Casimir energy of two plates inside a cylinder.

Valery N. Marachevsky [a*]

[a] *V. A. Fock Institute of Physics, St. Petersburg State University, 198504 St. Petersburg, Russia*

Abstract

The new exact formulas for the attractive Casimir force acting on each of the two perfectly conducting plates moving freely inside an infinite perfectly conducting cylinder with the same cross section are derived at zero and finite temperatures by making use of zeta function technique. The short and long distance behaviour of the plates' free energy is investigated.

1 Introduction

Recently a new geometry in the Casimir effect[1], a piston geometry, has been introduced in a $2D$ Dirichlet model [2]. Generally the piston is located in a semi-infinite cylinder closed at its head. The piston is perpendicular to the walls of the cylinder and can move freely inside it. The cross sections of the piston and cylinder coincide. Physically this means that the approximation is valid when the distance between the piston and the walls of a cylinder is small in comparison with the piston size.

In paper [3] a perfectly conducting square piston at zero temperature was investigated in $3D$ model in the electromagnetic and scalar case. The exact formula (Eq.(6) in [3]) for the force on a piston was written in the electromagnetic case. Also the limit of short distances was found for arbitrary cross sections (Eq.(7) in [3]). This result was generalized in [4]. In paper [4] the exact formula for the free energy of two perfectly conducting plates of an arbitrary cross section inside the waveguide (or infinite cylinder) with the same cross section was written (Eq.(118) in [4] or (30) here). In the zero temperature case and square cross section of the waveguide our general result for the force (35) coincides with Eq.(6) in [3]. It is interesting that our result resembles the result for the interquark potential in a rigid string model [5] .

A dilute circular piston and cylinder were studied perturbatively in [6]. In this case the force on two plates inside a waveguide and the force in a piston geometry differ essentially. The force in a piston geometry can even change sign in this approximation for thin enough walls of the material. Other examples of repulsive pistons were presented in [7].

In Sec.2 we derive the new exact result (30) for the free energy of two parallel plates inside an infinite cylinder by making use of the zeta function technique [8, 9]. We consider a perfectly conducting case, the plates move freely inside the cylinder with the same cross section, which is arbitrary. The plates are perpendicular to the walls of the cylinder. In Sec.3 we apply the heat kernel technique [10, 11, 4] to derive the leading short distance behaviour of the free energy. In the short distance limit we prove that there are no temperature corrections to the leading terms obtained in [3] (Eq.(7) in [3]). The long distance limit result (40) (the high temperature limit result) is new.

We take $\hbar = c = 1$.

[*] **e-mail**: maraval@mail.ru

2 Derivation

Our aim is to calculate the Casimir energy of interaction and the force between the two parallel plates of an arbitrary cross section inside an infinite cylinder of the same cross section (the plates are perpendicular to the walls of the cylinder).

TE and TM eigenfrequencies of the perfectly conducting cylindrical resonator with an arbitrary cross section can be written as follows. For TE modes ($E_z = 0$) inside the perfectly conducting cylindrical resonator of the length a with an arbitrary cross section the magnetic field $B_z(x, y, z)$ and eigenfrequencies ω_{TE} are determined by:

$$B_z(x,y,z) = \sum_{i=1,n=1}^{+\infty} B_{in} \sin\Big(\frac{\pi n z}{a}\Big) g_i(x,y), \tag{1}$$

$$\Delta^{(2)} g_i(x,y) = -\lambda_{iN}^2 g_i(x,y) \tag{2}$$

$$\frac{\partial g_i(x,y)}{\partial n}\Big|_{\partial M} = 0 \tag{3}$$

$$\omega_{TE}^2 = \Big(\frac{\pi n}{a}\Big)^2 + \lambda_{iN}^2, \quad n = 1.. + \infty, i = 1.. + \infty. \tag{4}$$

The other components of the magnetic and electric fields can be expressed via $B_z(x, y, z)$.

For the TM modes ($B_z = 0$) inside the perfectly conducting cylindrical resonator of the length a with an arbitrary cross section the electric field $E_z(x, y, z)$ and eigenfrequencies ω_{TM} are determined by:

$$E_z(x,y,z) = \sum_{n=0,k=1}^{+\infty} E_{kn} \cos\Big(\frac{\pi n z}{a}\Big) f_k(x,y), \tag{5}$$

$$\Delta^{(2)} f_k(x,y) = -\lambda_{kD}^2 f_k(x,y) \tag{6}$$

$$f_k(x,y)|_{\partial M} = 0 \tag{7}$$

$$\omega_{TM}^2 = \Big(\frac{\pi n}{a}\Big)^2 + \lambda_{kD}^2, \quad n = 0.. + \infty, k = 1.. + \infty \tag{8}$$

In ζ-function regularization scheme the Casimir energy is defined as follows:

$$E = \frac{1}{2}\Big(\sum \omega_{TE}^{1-s} + \sum \omega_{TM}^{1-s}\Big)\Big|_{s=0}, \tag{9}$$

where the sum has to be calculated for large positive values of s and after that an analytical continuation to the value $s = 0$ is performed.

Alternatively one can define the Casimir energy via a zero temperature one loop effective action W (T_1 is a time interval here):

$$W = -ET_1 \tag{10}$$

$$E = -\zeta'(0) \tag{11}$$

$$\zeta(s) = \frac{1}{\Gamma(\frac{s}{2})} \int_0^{+\infty} dt\, t^{\frac{s}{2}-1} \sum_{\omega_{TE},\omega_{TM}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \exp\Big(-t\Big(\frac{a}{\pi}\Big)^2 \Big(\omega^2 + p^2\Big)\Big) \tag{12}$$

After integration over p in (12) one can see that definitions (9) and (11) coincide.

In every Casimir sum it is convenient to write:

$$\sum_{n=1}^{+\infty} \exp(-tn^2) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \exp(-tn^2) - \frac{1}{2} = \frac{1}{2}\theta_3\Big(0, \frac{t}{\pi}\Big) - \frac{1}{2}. \tag{13}$$

For the first term on the right-hand side of (13) we use the property of the theta function $\theta_3(0,x)$:

$$\theta_3(0,x) = \frac{1}{\sqrt{x}}\,\theta_3\Big(0,\frac{1}{x}\Big) \tag{14}$$

and the value of the integral

$$\int_0^{+\infty} dt\, t^{\alpha-1}\exp\Big(-p\,t-\frac{q}{t}\Big) = 2\Big(\frac{q}{p}\Big)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{pq}) \tag{15}$$

for nonzero values of n to rewrite the Neumann zeta function $\zeta_N(s)$ (arising from TE modes) in the form:

$$\begin{aligned}\zeta_N(s) = \sum_{\lambda_{iN}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\Big[&\frac{\sqrt{\pi}\,\Gamma\big((s-1)/2\big)}{2\,\Gamma(s/2)}\Big(\frac{a\sqrt{\lambda_{iN}^2+p^2}}{\pi}\Big)^{1-s}+\\ &+\sum_{n=1}^{+\infty}\frac{2\sqrt{\pi}}{\Gamma(s/2)}\Big(\frac{\pi^2 n}{a\sqrt{\lambda_{iN}^2+p^2}}\Big)^{\frac{s-1}{2}}K_{\frac{s-1}{2}}\Big(2an\sqrt{\lambda_{iN}^2+p^2}\Big)\Big]-\\ &-\sum_{\lambda_{iN}}\frac{\sqrt{\pi}\,\Gamma\big((s-1)/2\big)}{4a\Gamma(s/2)}\Big(\frac{a\lambda_{iN}}{\pi}\Big)^{1-s}\end{aligned} \tag{16}$$

The Neumann part of the Casimir energy is given by:

$$\begin{aligned}E_N = -\zeta_N'(0) = &\sum_{\lambda_{iN}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\frac{1}{2}\ln\Big(1-\exp(-2a\sqrt{\lambda_{iN}^2+p^2})\Big)+\\ &+\frac{a}{2}\sum_{\lambda_{iN}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\Big(\lambda_{iN}^2+p^2\Big)^{\frac{1-s}{2}}\Big|_{s=0}-\frac{1}{4}\sum_{\lambda_{iN}}\lambda_{iN}^{1-s}\Big|_{s=0}.\end{aligned} \tag{17}$$

Here we used $K_{-1/2}(x) = \sqrt{\pi/(2\,x)}\,\exp(-x)$.

The Dirichlet part of the Casimir energy (from TM modes) is obtained by analogy:

$$\begin{aligned}E_D = &\sum_{\lambda_{kD}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\frac{1}{2}\ln\Big(1-\exp(-2a\sqrt{\lambda_{kD}^2+p^2})\Big)+\\ &+\frac{a}{2}\sum_{\lambda_{kD}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\Big(\lambda_{kD}^2+p^2\Big)^{\frac{1-s}{2}}\Big|_{s=0}+\frac{1}{4}\sum_{\lambda_{kD}}\lambda_{kD}^{1-s}\Big|_{s=0}.\end{aligned} \tag{18}$$

The electromagnetic Casimir energy of a perfectly conducting resonator of the length a and an arbitrary cross section is given by the sum of (17) and (18) :

$$E = \sum_{\lambda_{iN}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\frac{1}{2}\ln\Big(1-\exp(-2a\sqrt{\lambda_{iN}^2+p^2})\Big)+ \tag{19}$$

$$+\sum_{\lambda_{kD}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\frac{1}{2}\ln\Big(1-\exp(-2a\sqrt{\lambda_{kD}^2+p^2})\Big)+ \tag{20}$$

$$+\frac{a}{2}\sum_{\lambda_{iN}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\Big(\lambda_{iN}^2+p^2\Big)^{\frac{1-s}{2}}\Big|_{s=0}+ \tag{21}$$

$$+\frac{a}{2}\sum_{\lambda_{kD}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\Big(\lambda_{kD}^2+p^2\Big)^{\frac{1-s}{2}}\Big|_{s=0}+ \tag{22}$$

$$+\frac{1}{4}\sum_{\lambda_{kD}}\lambda_{kD}^{1-s}\Big|_{s=0}-\frac{1}{4}\sum_{\lambda_{iN}}\lambda_{iN}^{1-s}\Big|_{s=0}. \tag{23}$$

The terms

$$E_{waveguide} = \frac{1}{2}\sum_{\lambda_{iN}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \left(\lambda_{iN}^2 + p^2\right)^{\frac{1-s}{2}}\Bigg|_{s=0} + \tag{24}$$

$$+\frac{1}{2}\sum_{\lambda_{kD}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \left(\lambda_{kD}^2 + p^2\right)^{\frac{1-s}{2}}\Bigg|_{s=0} \tag{25}$$

yield the electromagnetic Casimir energy for a unit length of a perfectly conducting infinite cylinder with the same cross section as the resonator under consideration.

For the experimental check of the Casimir energy for the rectangular cavity one should measure the force somehow. We think about the following possibility: one should insert two parallel perfectly conducting plates inside an infinite perfectly conducting cylinder and measure the force acting on one of the plates as it is being moved through the cylinder. The distance between the inserted plates is a.

To calculate the force on each plate the following gedanken experiment is useful. Imagine that 4 parallel plates are inserted inside an infinite cylinder and then 2 exterior plates are moved to spatial infinity. This situation is exactly equivalent to 3 perfectly conducting cavities touching each other. From the energy of this system one has to subtract the Casimir energy of an infinite cylinder, only then do we obtain the energy of interaction between the interior parallel plates, the one that can be measured in the proposed experiment. Doing so we obtain the attractive force on each interior plate inside the cylinder:

$$F(a) = -\frac{\partial E_{arb}(a)}{\partial a}, \tag{26}$$

$$E_{arb}(a) = \sum_{\omega_{wave}} \frac{1}{2} \ln(1 - \exp(-2a\,\omega_{wave})), \tag{27}$$

the sum here is over all TE and TM eigenfrequencies ω_{wave} for the cylinder with an arbitrary cross section and an infinite length. Thus it can be said that *the exchange of photons with the eigenfrequencies of an infinite cylinder between the inserted plates always yields the attractive force between the plates.*

For rectangular boxes it was generally believed [12, 13] that the repulsive contribution to the force acting on two parallel opposite sides of a *single* box (separated by a distance a) and resulting here from (21-22) could be measured in experiment. However, it is not possible to use the expression (19-23) directly to calculate the force since it includes the self-energy Casimir parts (21-22) of the other sides of the resonator. Nevertheless the expression (19-23) can be used to derive a measurable force (26) between the freely moving parallel plates inserted inside an infinite cylinder of the same cross section as the plates.

To get the free energy $F_{arb}(a,\beta)$ for bosons at nonzero temperatures ($\beta = 1/T$) one has to make the substitutions:

$$p \to p_m = \frac{2\pi m}{\beta}, \tag{28}$$

$$\int_{-\infty}^{+\infty} \frac{dp}{2\pi} \to \frac{1}{\beta}\sum_{m=-\infty}^{+\infty}. \tag{29}$$

Thus the free energy describing the interaction of the two parallel perfectly conducting plates inside an infinite perfectly conducting cylinder of an arbitrary cross section has the form:

$$F_{arb}(a,\beta) = \frac{1}{\beta}\sum_{\lambda_{kD}} \sum_{m=-\infty}^{+\infty} \frac{1}{2}\ln\Big(1 - \exp(-2a\sqrt{\lambda_{kD}^2 + p_m^2})\Big) +$$

$$+\frac{1}{\beta}\sum_{\lambda_{iN}} \sum_{m=-\infty}^{+\infty} \frac{1}{2}\ln\Big(1 - \exp(-2a\sqrt{\lambda_{iN}^2 + p_m^2})\Big), \tag{30}$$

where λ_{kD}^2 and λ_{iNeum}^2 are eigenvalues of the two-dimensional Dirichlet and Neumann problems (a boundary here coincides with the boundary of each plate inside the cylinder):

$$\Delta^{(2)} f_k(x,y) = -\lambda_{kD}^2 f_k(x,y) \tag{31}$$
$$f_k(x,y)|_{\partial M} = 0, \tag{32}$$

$$\Delta^{(2)} g_i(x,y) = -\lambda_{iN}^2 g_i(x,y) \tag{33}$$
$$\left.\frac{\partial g_i(x,y)}{\partial n}\right|_{\partial M} = 0. \tag{34}$$

The attractive force between the plates inside an infinite cylinder of the same cross section at nonzero temperatures is given by:

$$F(a,\beta) = -\frac{\partial F_{arb}(a,\beta)}{\partial a} = \\ -\frac{1}{\beta}\sum_{\omega_{TD}} \frac{\omega_{TD}}{\exp(2a\omega_{TD})-1} - \frac{1}{\beta}\sum_{\omega_{TN}} \frac{\omega_{TN}}{\exp(2a\omega_{TN})-1}. \tag{35}$$

Here $\omega_{TD} = \sqrt{p_m^2 + \lambda_{kD}^2}$ and $\omega_{TN} = \sqrt{p_m^2 + \lambda_{iN}^2}$.

3 Asymptotic cases

It is convenient to apply the technique of the heat kernel to obtain the short distance behaviour of the free energy (30). It can be done by noting that if the heat kernel expansion

$$\sum_{\lambda_i} e^{-t\lambda_i^2} \underset{t\to 0}{\sim} \sum_{k=0}^{+\infty} t^{\frac{-n+k}{2}} c_k \tag{36}$$

exists (n is a dimension of the Riemannian space) then one can write the expansion

$$\sum_{\lambda_i} e^{-\sqrt{t}\lambda_i} \underset{t\to 0}{\sim} \sum_{k=0}^{n-1} \frac{2\,\Gamma(n-k)}{\Gamma((n-k)/2)}\; t^{\frac{-n+k}{2}} c_k \tag{37}$$

by making use of the analytical structure of the zeta function. The strategy is the following: one expands the logarithms in the formula (30) in series and then applies the expansion (37) to each term.

For $a \ll \beta/(4\pi)$ one obtains from (30) and (37) the leading terms for the free energy:

$$F_{arb}(a,\beta)|_{a\ll\beta/(4\pi)} = -\frac{\zeta_R(4)}{8\pi^2}\frac{S}{a^3} - \frac{\zeta_R(2)}{4\pi a}(1-2\chi) + O(1), \tag{38}$$

where

$$\chi = \sum_i \frac{1}{24}\left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right) + \sum_j \frac{1}{12\pi}\int_{\gamma_j} L_{aa}(\gamma_j)d\gamma_j. \tag{39}$$

Here α_i is the interior angle of each sharp corner and $L_{aa}(\gamma_j)$ is the curvature of each smooth section described by the curve γ_j. The force calculated from (38) coincides with F_C in [3], (Eq.7).

In the opposite long distance limit $a \gg \beta/(4\pi)$ one has to keep only $m = 0$ term in (30). Thus the free energy of the plates inside a cylinder in this limit (the high temperature limit) is equal to:

$$F_{arb}(a,\beta)|_{a\gg\beta/(4\pi)} = \frac{1}{2\beta}\sum_{\lambda_{kD}} \ln\Big(1-\exp(-2a\lambda_{kD})\Big) + \\ + \frac{1}{2\beta}\sum_{\lambda_{iN}} \ln\Big(1-\exp(-2a\lambda_{iN})\Big) \quad (40)$$

This result is new.

One can check that the limit $a \to 0$ in (40) immediately yields the high temperature result for two parallel perfectly conducting plates separated by a distance a. One expands logarithms in series and uses (37) and $c_{0D} = c_{0N} = S/(4\pi)$ in two dimensions ($n = 2$) to obtain:

$$F_{arb}(a,\beta)|_{a\gg\beta/(4\pi), a\to 0} = \\ = -\sum_{\lambda_{kD}} \frac{1}{2\beta}\sum_{n=1}^{+\infty} \frac{\exp(-2an\lambda_{kD})}{n}\Big|_{a\to 0} - \sum_{\lambda_{iN}} \frac{1}{2\beta}\sum_{n=1}^{+\infty} \frac{\exp(-2an\lambda_{iN})}{n}\Big|_{a\to 0} = \\ = \sum_{n=1}^{+\infty} -\frac{1}{2\beta}\frac{1}{n}\frac{1}{(2an)^2} 2(c_{0D}+c_{0N}) = -\frac{\zeta_R(3)}{\beta a^2}\frac{S}{8\pi}, \quad (41)$$

which is a well known result [14, 15, 16].

Acknowledgements

V.M. thanks D.V.Vassilevich, Yu.V.Novozhilov and V.Yu.Novozhilov for suggestions during the preparation of the paper. V.M. thanks R.L.Jaffe and M.P.Hertzberg for correspondence and discussions. This work has been supported by grants RNP 2.1.1.1112 and SS .5538.2006.2.

References

[1] H.B.G.Casimir, *K.Ned.Akad.Wet.Proc.***51**, 793 (1948).

[2] R.M.Cavalcanti, *Phys.Rev.D* **72**, 065015 (2004) [arXiv: quant-ph/0310184].

[3] M. P. Hertzberg, R. L. Jaffe, M. Kardar and A. Scardicchio, *Phys. Rev. Lett.* **95**, 250402 (2005) [arXiv: quant-ph/0509071].

[4] V.N.Marachevsky, *One loop boundary effects: techniques and applications* [arXiv: hep-th/0512221].

[5] V.V.Nesterenko and I.G.Pirozhenko, *J.Math.Phys.***38**, 6265 (1997) [arXiv: hep-th/9703097].

[6] G.Barton, *Phys.Rev.D* **73**, 065018 (2006).

[7] S.A.Fulling and J.H.Wilson, *Repulsive Casimir Pistons* [arXiv: quant-ph/0608122] .

[8] E.M.Santangelo, *Theor.Math.Phys.* **131**, 527 (2002); *Teor.Mat.Fiz.* **131**, 98 (2002) [arXiv: hep-th/0104025].

[9] E. Elizalde, S. D. Odintsov, A. Romeo, A.Bytsenko and S. Zerbini, ”Zeta regularization with applications”, World Sci., Singapore, 1994.

[10] D. V. Vassilevich, *Phys.Rept.* **388**, 279 (2003) [arXiv:hep-th/0306138].

[11] P. B. Gilkey, "Invariance theory, the heat equation, and the Atiyah-Singer index theorem", CRC Press, 1994.

[12] W.Lukosz, *Physica(Amsterdam)* **56**, 109 (1971).

[13] M.Bordag, U.Mohideen, V.M.Mostepanenko, *Phys.Rept.* **353**, 1 (2001) [arXiv: quant-ph/0106045].

[14] I.Brevik, S.A.Ellingsen and K.A.Milton, *Thermal corrections to the Casimir effect* [arXiv: quant-ph/0605005].

[15] F.Sauer, *PhD Thesis Gottingen*, 1962.

[16] E.M.Lifshitz, *Zh.Eksp.Teor.Fiz.* **29**, 94 (1956) [*Soviet Phys. JETP* **2**, 73]; I.D.Dzyaloshinskii, E.M.Lifshitz and L.P.Pitaevskii, *Usp.Fiz.Nauk* **73**, 381 (1961) [*Soviet Phys.Usp.***4**, 153].