Quantum effective action in spacetimes with branes calculational peculiarities

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Abstract

The main goal is to discuss the effective action formalism for field-theoretical models in the presence of branes and peculiarities therein. First, the crucial definitions are given, then different calculational schemes are discussed. As a byproduct of the formalism developed we obtain a rather elegant and handy technique for calculating *heat kernel trace expansion* for differential operators with nontrivial boundary conditions on manifolds with boundaries.

1 Braneworld Setup.

The field theoretical setup for models with branes, motivated from string theory can be realized by the fundamental action which is the sum of bulk and brane actions

$$\mathbf{S}[\Psi,\varphi] = \mathbf{S}_{Bulk}[\Psi] + \mathrm{S}_{brane}[\varphi] \tag{1}$$

Bulk action is the integral over bulk manifold of the local invariants of fields, propagating in the bulk. Set of fields which are dynamical in the bulk we shall denote as Ψ. Brane action is the integral over brane d-dimensional manifold of invariants built of boundary values of fundamental bulk fields $\Psi\big|_{brane} = \varphi$. The latter are called induced or brane fields. In the theory various additional matter and gauge fields which are binded to branes can be present. But they do not play crucial role in what follows, so we omit them and concentrate on fields, which propagate not only on branes, but in the bulk as well. The main goal is to discuss the effective dynamics of such fields as it seen by the observers living on branes.

$$\Psi: \qquad G_{AB},... \quad - \quad \text{Bulk fields}$$
$$\varphi \equiv \Psi\big|_{brane}: \qquad g_{\alpha\beta},... \quad - \quad \text{induced (brane) fields}$$

Branes can be considered as boundaries of the bulk, no matter they really bound the bulk manifold or they are inside the bulk manifold (so-called fixed points). The only thing one has to do is to select appropriate boundary or junction conditions . So some times we refer to brane fields as boundary fields [1].

In what follows we restrict ourselves to the case when branes are d-dimensional manifolds embedded in the bulk of dimension $(d+1)$. Or, in other words, we will concentrate on the case of codimension 1 branes, which is motivated by interest in applications to braneworld scenarios such as Randall-Sundrum or Dvali-Gabadadze-Porrati with one large extra dimension [2],[3].

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The brane effective action approach allows us to introduce the brane functional of only induced fields ϕ, which incorporate the dynamics of correspondent fields Ψ in the bulk [1],[4]

$$\mathbf{S}[\Psi,\varphi] \to S^{eff}[\varphi] \tag{2}$$

The main advantage of this approach, when one is allowed to deal with functional of only brane fields, compared for example with Kaluza-Klein approach, is that these induced fields can be "measured" directly by observers on the brane due to local interaction of brane gauge and matter fields with these induced fields [1],[5].

The price one pays for this is that free action for induced fields appears to be nonlocal. This means that from the viewpoint of brane observer the dynamics of induced fields along branes is essentially nonlocal. On the other hand such properties as local causality, unitarity are preserved by construction and all this justifies and suggests new ideas for the search of consistent modifications of Einstein gravity since one accepts the possibility that our 4-dimensional universe can be the brane embedded in multidimensional bulk.

Passing on to working with brane effective action one does not completely lose the information about bulk dynamics of fundamental fields – the geometry, topology of the bulk and information of brane embedding is incorporated in nonlocality of induced fields dynamics, particularly in operator of small fluctuations. Thus brane observers are able to obtain some information about the bulk by measuring deviations from local d-dimensional behavior of fundamental fields, including gravity.

For example one can study the deviations from the Newton law on very large and very small distances. Alternative way is to observe the deformations of gravitational waves and signals from distant astrophysical objects. For the class of Randall-Sundrum type models that was analyzed in [6].

Another argument in favor of efficiency of the effective action approach is that nonlocalities governing the propagation of induced field perturbations are straightforwardly and explicitly calculable, at least in models possessing symmetric bulk backgrounds. So one has a handy tool for explicit analysis of induced fields behavior on arbitrary scales. It is very helpful that by construction all objects appear to be d-dimensionally covariant[1].

2 Classical and Quantum Brane Effective Actions.

The first useful object in effective action approach to theories with branes is the functional called brane effective action or classical brane effective action $S^{eff}[\varphi]$:

$$e^{-S^{eff}[\varphi]} = e^{-\mathrm{S}_{brane}[\varphi]} \int\limits_{\Psi\big|_{brane}=\varphi} d\Psi\; e^{-\mathbf{S}_{Bulk}[\Psi]} \tag{3}$$

The initial fundamental action (1) for bulk and brane fields one has to integrate over all configurations of Ψ in the bulk with fixed boundary values $\Psi\big|_{brane} = \varphi$ – with fixed but arbitrary brane configurations. The resulting functional is the nonlocal functional only of brane (induced) fields φ, which defines the classical propagation of the letter in d-dimensional world of observers living on branes.

Another reason to call S^{eff} the classical brane effective action is the fact that when calculating expectation values of quantum operators of brane and induced fields via path integral

[1]For the sake of simplicity in what follows we skip analysis of the gauge aspects of the formalism, which can be found elsewhere.

one has to integrate the operator over brane field configurations weighted by the exponent of S^{eff}:

$$< \hat{O}(\varphi) > \equiv \int d\Psi \; O(\varphi) \; e^{-\mathbf{S}[\Psi,\varphi]} = \int d\varphi \; O(\varphi) \; e^{-S^{eff}[\varphi]} \tag{4}$$

The second object to be introduced here is the *braneworld quantum effective action* $\Gamma[\bar{\varphi}]$. It is by definition the functional of the mean brane field emergent as the response to arbitrary sources concentrated on branes.

$$e^{-\Gamma[\bar{\varphi}]} \equiv \int d\Psi \; e^{-\mathbf{S}[\Psi,\varphi] - \int\limits_{brane} (\varphi-\bar{\varphi})\, j} \Big|_{j=j[\bar{\varphi}]} = \int d\varphi \; e^{-S^{eff}[\varphi] - \int\limits_{brane} (\varphi-\bar{\varphi})\, j} \Big|_{j=j[\bar{\varphi}]} \tag{5}$$

Its construction differs from construction of quantum effective action of the whole theory (1) in the bulk only by the fact that one introduces sources (j) only on branes. Owing to that it becomes the functional only of mean values of induced fields $\bar{\varphi} \equiv < \varphi >$. Relevance of that object is again justified by the braneworld philosophy that observers living on branes are able to investigate the world only by means of local interactions on branes.

3 One-loop Brane Effective Action and Duality Relations.

For realistic models which are nonlinear in fields, especially for models such as gravity which is inevitably present in braneworld scenarios, one is unable to calculate explicitly functionals S^{eff} or $\Gamma[\bar{\varphi}]$. So, one has to use some perturbation technique. When weak field approximation (over some background) is relevant the most useful is background field method, where one expands the action functional in powers of field deviations from some background configuration, such as mean field or some classical solution

$$\Gamma[\bar{\varphi}] = S^{tree}[\bar{\varphi}] + \Gamma^{1-loop}[\bar{\varphi}] + ... \tag{6}$$

Interesting and nontrivial observations can be found already at one-loop order, when in path integral formalism one deals with calculable gaussian functional integrals. In one-loop approximation the effective action (with auxiliary sources for induced fields fluctuations) is given by the following path integral:

$$e^{-\Gamma^{1-loop}[\bar{\varphi}]} \equiv \int d\Psi \; e^{-\frac{1}{2}\int\limits_{Bulk} \Psi \overleftrightarrow{\mathbf{F}} \Psi - \frac{1}{2}\int\limits_{brane} \phi f \phi} \tag{7}$$

where we perform integration over fluctuations Ψ: $\Psi\Big|_{brane} = \phi$, over background configuration $\Psi_0 = \Psi[\bar{\varphi}]$, which is the solution of:

$$\begin{cases} \frac{\delta \mathbf{S}_{Bulk}[\Psi]}{\delta \Psi} = 0, \\ \Psi|_{brane} = \bar{\varphi}. \end{cases} \tag{8}$$

i.e. the on-shell continuation of brane background to the bulk.

In (7) background $\bar{\varphi}$ shows itself through the operators of small fluctuations in the bulk $\mathbf{F}$ and on branes f:

$$\mathbf{F}\delta^{d+1}(X,X') \equiv \frac{\delta^2 \mathbf{S}_{Bulk}[\Psi]}{\delta\Psi(X)\delta\Psi(X')}\Big|_{\Psi=\Psi[\bar{\varphi}]}, \qquad f\delta^d(x,x') \equiv \frac{\delta^2 \mathrm{S}_{brane}[\varphi]}{\delta\varphi(x)\delta\varphi(x')}\Big|_{\varphi=\bar{\varphi}}. \tag{9}$$

It can be proved [4] that r.h.s of (7) gives the square root of the functional determinant of bulk Neumann Green function for bulk operator F of small fluctuations Ψ over background

$\Psi_0 \equiv \Psi[\bar{\varphi}]$. Which is the same as the determinant of operator F to the power minus one half on the space of functions with Neumann boundary conditions $(-\nabla_n + f)\Psi|_{brane} = 0$:

$$\int d\Psi \, e^{-\frac{1}{2}\int\limits_{Bulk} \Psi \overleftrightarrow{\mathbf{F}} \Psi - \frac{1}{2}\int\limits_{brane} \phi f \phi + \int\limits_{brane} \phi j} = \mathbf{Det}^{1/2}\mathbf{G}_N \cdot e^{\frac{1}{2}\int\limits_{brane} j |\mathbf{G}_N| j} . \tag{10}$$

We introduced auxiliary sources j in path integral to pick up another useful (so-called *tree*) contribution [1]. Notation $|\mathbf{G}_N|$ stands for boundary operator whose kernel is the restriction of bulk Neumann Green function $\mathbf{G}_N(X, X')$

$$\begin{cases} \mathbf{F}\,\mathbf{G}_N(X, X') = \delta^{d+1}(X, X'), \\ (-\nabla_n + f)\,\mathbf{G}_N(X, X')|_{X \in brane} = 0, \end{cases} \tag{11}$$

to boundary (brane):

$$|\mathbf{G}_N|(x, x') \equiv \mathbf{G}_N(X, X')\Big|_{X=x,\, X'=x'}.$$

∇_n – is the covariant derivative normal to brane.

Note, that there were made *no* special assumptions about boundary operator f but it selfadjointness! So we covered all possible *generalized* Neumann boundary conditions such as Robin, oblique or even more involved ones, arising for example in Dvali-Gabadadze-Porrati model (where f is brane d-dimensional D'Alembertian). More detailed discussion of this subject can be found in the next section.

The alternative way to treat path integral above is to split integration

$$\int d\Psi ... \to \int d\phi \int\limits_{\Psi\big|_{brane}=\phi} d\Psi.$$

Performing first integration over bulk fields with fixed boundary values generate determinant of Dirichlet operator in the bulk. Finally this way leads to [4]

$$\int d\Psi \, e^{-\frac{1}{2}\int\limits_{Bulk} \Psi \overleftrightarrow{\mathbf{F}} \Psi - \frac{1}{2}\int\limits_{brane} \phi f \phi + \int\limits_{brane} \phi j} = \mathbf{Det}^{1/2}\mathbf{G}_D \cdot \mathrm{Det}^{-1/2}(F^{ind} + f) \cdot e^{\frac{1}{2}\int\limits_{brane} j\, |F^{ind} + f|^{-1} j} . \tag{12}$$

where $\mathbf{G}_D$ is the Dirichlet Green function:

$$\begin{cases} \mathbf{F}\mathbf{G}_D(X, X') = \delta^{d+1}(X, X'), \\ \mathbf{G}_D(X, X')|_{X \in brane} = 0, \end{cases} \tag{13}$$

and F^{ind} is the (nonlocal) effective inverse propagator for field fluctuations on the boundary generated by bulk dynamics of Ψ, which kernel is defined in terms of Dirichlet Green function as follows:

$$F^{ind}\delta^d(x, x') \equiv \big(- \overrightarrow{\nabla}_n \mathbf{G}_D(X, X') \overleftarrow{\nabla}_{n'} \big)\Big|_{X=x,\, X'=x'}. \tag{14}$$

Comparison of prefactors and j-dependent exponents in (10) and (12) gives justification for the following duality relations between Green functions for different boundary value problems and between functional determinants of operators with different boundary conditions [1],[4]:

$$-\Big|\overrightarrow{\nabla}_n \mathbf{G}_D \overleftarrow{\nabla}_n\Big| + f = \Big|\mathbf{G}_N\Big|^{-1}, \tag{15}$$

$$\mathbf{Det}\mathbf{G}_D \cdot \mathrm{Det}^{-1}(F^{ind} + f) = \mathbf{Det}\mathbf{G}_N . \tag{16}$$

Surely such formal path integral manipulations can not serve as the rigorous prove for relations above. But in fact more pedantic considerations show that (15,16) are correct for rather general setup [1],[4]. Being written in such form duality relations turn out to be valid when for both complimentary boundary value problems zero modes are absent, so that operators F in the bulk with appropriate boundary conditions are selfadjoint.

4 New Heat Kernel Technique.

One of interesting byproducts of the brane effective action formalism briefly discussed above is a new technique for calculating the heat kernel expansion on manifolds with boundaries.

In quantum field theory heat kernel $K(s|X,Y) \equiv e^{-s\mathbf{F}}\delta(X,Y)$ for differential operator $\mathbf{F}$ is a well-known and convenient tool for studying one-loop divergences, counterterms, quantum anomalies, Casimir effect and various asymptotics of effective actions and propagators [7]. In the previous century heat kernel theory became a classical subject in mathematics, particularly in spectral theory of operators. Convenience and efficiency of heat kernel in many respects is based on the possibility of expanding it and its trace in asymptotic series in (integer) powers of proper time s in the (ultraviolet) limit $s \to 0$:

$$e^{-s\mathbf{F}}\delta(X,Y) = \frac{1}{(4\pi s)^{\mathrm{D}/2}}\sum_{n=0}^{\infty} s^n a_n(X,Y) \tag{17}$$

with easily calculable coefficients $a_n(X,Y)$, since they satisfy recurrent equations. These coefficients in physics known as Schwinger-DeWitt coefficients are universal and can be explicitly found in the coincidence limit $(X \to Y)$ as local invariants of space-time curvature, fibre-bundle connection and background external fields [7]. D here is the spacetime dimension.

In particular this expansion gives rise to local low-energy expansion of the effective action in inverse powers of large mass parameter M^2 when the inverse propagator is supplied with that mass term $\mathbf{F} \to \mathbf{F} + M^2$

$$\mathit{\Gamma}_M \equiv \frac{1}{2}\ln\mathbf{Det}(\mathbf{F}+M^2) = -\frac{1}{2}\int^{\infty}\frac{ds}{s}\,\frac{1}{(4\pi s)^{\mathrm{D}/2}}\,e^{-sM^2}\sum_{n=0}^{\infty} s^n \int_{Bulk} a_n(X,X). \tag{18}$$

On spacetimes with boundaries ($\mathrm{D} = d{+}1$) the heat kernel theory becomes complicated. The apparent modification consists in that while bulk coefficients of expansion remain unchangeable, the additional series of half-integer powers of proper time (and possible logarithmic terms) with coefficients built up of boundary invariants arise

$$\mathbf{Tr}\, e^{-s\mathbf{F}-sM^2} = \frac{1}{(4\pi s)^{(\mathrm{d}+1)/2}}\,e^{-sM^2}\sum_{n=0}^{\infty}\Big[s^n\,\mathbf{A}_n + s^{n/2}\,\mathrm{B}_{n/2} + \ln(s\mu^2)\,s^{n/2}\,\mathrm{C}_{n/2}\Big] \tag{19}$$

where

$$\mathbf{A}_n = \int_{Bulk} a_n(X,X); \qquad \mathrm{B}_n, \mathrm{C}_n = \int_{brane} b_n, c_n(x,x). \tag{20}$$

The crucial difficulty consists in the absence of recurrent local scheme for boundary coefficients B_n (and C_n). In contrast to the bulk coefficients $\mathbf{A}_n$ which are universal and independent on boundary conditions the surface (boundary) coefficients essentially depend on the latter. Surface coefficients can be somehow regularly found (by the method of images) only for Dirichlet and homogeneous Neumann boundary conditions for particular cases of bounded spacetimes. For more complicated boundary conditions (such as Robin and oblique cases) the problem of calculating the surface contributions in (19) becomes highly involved or needs the use of methods (e.g. based on conformal properties of geometric invariants) which are not universally applicable (see [7],[8] and references therein).

Surprisingly that obtained one-loop duality suggests new elegant and rather universal way for explicit covariant calculation of Schwinger-DeWitt boundary coefficient for generalized Neumann boundary conditions. Boundary conditions are parameterized by the boundary operator $W = -\nabla_n + f$ which determine the selfadjoint extension of bulk symmetric operator $\mathbf{F}$. In our approach, beside pure Neumann case $f = 0$, which arise for example in Randall-Sundrum type models, one can deal with arbitrary local boundary conditions, in particular with the following generalized Neumann boundary conditions:

$$\begin{array}{ll} f = v(x) & \text{– inhomogeneous Neumann (Robin) case;} \\ f = \frac{1}{2}(\Gamma^\mu \nabla_\mu + \nabla_\mu \Gamma^\mu) & \text{– oblique boundary conditions (open string, Chern-Simons);} \\ f = -\frac{1}{m}\Box \equiv -\frac{1}{m}\, g^{\alpha\beta}\nabla_\alpha \nabla_\beta & \text{– Dvali-Gabadadze-Porrati model.} \end{array} \tag{21}$$

The key point of new technique is that one-loop duality relation (16) being rewritten in the following form

$$\mathbf{Tr}\,\ln(\mathbf{F}_N + M^2) - \mathbf{Tr}\,\ln(\mathbf{F}_D + M^2) = \mathrm{Tr}\,\ln\left(F^{ind}_M(\Box) + f(\nabla)\right) \tag{22}$$

and being expanded in inverse powers of M

$$\begin{aligned} &\mathbf{Tr}\,\ln(\mathbf{F}_{N,D} + M^2) \\ &\cong \int_0^\infty \frac{ds}{s}\, s^{-\frac{d+1}{2}}\, e^{-sM^2} \sum_{n=0}^\infty \left[s^n\,\mathbf{A}_n + s^{n/2}\,B^{N,D}_{n/2} + \ln(s\mu^2)\, s^{n/2}\, C^N_{n/2}\right] \\ &\cong M^{d+1} \sum_{n=0}^\infty \left[M^{-2n}\,\mathbf{A}_n + M^{-n}\left(B^{N,D}_{n/2} + C^N_{n/2}\right) - \ln(M^2/\mu^2)\, M^{-n}\, C^N_{n/2}\right] \end{aligned} \tag{23}$$

gives the difference[2] of Schwinger-DeWitt coefficients of heat kernel trace of generalized Neumann operator $\mathbf{F}_N$ and Dirichlet operator $\mathbf{F}_D$ in terms of expansion coefficients of *boundary* operator $\left(F^{ind}_M + f\right)$

$$\mathrm{Tr}\,\ln\left(F^{ind}_M(\Box) + f(\nabla)\right) \cong M^{d+1} \sum_{n=0}^\infty M^{-n} \int\limits_{brane} \left[\left(b^N_{n/2} + c^N_{n/2}\right) - \ln(M^2/\mu^2)\, c^N_{n/2} - b^D_{n/2}\right]. \tag{24}$$

Boundary coefficients $b^D_{n/2}$ for Dirichlet case are considered as reference ones since there exist involved but explicit calculational schemes and they were calculated to rather high orders n (see e.g. [7] and references therein). The Dirichlet case is dedicated since it is unique – it does not depend of initial boundary operator f and for the same reason it is the simplest case.

The only thing one needs in our scheme to obtain generalized Neumann Schwinger-DeWitt coefficients is the handy tool to calculate the large M expansion for trace of logarithm of boundary operator in l.h.s of (24). Despite the nonlocality of boundary operator F^{ind} great simplification comes from the fact that boundary (brane) itself is the d-dimensional manifold *without* boundary. For most of known realistic braneworld scenarios ([2],[3],...) where branes are symmetric manifolds (and if there are more then one brane they are parallel) operator $\left(F^{ind}_M + f\right)$ can be expressed explicitly as the function of intrinsic and embedding geometric quantities and tangential boundary covariant derivative ∇_α. This allows one to use straightforwardly the theory of asymptotic expansion of integrals with weak peculiarity to obtain the expansion in inverse powers of M. Finally identifying terms at the same powers of M in both sides of (24) one attains the aim.

This elegant scheme really works. In particular we have tested it for few first coefficients for Robin and oblique boundary conditions [4],[9] know in the literature (see e.g. [7] and references therein). The case originating from Dvali-Gabadadze-Porrati model (21) is tractable as well and gives an interesting example of nonanalytic expansion (24) [9].

[2]Note that in l.h.s. of (22) bulk contributions ($\mathbf{A}_n$) from expansion of generalized Neumann and Dirichlet operators are identical and therefore cancel out.

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