superspace. The direct computations will specify these functions $\mathcal{F}_\star$ and $\mathcal{H}_\star$.

For the further considerations it will be more convenient to study the variation of effective action $\delta\Gamma$ rather then Γ itself. In particular, given the holomorphic part of the action (17)

$$
\Gamma_{hol} = \int d^4x d^4\theta \, \mathcal{F}_\star(W), \qquad (20)
$$

using the same steps as in the undeformed non-Abelian $\mathcal{N} = 2$ supergauge theory [19], one can write its variation in the full superspace

$$
\delta\Gamma_{hol} = \int d^{12}z du \, \delta V^{++} \star V^{--} \star \frac{1}{W} \star \mathcal{F}'_\star(W). \qquad (21)
$$

The one-loop effective action in the model (4) is defined by the following formal expression [2]

$$
\Gamma = \mathrm{Tr}\ln \frac{\delta^2 S}{\delta\breve q^+(1)\delta q^+(2)} = \mathrm{Tr}\ln(\nabla^{++}\star) = -\mathrm{Tr}\ln\, G^{(1,1)}(1|2), \qquad (22)
$$

[2] Note that the one-loop effective action in the Euclidean space is given by $\Gamma = \mathrm{Tr}\ln S^{(2)}_{\phi\bar\phi}$ rather then the Minkowski space expression $\Gamma = i\mathrm{Tr}\ln S^{(2)}_{\phi\bar\phi}$. Here $S^{(2)}_{\phi\bar\phi}$ is the second mixed functional derivative of a classical action $S[\phi, \bar\phi]$.

where $G^{(1,1)}(1|2)$ is given by eq. (14). It is easy to find the variation of (22)

$$\delta\Gamma = \mathrm{Tr}[\delta V^{++} \star G^{(1,1)}] = \int d\zeta du\, \delta V^{++}(1) \star G^{(1,1)}(1|2)|_{(1)=(2)}. \tag{23}$$

There is an important relation between full and free hypermultiplet propagators

$$G^{(1,1)}(1|3) = G_0^{(1,1)}(1|3) - \int d\zeta_2 du_2\, G_0^{(1,1)}(1|2) \star V^{++}(2) \star G^{(1,1)}(2|3) \tag{24}$$

which can be checked directly to satisfy (13). Substituting (24) into (23), we find

$$\delta\Gamma = -\int d\zeta_1 du_1 d\zeta_2 du_2\, \delta V^{++}(1) \star G_0^{(1,1)}(1|2) \star V^{++}(2) \star G^{(1,1)}(2|1). \tag{25}$$

Taking into account the exact form of the propagators (12,14), we rewrite eq. (25) as follows

$$\begin{aligned}\delta\Gamma &= -\int d^{12}z_1 d^{12}z_2 du_1 du_2\, \delta V^{++}(1) \star \frac{1}{\Box}\frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3} \\ &\quad \times V^{++}(2) \star \frac{1}{\hat{\Box}_{\star(2)}} \star (D_1^+)^4 (D_2^+)^4 \left\{ e_\star^{\Omega(2)} \star e_\star^{-\Omega(1)} \star \frac{\delta^{12}(z_2 - z_1)}{(u_2^+ u_1^+)^3} \right\}.\end{aligned} \tag{26}$$

The equation (26) is a starting point for further calculations of different contributions to the effective action.

3.1 Divergent part of effective action

To derive the divergent part of the effective action it is sufficient to consider the approximation

$$\frac{1}{\hat{\Box}_\star} \approx \frac{1}{\Box} \tag{27}$$

since all other terms in the operator $\hat{\Box}_\star$ result to higher powers of momenta in the denominator. Upon the condition (27), the variation of effective action (26) simplifies essentially

$$\begin{aligned}\delta\Gamma_{div} &= \int d^{12}z_1 d^{12}z_2 \frac{du_1 du_2}{(u_1^+ u_2^+)^6} \delta V^{++}(1) \star \frac{1}{\Box}\delta^{12}(z_1 - z_2) \\ &\quad \times V^{++}(2) \star \frac{1}{\Box}(D_1^+)^4 (D_2^+)^4 \left\{ e_\star^{\Omega(2)} \star e_\star^{-\Omega(1)} \star \delta^{12}(z_2 - z_1) \right\}.\end{aligned} \tag{28}$$

We have to apply the identity

$$\delta^8(\theta_1 - \theta_2)(D_1^+)^4 (D_2^+)^4 \delta^{12}(z_1 - z_2) = (u_1^+ u_2^+)^4 \delta^{12}(z_1 - z_2) \tag{29}$$

to shrink the integration over the Grassmann variables to a point. Note that all the derivatives D^+ in eq. (28) must hit the delta function, otherwise the result is zero since there are exactly eight such derivatives to apply (29). Moreover, the presence of star-product can not modify the relation (29). Calculating the divergent momentum integral and applying the equation (29), the integration over $d^{12}z_2$ in (28) can be performed resulting to

$$\delta\Gamma_{div} = \frac{1}{16\pi^2\varepsilon} \int d^{12}z du_1\, \delta V^{++}(z, u_1) \star \int du_2 \frac{V^{++}(z, u_2) \star e_\star^{\Omega(z,u_2)} \star e_\star^{-\Omega(z,u_1)}}{(u_1^+ u_2^+)^2}. \tag{30}$$

Using the relation (11) we obtain finally

$$\delta\Gamma_{div} = \frac{1}{16\pi^2\varepsilon} \int d^{12}z du\, \delta V^{++} \star V^{--}. \tag{31}$$

The variation (31) can be easily integrated with the help of eq. (21)

$$\Gamma_{div} = \frac{1}{32\pi^2\varepsilon}\int d^4x d^4\theta\, W^2. \tag{32}$$

We see that the divergent part of effective action is proportional to the classical action in $\mathcal{N}=(1,0)$ SYM model. In this sense the non-anticommutative charged hypermultiplet model (4) is renormalizable.

3.2 Finite part of the effective action

Now we will derive the finite part of the effective action of deformed hypermultiplet model. We start with the expression (26) applying the following approximation

$$\frac{1}{\hat{\Box}_\star} \approx \frac{1}{\Box - \frac{1}{2}\{W,\bar{W}\}_\star}. \tag{33}$$

This means that we neglect all spatial and spinor covariant derivatives of strength superfields in the decomposition (16)

$$\partial_m W = \partial_m \bar{W} = 0, \quad \nabla^{+\alpha}\star W = 0, \quad \bar{\nabla}^+_{\dot{\alpha}}\star\bar{W} = 0. \tag{34}$$

Exactly such an approximation (34) is sufficient for deriving the (anti)holomorphic contributions. Therefore the effective action (26) can be rewritten as follows

$$\begin{aligned}\delta\Gamma &= \int d^{12}z_1 d^{12}z_2 \frac{du_1 du_2}{(u_1^+ u_2^+)^6}\delta V^{++}(1)\star\frac{1}{\Box}\delta^{12}(z_1-z_2)\\ &\times V^{++}(2)\star\frac{1}{\Box-\frac{1}{2}\{W,\bar{W}\}_\star}(D_1^+)^4(D_2^+)^4\left\{e_\star^{\Omega(2)}\star e_\star^{-\Omega(1)}\star\delta^{12}(z_2-z_1)\right\}.\end{aligned} \tag{35}$$

Once again, all the derivatives D^+ in the second line of (35) must hit the delta-function. Applying the identity (29) the integration over $d^8\theta_2$ is performed

$$\begin{aligned}\delta\Gamma &= \int d^{12}z_1 d^4x_2 \frac{du_1 du_2}{(u_1^+ u_2^+)^2}\delta V^{++}(x_1,\theta,u_1)\star V^{++}(x_2,\theta,u_2)\star e_\star^{\Omega(2)}\star e_\star^{-\Omega(1)}\\ &\star\frac{1}{\Box}\delta^4(x_1-x_2)\frac{1}{\Box-\frac{1}{2}\{W,\bar{W}\}_\star}\delta^4(x_2-x_1).\end{aligned} \tag{36}$$

In the momentum space the second line of (36) reads

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{k^2+\frac{1}{2}\{W,\bar{W}\}_\star} = -\frac{1}{16\pi^2}\ln_\star\left[\frac{\{W,\bar{W}\}_\star}{2\Lambda^2}\right] + (\text{divergent term}), \tag{37}$$

where Λ is an arbitrary constant of dimension $+1$. Note that the integral (37) has the logarithmic divergence. We consider here only its finite part since the divergent contribution have been calculated above. As a result, the finite part of (36) is given by

$$\begin{aligned}\delta\Gamma &= -\frac{1}{16\pi^2}\int d^{12}z du_1\,\delta V^{++}(x,\theta,u_1)\star\int du_2\frac{V^{++}(x,\theta,u_2)\star e_\star^{\Omega(x,\theta,u_2)}\star e_\star^{-\Omega(x,\theta,u_1)}}{(u_1^+u_2^+)^2}\\ &\star\ln_\star\frac{\{W,\bar{W}\}_\star}{2\Lambda^2}.\end{aligned} \tag{38}$$

Applying the identity (11), we conclude

$$\delta\Gamma = -\frac{1}{16\pi^2}\int d^{12}z du\,\delta V^{++}\star V^{--}\star\ln_\star\frac{\{W,\bar{W}\}_\star}{2\Lambda^2}. \tag{39}$$

The variation of the effective action (39) is one of the main results of the present work. It gives us the part of the low-energy effective action depending on the strength superfields without derivatives.

If the parameter of non-anticommutativity tends to zero, $I \to 0$, the equation (39) reduces to the holomorphic and antiholomorphic parts of the effective action

$$\delta\Gamma_{(I=0)} = -\frac{1}{16\pi^2}\int d^{12}zdu\,\delta V^{++}V^{--}\left(\ln\frac{W}{\Lambda}+\ln\frac{\bar{W}}{\Lambda}\right). \tag{40}$$

The variation (40) exactly corresponds to the usual holomorphic potential and its conjugate.

3.3 Holomorphic contribution

Let us single out purely holomorphic part from the expression (39). For this purpose we restrict the background strength superfield $\bar{W}$ to be constant

$$\bar{W} = \bar{\mathbf{W}} = const, \tag{41}$$

and the log function in (39) simplifies to

$$\ln_\star \frac{\{W,\bar{W}\}_\star}{2\Lambda^2} = \ln_\star\frac{W}{\Lambda} + \ln\frac{\bar{\mathbf{W}}}{\Lambda}. \tag{42}$$

The holomorphic part is now given by

$$\delta\Gamma_{hol} = -\frac{1}{16\pi^2}\int d^{12}zdu\,\delta V^{++}\star V^{--}\star\ln_\star\frac{W}{\Lambda}. \tag{43}$$

According to the equation (21), the variation (43) can easily be integrated:

$$\Gamma_{hol} = -\frac{1}{32\pi^2}\int d^4xd^4\theta\,W\star W\star\ln_\star\frac{W}{\Lambda}. \tag{44}$$

As a result, we proved that the holomorphic part of effective action in the hypermultiplet model is nothing but a star-product generalization of a standard holomorphic potential.

3.4 Antiholomorphic contribution

Similarly, the antiholomorphic part of the effective action can be found from (39) when we restrict the strength W to be constant

$$W = \mathbf{W} = const. \tag{45}$$

The antiholomorphic part now reads

$$\delta\Gamma_{antihol} = \frac{1}{16\pi^2}\int d^{12}zdu\,\delta V^{++}\star V^{--}\star\ln_\star\frac{\bar{W}}{\Lambda}. \tag{46}$$

In contrast to the variation (43), the expression (46) can not be so easily integrated since there is no antiholomorphic potential written in the antichiral superspace. However, one can readily find the (part of) effective equation of motion corresponding to the variation (46)

$$\frac{\delta\Gamma_{antihol}}{\delta V^{++}} = \frac{1}{16\pi^2}(\bar{D}^+)^2\left[\bar{W}\star\ln_\star\frac{\bar{W}}{\Lambda}\right]. \tag{47}$$

The variation (46) will be integrated in the next section only for some particular choice of background gauge superfields.

4 Component structure of low-energy effective action

We are interested in the leading component terms of the (anti)holomorphic effective actions (44) and (46) in the bosonic sector. Here we will follow the works [8, 9], using the same conventions and notations for the component fields and sigma-matrices.

The scalar and vector fields enter the prepotential V^{++} in the Wess-Zumino gauge as follows

$$V^{++}_{WZ} = (\theta^+)^2\bar\phi + (\bar\theta^+)\phi + (\theta^+\sigma_m\bar\theta^+)A_m - 2i(\bar\theta^+)^2(\theta^+\theta^-)\partial_m A_m - (\bar\theta^+)^2(\theta^-\sigma_{mn}\theta^+)F_{mn}. \quad (48)$$

The prepotential V^{--} is defined as a solution of zero-curvature equation (9). Unfolding the star-product in (9), we have

$$\begin{aligned}&D^{++}V^{--} - D^{--}V^{++}_{WZ} + 2I[\partial^\alpha_+ V^{++}_{WZ}\partial_{-\alpha}V^{--} - \partial^\alpha_- V^{++}_{WZ}\partial_{+\alpha}V^{--}]\\ &+\frac{I^3}{2}[\partial^\alpha_-(\partial_+)^2V^{++}_{WZ}\partial_{+\alpha}(\partial_-)^2V^{--} - \partial^\alpha_+(\partial_-)^2V^{++}_{WZ}\partial_{-\alpha}(\partial_+)^2V^{--}] = 0,\end{aligned} \quad (49)$$

where

$$\partial_{+\alpha} = \frac{\partial}{\partial\theta^{+\alpha}}, \qquad \partial_{-\alpha} = \frac{\partial}{\partial\theta^{-\alpha}}. \quad (50)$$

One can look for the prepotential V^{--} in the following form

$$\begin{aligned}V^{--} &= v^{--} + \bar\theta^-_{\dot\alpha}v^{-\dot\alpha} + (\bar\theta^-)^2A + (\bar\theta^+\bar\theta^-)\varphi^{--}\\ &+(\bar\theta^+\tilde\sigma^{mn}\bar\theta^-)\varphi^{--}_{mn} + (\bar\theta^-)^2\bar\theta^+_{\dot\alpha}\tau^{-\dot\alpha} + (\bar\theta^+)^2(\bar\theta^-)^2\tau^{--},\end{aligned} \quad (51)$$

where all fields in the rhs of eq. (51) depend only on θ^+_α, θ^-_α variables. The superfields v^{--}, $v^{-\dot\alpha}$, φ^{--}, A, $\tau^{-\dot\alpha}$, τ^{--} should be found from the eq. (49). The iterative procedure of solving eq. (49) is given in [8]. Following the same steps we find

$$v^{--} = (\theta^-)^2\frac{\bar\phi}{1+4I\bar\phi}, \quad (52)$$

$$v^{-\dot\alpha} = \frac{(\theta^-\sigma_m)^{\dot\alpha}A_m}{1+4I\bar\phi}, \quad (53)$$

$$\varphi^{--} = -\frac{2i(\theta^-)^2\partial_m A_m}{1+4I\bar\phi}, \quad (54)$$

$$A = \phi + \frac{4IA_mA_m}{1+4I\bar\phi} + (\theta^+\sigma_{mn}\theta^-)F_{mn}, \quad (55)$$

$$\tau^{-\dot\alpha} = \frac{4I(\theta^-\sigma_{mn})^\alpha F_{mn}\sigma_{r\alpha}{}^{\dot\alpha}A_r}{1+4I\bar\phi}, \quad (56)$$

$$\tau^{--} = \frac{4I(\theta^-)^2(F_{mn}F_{mn} + F_{mn}\tilde F_{mn})}{1+4I\bar\phi}. \quad (57)$$

Now we obtain the component structure of the strength superfields

$$W = -\frac{1}{4}(\bar D^+)^2V^{--} = \phi + \frac{4IA_mA_m}{1+4I\bar\phi} + (\theta^+\sigma_{mn}\theta^-)F_{mn}, \quad (58)$$

$$\bar W = -\frac{1}{4}(D^+)^2V^{--} = \frac{\bar\phi}{1+4I\bar\phi} + (\bar\theta^+\tilde\sigma_{mn}\bar\theta^-)\frac{F_{mn}}{1+4I\bar\phi}. \quad (59)$$

Note that the strength superfields (58) and (59) are deformed differently.

Introducing the notations

$$\Phi = \phi + \frac{4IA_mA_m}{1+4I\bar\phi}, \qquad \bar\Phi = \frac{\bar\phi}{1+4I\bar\phi}, \qquad \mathbf{F}_{mn} = \frac{F_{mn}}{1+4I\bar\phi}, \quad (60)$$

the eqs. (58,59) can be written as follows

$$W = \Phi + (\theta^+\sigma_{mn}\theta^-)F_{mn}, \qquad \bar{W} = \bar{\Phi} + (\bar{\theta}^+\tilde{\sigma}_{mn}\bar{\theta}^-)\mathbf{F}_{mn}. \tag{61}$$

To find the component structure of the holomorphic potential we substitute the superfields (61) into (44) and compute all star-products. The result is given by

$$\Gamma_{hol} = -\frac{1}{32\pi^2}\int d^4x(F^2 + F\tilde{F})\left[\ln\frac{\Phi}{\Lambda} + \Delta(X(\Phi, F_{mn}))\right], \tag{62}$$

where

$$\Delta(X) = \frac{1}{2}(1-X)^2\ln(X-1) + \frac{1}{2}(1+X)^2\ln(1+X) - (1+X^2)\ln X, \tag{63}$$

$$X(\Phi, F_{mn}) = \frac{\Phi}{2I\sqrt{2(F^2 + F\tilde{F})}}. \tag{64}$$

The equation (62) shows that the function $\Delta(X)$ is a correction due to non-anticommutativity to the standard holomorphic effective action given by the logarithmic term.

Let us consider the antiholomorphic potential when the strength superfield $\bar{W}$ is defined by the component expression (59) with the fields F_{mn} and $\bar{\phi}$ being constant. Inserting this superfield into the equation (46) we obtain the following component structure of the antiholomorphic potential

$$\begin{aligned}\Gamma_{antihol} &= -\frac{1}{32\pi^2}\int d^4x d^4\bar{\theta}\,\bar{W}^2\ln\frac{\bar{W}}{\Lambda} = -\frac{1}{32\pi^2}\int d^4x(\mathbf{F}^2 + \mathbf{F}\tilde{\mathbf{F}})\left(\ln\frac{\bar{\Phi}}{\Lambda} + \frac{3}{2}\right) \\ &= -\frac{1}{32\pi^2}\int d^4x\frac{(F^2 + F\tilde{F})}{(1+4I\bar{\phi})^2}\left(\ln\frac{\bar{\phi}}{\Lambda(1+4I\bar{\phi})} + \frac{3}{2}\right).\end{aligned} \tag{65}$$

We see that the non-anticommutativity manifests itself here by a simple rescaling of fields by the factor $1/(1+4I\bar{\phi})$. In the limit $I \to 0$ the expression (65) reduces to the standard one for the antiholomorphic potential.

5 Conclusions

In this paper we studied the low-energy effective action and renormalizability of $\mathcal{N} = (1,0)$ non-anticommutative charged hypermultiplet theory. This model describes the interaction of a hypermultiplet with an Abelian vector superfield under the singlet chiral deformation of supersymmetry. Let us summarize the basic results obtained in the present work.

1. The divergent part of the effective action is calculated and is shown to be proportional to the classical action of $\mathcal{N} = (1,0)$ non-anticommutative SYM theory. In this sense the present model is renormalizable.

2. The general structure of the low-energy effective action of this theory is revealed. Away from the undeformed limit, the antiholomorphic piece no longer exists by itself but is incorporated in a *full* $\mathcal{N} = (1,1)$ superspace integral.

3. The holomorphic effective action is calculated and remains a *chiral* superspace integral. It is shown to be given by the holomorphic potential which is a star-generalization of the undeformed one. The contribution to (the variation of) the effective action that corresponds to the antiholomorphic potential is also found as an expression written in full $\mathcal{N} = (1,1)$ superspace.

4. The component structure of the (anti)holomorphic effective action is studied in the bosonic sector in the constant-fields approximation. It is shown that the holomorphic and antiholomorphic potentials still get deformed differently. In this approximation the deformed holomorphic potential (62) acquires the extra terms given by the function (63). For the antiholomorphic piece, it is shown that the deformation merely effects a rescaling of component fields by a factor of $(1 + 4I\bar{\phi})^{-1}$, where $\bar{\phi}$ is one of the two scalar fields of the vector multiplet.

In the light of the present results, it would be rewarding to solve the following problems concerning the quantum aspects of non-anticommutative theories with extended supersymmetry. First, it is tempting to determine for the hypermultiplet model the deformation of the next-to-leading terms in the effective action, which are necessarily non-holomorphic. Also, one should develop the non-Abelian generalization. Next, it is important to perform an analogous investigation for the pure SYM theory, since in the undeformed case this model (the Seiberg-Witten theory) plays an important role in modern theoretical physics. Finally, it would be interesting to extend the quantum studies of non-anticommutative theories to the case of non-singlet (i.e. more general) deformations of supersymmetry, as considered particularly in [6] on the classical level.

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