

BRST approach to higher spin field theories

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Abstract

We develop the BRST approach to Lagrangian formulation for massive bosonic and massless fermionic higher spin fields on a flat space-time of arbitrary dimension. General procedure of gauge invariant Lagrangian construction describing the dynamics of the fields with any spin is given. No off-shell constraints on the fields (like tracelessness) and the gauge parameters are imposed. The procedure is based on construction of new representations for the closed algebras generated by the constraints defining irreducible representations of the Poincare group. We also construct Lagrangians describing propagation of all massive bosonic fields and massless fermionic fields simultaneously.

1 Introduction

Construction of higher spin field theory is one of the fundamental problems of high energy theoretical physics. At present, there exist the various approaches to this problem (see e.g. [1] for reviews). This paper is a brief review of recent development of BRST approach to free higher spin field theory. It is based on two papers [2, 3] (see also [4, 5] for recent development) devoted to Lagrangian construction of free fermionic massless higher spin fields and Lagrangian construction of free bosonic massive higher spin fields respectively. The main motivation for using BRST approach is to try to construct the theory of interacting higher spin fields analogously to string field theory. The first natural step in constructing a higher spin interacting model is formulation of the corresponding free theory.

The paper is organized as follows. In sections 2 and 3 we discuss operator algebras generated by primary constraints which define irreducible representations of the Poincare group both in the massless fermionic and massive bosonic cases respectively. The structure of the algebras proved to be the same and the method of Lagrangian construction is explained in section 4 on the base of a toy model. Then in sections 5 and 6 we apply this method for Lagrangian construction both for massless fermionic and massive bosonic fields respectively. Section 7 summarizes the obtained results.

2 Massless fermionic theory. Algebra of the constraints.

It is well known that the totally symmetrical tensor-spinor field $\Psi_{\mu_1 \dots \mu_n}$ (the Dirac index is suppressed), describing the irreducible spin $s = n + 1/2$ representation must satisfy the following constraints (see e.g. [6])

$$\gamma^\nu \partial_\nu \Phi_{\mu_1 \dots \mu_n} = 0, \quad \gamma^\mu \Phi_{\mu \mu_2 \dots \mu_n} = 0. \quad (1)$$

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Here γ^μ are the Dirac matrices $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, $\eta_{\mu\nu} = (+, -, \dots, -)$.

In order to describe all higher tensor-spinor fields together it is convenient to introduce Fock space generated by creation and annihilation operators a_μ^+, a_μ with vector Lorentz index $\mu = 0, 1, 2, \dots, D-1$ satisfying the commutation relations

$$[a_\mu, a_\nu^+] = -\eta_{\mu\nu}. \quad (2)$$

These operators act on states in the Fock space

$$|\Phi\rangle = \sum_{n=0}^{\infty} \Phi_{\mu_1 \dots \mu_n}(x) a^{+\mu_1} \dots a^{+\mu_n} |0\rangle \quad (3)$$

which describe all half-integer spins simultaneously if the following constraints are taken into account

$$T_0|\Phi\rangle = 0, \quad T_1|\Phi\rangle = 0, \quad (4)$$

where

$$T_0 = \gamma^\mu p_\mu, \quad T_1 = \gamma^\mu a_\mu, \quad (5)$$

with $p_\mu = -i\frac{\partial}{\partial x^\mu}$. If constraints (4) are fulfilled for the general state (3) then constraints (1) are fulfilled for each component $\Phi_{\mu_1 \dots \mu_n}(x)$ in (3) and hence the relations (4) describe all free higher spin fermionic fields together. The constraints T_0, T_1 are primary constraints. They generate all the constraints on the space of ket-vectors (3). Thus we get three more constraints

$$L_0|\Phi\rangle = 0, \quad L_1|\Phi\rangle = 0, \quad L_2|\Phi\rangle = 0, \quad (6)$$

where

$$L_0 = -p^2, \quad L_1 = a^\mu p_\mu, \quad L_2 = \frac{1}{2} a_\mu a^\mu. \quad (7)$$

Our purpose is to construct Lagrangian for the massless fermionic higher spin fields on the base of BRST approach, therefore we must construct Hermitian BRST operator. In the case under consideration the constraints T_0, L_0 are Hermitian, $T_0^+ = T_0, L_0^+ = L_0$, however the constraints T_1, L_1, L_2 are not Hermitian. Therefore we extend the set of the constraints adding three new operators

$$T_1^+ = \gamma^\mu a_\mu^+, \quad L_1^+ = a^{+\mu} p_\mu, \quad L_2^+ = \frac{1}{2} a_\mu^+ a^{+\mu} \quad (8)$$

to the initial constraints (5) and (7). As a result, the set of operators $T_0, T_1, T_1^+, L_0, L_1, L_2, L_1^+, L_2^+$ is invariant under Hermitian conjugation. Taking hermitian conjugation of (4) and (6) we see that the operators (8) together with T_0 and L_0 are constraints on the space of ket-vectors

$$\langle\Phi|T_0 = \langle\Phi|T_1^+ = \langle\Phi|L_0 = \langle\Phi|L_1^+ = \langle\Phi|L_2^+ = 0. \quad (9)$$

Algebra of operators (5), (7), (8) is open in terms of commutators of these operators. We will suggest the following procedure of consideration. We want to use the BRST construction in the simplest (minimal) form corresponding to closed algebras. To get such an algebra we add to the above set of operators, all the operators generated by the commutators of (5), (7), (8). Doing such a way we obtain one new operator

$$G_0 = -a_\mu^+ a^\mu + \frac{D}{2}, \quad (10)$$

which arises from the commutators

$$-\frac{1}{2}[T_1, T_1^+] = [L_2, L_2^+] = G_0, \quad (11)$$

and which is not a constraint neither in the space of ket-vectors nor in the space of bra-vectors. The resulting operators algebra may be found in [2].

Let us summarize what we have at the moment. The structure of the operator algebra in the fermionic case is as follows. First we have hermitian operators T_0 , L_0 , G_0 . Two of them T_0 and L_0 are constraints both in the space of ket-vectors and in the space of bra-vectors, another G_0 is not a constraint neither in the space of ket-vectors nor in the space of bra-vectors. Then we have pairs of mutually conjugated operators (T_1, T_1^+) , (L_1, L_1^+) , (L_2, L_2^+) . One representative from the pairs is a constraint in the space of ket-vectors another representative is a constraint on the space of bra-vectors. The problem is to find BRST operator which reproduce equations of motion (1) up to gauge transformations.

Let us turn to the massive bosonic case.

3 Massive bosonic theory. Algebra of the constraints.

It is well known that the totally symmetric tensor field $\Phi_{\mu_1 \dots \mu_s}$, describing the irreducible spin- s massive representation of the Poincare group must satisfy the following constraints (see e.g. [6])

$$(\partial^2 + m^2)\Phi_{\mu_1 \dots \mu_s} = 0, \quad \partial^{\mu_1} \Phi_{\mu_1 \mu_2 \dots \mu_s} = 0, \quad \eta^{\mu_1 \mu_2} \Phi_{\mu_1 \dots \mu_s} = 0. \quad (12)$$

Analogously to the fermionic case, in order to describe all higher integer spin fields simultaneously we introduce Fock space generated by creation and annihilation operators a_μ^+ , a_μ satisfying the commutation relations (2) and define the operators

$$L_0 = -p^2 + m^2, \quad L_1 = a^\mu p_\mu, \quad L_2 = \frac{1}{2} a^\mu a_\mu, \quad (13)$$

where $p_\mu = -i \frac{\partial}{\partial x^\mu}$. These operators act on states in the Fock space

$$|\Phi\rangle = \sum_{s=0}^{\infty} \Phi_{\mu_1 \dots \mu_s}(x) a^{\mu_1+} \dots a^{\mu_s+} |0\rangle \quad (14)$$

which describe all integer spin fields simultaneously if the following constraints on the states take place

$$L_0|\Phi\rangle = 0, \quad L_1|\Phi\rangle = 0, \quad L_2|\Phi\rangle = 0. \quad (15)$$

If constraints (15) are fulfilled for the general state (14) then constraints (12) are fulfilled for each component $\Phi_{\mu_1 \dots \mu_s}(x)$ in (14) and hence the relations (15) describe all free massive higher spin bosonic fields simultaneously.

Constraints (13) are all constraints in the space of ket-vectors (14). Again, as in the fermionic case, in order to be possible to construct hermitian BRST operator we must add to the constraints (13) their hermitian conjugated operators. Since $L_0^+ = L_0$ we add two operators

$$L_1^+ = a^{+\mu} p_\mu, \quad L_2^+ = \frac{1}{2} a_\mu^+ a^{+\mu} \quad (16)$$

to the initial constraints (13). As a result, the set of operators $L_0, L_1, L_2, L_1^+, L_2^+$ is invariant under Hermitian conjugation. Taking hermitian conjugation of (15) we see that the operators (16) together with L_0 are constraints in the space of ket-vectors

$$\langle \Phi | L_0 = \langle \Phi | L_1^+ = \langle \Phi | L_2^+ = 0. \quad (17)$$

Algebra of the constraints (13), (16) is not closed and in order to construct BRST operator we must include in the algebra all the operators generated by (13), (16). Thus we have to include in the algebra two more hermitian operator

$$m^2 \quad \text{and} \quad G_0 = -a_\mu^+ a^\mu + \frac{D}{2}. \quad (18)$$

which are obtained from the commutators

$$[L_1, L_1^+] = L_0 - m^2, \quad [L_2, L_2^+] = G_0, \quad (19)$$

and which are not constraints neither in the space of ket-vectors nor in the space of bra-vectors. The resulting operator algebra can be found in [3].

Let us summarize the structure of the operator algebra in the bosonic case. It is the same as in the fermionic case. First we have hermitian operators T_0 , L_0 , m^2 , G_0 . Two of them T_0 and L_0 are constraints both in the space of ket-vectors and in the space of bra-vectors, another m^2 and G_0 are not constraints neither in the space of ket-vectors nor in the space of bra-vectors. Then we have pairs of mutually conjugated operators (L_1, L_1^+) , (L_2, L_2^+) . One representative from the pairs is a constraint in the space of ket-vectors another representative is a constraint on the space of bra-vectors.

In order to understand better the method used for construction of BRST operator leading to the proper equations of motion (1), (15) it is useful to consider a toy model.

4 A simplified model

Let us consider a model where the 'physical' states are defined by the equations

$$L_0|\Phi\rangle = 0, \quad L_1|\Phi\rangle = 0, \quad (20)$$

with some operators L_0 and L_1 . Let us also suppose that some scalar product $\langle\Phi_1|\Phi_2\rangle$ is defined for the states $|\Phi\rangle$ and let L_0 be a Hermitian operator $(L_0)^+ = L_0$ and let L_1 be non-Hermitian $(L_1)^+ = L_1^+$ with respect to this scalar product. In this section we show how to construct Lagrangian which will reproduce (20) as equations of motion up to gauge transformations.

In order to get the Lagrangian within BRST approach we should begin with the Hermitian BRST operator. However, the standard prescription does not allow to construct such a Hermitian operator on the base of operators L_0 and L_1 if L_1 is non-Hermitian. We assume to define the nilpotent Hermitian operator in the case under consideration as follows.

Let us consider the algebra generated by the operators L_0 , L_1 , L_1^+ and let this algebra takes the form

$$[L_0, L_1] = [L_0, L_1^+] = 0, \quad (21)$$

$$[L_1, L_1^+] = L_0 + C, \quad C = const \neq 0. \quad (22)$$

It is known (see e.g. [7]) that in the case $C = 0$ if we construct Hermitian BRST operators as if all the operators L_0 , L_1 , L_1^+ were the first class constraints then this BRST operator will reproduce the proper equations of motion (20) up to gauge transformations.

Now let us consider the case $C \neq 0$. In this case the central charge C plays the role analogous to m^2 and G_0 in the algebras of the two previous sections. If we construct BRST operator as if the operators L_0 , L_1 , L_1^+ , C are the first class constraints we get a solution $|\Phi\rangle = 0$ [3] what contradicts to (20). This happens because we treat the operator C as a constraint.

But the case $C = 0$ may serve as a hint about solution to our problem. Namely, we construct new representation of the algebra (21), (22) with operator $C_{new} = 0$ in this representation.

Thus the solution is as follows. We enlarge the representation space of the operator algebra (21), (22) by introducing the additional (new) creation and annihilation operators and construct a new representation of the algebra bringing into it an arbitrary parameter h . The basic idea is to construct such a representation where the new operator C_{new} has the form $C_{new} = C + h$. Since parameter h is arbitrary and C is a central charge, we can choose $h = -C$ and the operator C_{new} will be zero in the new representation. After this we proceed as if operators L_{0new} , L_{1new} , L_{1new}^+ are the first class constraints.

For example, we can construct new representation of the operator algebra (21), (22) as follows

$$L_{0new} = L_0, \quad C_{new} = C + h, \quad (23)$$

$$L_{1new} = L_1 + hb, \quad L_{1new}^+ = L_1^+ + b^+. \quad (24)$$

Here we have introduced the new bosonic creation and annihilation operators b^+ , b with the standard commutation relations $[b, b^+] = 1$.

In principle, we could set $h = -C$ and get $C_{new} = 0$, but there is one more equivalent scheme. Namely we still consider C_{new} as nonzero operator including the arbitrary parameter h , but demand for state vectors and gauge parameters to be independent on ghost η_C as before. It can be shown [3] that these conditions reproduce that h should be equal to $-C$.

Now if we introduce the BRST construction taking the operators in new representation as if they were the first class constraints

$$Q_h = \eta_0 L_0 + \eta_C C_{new} + \eta_1^+ L_{1new} + \eta_1 L_{1new}^+ - \eta_1^+ \eta_1 (\mathcal{P}_0 + \mathcal{P}_C), \quad Q_h^2 = 0. \quad (25)$$

we shall get [3] that equation $Q_h |\Psi\rangle = 0$, where

$$|\Psi\rangle = \sum_{k=0}^{\infty} \sum_{k_i=0}^1 (\eta_0)^{k_1} (\eta_1^+)^{k_2} (\mathcal{P}_1^+)^{k_3} (b^+)^k |\Phi_{kk_1 k_2 k_3}\rangle. \quad (26)$$

reproduces (20) up to gauge transformations.

Let us pay attention that operators L_{1new} and L_{1new}^+ are not mutually conjugate in the new representation if we use the usual rules for Hermitian conjugation of the additional creation and annihilation operators $(b)^+ = b^+$, $(b^+)^+ = b$. To consider the operators L_{1new} , L_{1new}^+ as conjugate to each other we change a definition of scalar product for the state vectors (26) $\langle \Psi_1 | \Psi_2 \rangle_{new} = \langle \Psi_1 | K_h | \Psi_2 \rangle$, with

$$K_h = \sum_{n=0}^{\infty} |n\rangle \frac{h^n}{n!} \langle n|, \quad |n\rangle = (b^+)^n |0\rangle. \quad (27)$$

Now the new operators L_{1new} , L_{1new}^+ are mutually conjugate and the operator Q_h is Hermitian relatively the new scalar product (4) since the following relations take place

$$K_h L_{1new}^+ = (L_{1new})^+ K_h, \quad K_h L_{1new} = (L_{1new}^+)^+ K_h, \quad Q_h^+ K_h = K_h Q_h. \quad (28)$$

Finally we note that the proper equations of motion may be derived using the following Lagrangian

$$\mathcal{L} = \int d\eta_0 \langle \Psi | K_{-C} \Delta Q_{-C} | \Psi \rangle \quad (29)$$

where subscripts $-C$ means that we substitute $-C$ instead of h . Here the integral is taken over Grassmann odd variable η_0 .

5 Lagrangians for massless fermionic fields

5.1 New representation

Let us first construct new representation for the operator algebra. One finds

$$L_{2new}^+ = \frac{1}{2} a_\mu a^\mu + b^+, \quad L_{2new} = \frac{1}{2} a_\mu^+ a^{+\mu} + (b^+ b + d^+ d + h) b, \quad (30)$$

$$T_{1new}^+ = \gamma^\mu a_\mu + 2b^+ d + d^+, \quad T_{1new} = \gamma^\mu a_\mu^+ - 2(b^+ b + h) d - d^+ b, \quad (31)$$

$$G_{0new} = -a_\mu^+ a^\mu + \frac{D}{2} + 2b^+ b + d^+ d + h, \quad (32)$$

with the other operators being unchanged. Here b^+ , b are bosonic creation and annihilation operators and d^+ , d are fermionic ones with the standard commutation relations $[b, b^+] = 1$, $\{d, d^+\} = 1$. Then we introduce the scalar product in the Fock space so that $\langle \Phi_1 | \Phi_2 \rangle_{new} = \langle \Phi_1 | K | \Phi_2 \rangle$, with operator K

$$K = \sum_{n=0}^{\infty} \frac{1}{n!} \left(|n\rangle \langle n| C(n, h) - 2d^+ |n\rangle \langle n| d C(n+1, h) \right), \quad |n\rangle = (b^+)^n |0\rangle, \quad (33)$$

$$C(n, h) = h(h+1)(h+2) \dots (h+n-1), \quad C(0, h) = 1. \quad (34)$$

Now we construct BRST operator as if all the operators were the first class constraints

$$\begin{aligned} \tilde{Q} = & q_0 T_0 + q_1^+ T_{1new} + q_1 T_{1new}^+ + \eta_0 L_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_{2new} + \eta_2 L_{2new}^+ \\ & + \eta_G G_{0new} + i(\eta_1^+ q_1 - \eta_1 q_1^+) p_0 - i(\eta_G q_1 + \eta_2 q_1^+) p_1^+ + i(\eta_G q_1^+ + \eta_2^+ q_1) p_1 \\ & + (q_0^2 - \eta_1^+ \eta_1) \mathcal{P}_0 + (2q_1 q_1^+ - \eta_2^+ \eta_2) \mathcal{P}_G + (\eta_G \eta_1^+ + \eta_2^+ \eta_1 - 2q_0 q_1^+) \mathcal{P}_1 \\ & + (\eta_1 \eta_G + \eta_1^+ \eta_2 - 2q_0 q_1) \mathcal{P}_1^+ + 2(\eta_G \eta_2^+ - q_1^{+2}) \mathcal{P}_2 + 2(\eta_2 \eta_G - q_1^2) \mathcal{P}_2^+. \end{aligned} \quad (35)$$

Let us notice that the BRST operator (35) is selfconjugate in the following sense $\tilde{Q}^+ K = K \tilde{Q}$, with operator K (33).

5.2 Lagrangians for the free fermionic fields of single spin

It can be shown [2] that from equation $\tilde{Q}|\Psi\rangle = 0$ using gauge transformations we can remove dependence of the fields and the gauge parameters on the ghost fields η_0 , \mathcal{P}_0 , q_0 , p_0 and obtain equations of motion for field with given spin $s = n + 1/2$

$$\Delta Q_\pi |\chi_0^0\rangle_n + \frac{1}{2} \{ \tilde{T}_0, \eta_1^+ \eta_1 \} |\chi_0^1\rangle_n = 0, \quad \tilde{T}_0 |\chi_0^0\rangle_n + \Delta Q_\pi |\chi_0^1\rangle_n = 0. \quad (36)$$

Here $|\chi_0^0\rangle_n$ and $|\chi_0^1\rangle_n$ are states with ghost numbers 0 and -1 respectively and subscript n indicates that the corresponding field obeying the condition

$$\pi |\chi\rangle_n = (n + (D-4)/2) |\chi\rangle_n, \quad (37)$$

with

$$\pi = G_0 + 2b^+ b + d^+ d - i q_1 p_1^+ + i q_1^+ p_1 + \eta_1^+ \mathcal{P}_1 - \eta_1 \mathcal{P}_1^+ + 2\eta_2^+ \mathcal{P}_2 - 2\eta_2 \mathcal{P}_2^+. \quad (38)$$

Next $\tilde{T}_0 = T_0 - 2q_1^+ \mathcal{P}_1 - 2q_1 \mathcal{P}_1^+$, $\{A, B\} = AB + BA$ and Q_π is the part of \tilde{Q} (35) which independent of η_G , \mathcal{P}_G , η_0 , \mathcal{P}_0 , q_0 , p_0 with substitution $h \rightarrow -\pi$ [2].

These field equations (36) can be deduced from the following Lagrangian

$$\begin{aligned} \mathcal{L}_n = & {}_n \langle \chi_0^0 | K_\pi \tilde{T}_0 | \chi_0^0 \rangle_n + \frac{1}{2} {}_n \langle \chi_0^1 | K_\pi \{ \tilde{T}_0, \eta_1^+ \eta_1 \} | \chi_0^1 \rangle_n \\ & + {}_n \langle \chi_0^0 | K_\pi \Delta Q_\pi | \chi_0^1 \rangle_n + {}_n \langle \chi_0^1 | K_\pi \Delta Q_\pi | \chi_0^0 \rangle_n, \end{aligned} \quad (39)$$

where the standard scalar product for the creation and annihilation operators is assumed and the operator K_π is the operator K (33) where the following substitution is done $h \rightarrow -\pi$ [2].

The equations of motion (36) and the Lagrangian (39) are invariant under the gauge transformations

$$\delta |\chi_0^0\rangle_n = \Delta Q_\pi |\Lambda_0^0\rangle_n + \frac{1}{2} \{ \tilde{T}_0, \eta_1^+ \eta_1 \} |\Lambda_0^1\rangle_n, \quad \delta |\chi_0^1\rangle_n = \tilde{T}_0 |\Lambda_0^0\rangle_n + \Delta Q_\pi |\Lambda_0^1\rangle_n, \quad (40)$$

which are reducible

$$\delta |\Lambda^{(i)0}\rangle_n = \Delta Q_\pi |\Lambda^{(i+1)0}\rangle_n + \frac{1}{2} \{ \tilde{T}_0, \eta_1^+ \eta_1 \} |\Lambda^{(i+1)1}\rangle_n, \quad |\Lambda^{(0)0}\rangle_n = |\Lambda_0^0\rangle_n, \quad (41)$$

$$\delta |\Lambda^{(i)1}\rangle_n = \tilde{T}_0 |\Lambda^{(i+1)0}\rangle_n + \Delta Q_\pi |\Lambda^{(i+1)1}\rangle_n, \quad |\Lambda^{(0)1}\rangle_n = |\Lambda_0^1\rangle_n, \quad (42)$$

with finite number of reducibility stages $i_{max} = n - 1$ for spin $s = n + 1/2$. It can be shown [2] that the Lagrangian (39) can be transformed to the Fang-Fronsdal Lagrangian [8] in four dimensions after eliminating the auxiliary fields.

5.3 Lagrangian for all half-integer spin fields

Now we turn to construction of Lagrangian describing propagation of all half-integer spin fields simultaneously. It can be shown [2] that it looks like

$$\begin{aligned} \mathcal{L} = & \langle \chi_0^0 | K_\pi \tilde{T}_0 | \chi_0^0 \rangle + \frac{1}{2} \langle \chi_0^1 | K_\pi \{ \tilde{T}_0, \eta_1^+ \eta_1 \} | \chi_0^1 \rangle \\ & + \langle \chi_0^0 | K_\pi \Delta Q_\pi | \chi_0^1 \rangle + \langle \chi_0^1 | K_\pi \Delta Q_\pi | \chi_0^0 \rangle, \end{aligned} \quad (43)$$

where $|\chi_0^0\rangle$ and $|\chi_0^1\rangle$ are states with ghost numbers 0 and -1 respectively. Then we have the following gauge transformations for the fields

$$\delta|\chi_0^0\rangle = \Delta Q_\pi |\Lambda_0^0\rangle + \frac{1}{2} \{ \tilde{T}_0, \eta_1^+ \eta_1 \} |\Lambda_0^1\rangle, \quad \delta|\chi_0^1\rangle = \tilde{T}_0 |\Lambda_0^0\rangle + \Delta Q_\pi |\Lambda_0^1\rangle. \quad (44)$$

which are also reducible

$$\delta|\Lambda^{(i)0}_0\rangle = \Delta Q_\pi |\Lambda^{(i+1)0}_0\rangle + \frac{1}{2} \{ \tilde{T}_0, \eta_1^+ \eta_1 \} |\Lambda^{(i+1)1}_0\rangle, \quad |\Lambda^{(0)0}_0\rangle = |\Lambda_0^0\rangle, \quad (45)$$

$$\delta|\Lambda^{(i)1}_0\rangle = \tilde{T}_0 |\Lambda^{(i+1)0}_0\rangle + \Delta Q_\pi |\Lambda^{(i+1)1}_0\rangle, \quad |\Lambda^{(0)1}_0\rangle = |\Lambda_0^1\rangle. \quad (46)$$

Since the fields $|\chi_0^0\rangle$ and $|\chi_0^1\rangle$ contain infinite number of spins and since the order of reducibility grows with the spin value, then the order of reducibility of the gauge symmetry will be infinite.

6 Lagrangians for massive bosonic fields

6.1 New representation for the algebra

To construct new representation, we introduce two pairs of additional bosonic annihilation and creation operators b_1, b_1^+, b_2, b_2^+ with the standard commutation relations $[b_1, b_1^+] = [b_2, b_2^+] = 1$ and construct new representation as follows

$$m_{new}^2 = 0, \quad G_{0new} = -a_\mu^+ a^\mu + \frac{D}{2} + b_1^+ b_1 + \frac{1}{2} + 2b_2^+ b_2 + h, \quad (47)$$

$$L_{1new}^+ = a^\mu p_\mu + m b_1^+, \quad L_{1new} = a^{+\mu} p_\mu + m b_1, \quad (48)$$

$$L_{2new}^+ = \frac{1}{2} a^\mu a_\mu - \frac{1}{2} b_1^{+2} + b_2^+, \quad L_{2new} = \frac{1}{2} a^{+\mu} a_\mu - \frac{1}{2} b_1^2 + (b_2^+ b_2 + h) b_2, \quad (49)$$

with the other operators being unchanged. Then we change the definition of scalar product of vectors in the new representation $\langle \Phi_1 | \Phi_2 \rangle_{new} = \langle \Phi_1 | K | \Phi_2 \rangle$, with operator K in the form

$$K = \sum_{n=0}^{\infty} |n\rangle \frac{C(n, h)}{n!} \langle n|, \quad |n\rangle = (b_2^+)^n |0\rangle. \quad (50)$$

with $C(n, h)$ given in (34).

Next we introduce the operator \tilde{Q} as if all the operators were the first class constarints

$$\begin{aligned} \tilde{Q} = & \eta_0 L_0 + \eta_1^+ L_{1new} + \eta_1 L_{1new}^+ + \eta_2^+ L_{2new} + \eta_2 L_{2new}^+ + \eta_G G_{0new} - \eta_1^+ \eta_1 \mathcal{P}_0 - \eta_2^+ \eta_2 \mathcal{P}_G \\ & + (\eta_G \eta_1^+ + \eta_2^+ \eta_1) \mathcal{P}_1 + (\eta_1 \eta_G + \eta_1^+ \eta_2) \mathcal{P}_1^+ + 2\eta_G \eta_2^+ \mathcal{P}_2 + 2\eta_2 \eta_G \mathcal{P}_2^+, \end{aligned} \quad (51)$$

One can show that the operator (51) satisfy the relation $\tilde{Q}^+ K = K \tilde{Q}$, which means that this operator is Hermitian relatively the new scalar product with operator K (50).

6.2 Lagrangians for the massive bosonic field with given spin

It can be shown [3] that we can construct Lagrangian for the field with given spin as

$$\mathcal{L}_n = \int d\eta_0 \, {}_n\langle\chi|K_\sigma Q_\sigma|\chi\rangle_n. \quad (52)$$

Here field $|\chi\rangle_n$ subject to the condition

$$\sigma|\chi\rangle_n = (n + (D - 6)/2)|\chi\rangle_n. \quad (53)$$

with operator σ being

$$\sigma = G_0 + b_1^+ b_1 + 2b_2^+ b_2 + \eta_1^+ \mathcal{P}_1 - \eta_1 \mathcal{P}_1^+ + 2\eta_2^+ \mathcal{P}_2 - 2\eta_2 \mathcal{P}_2^+. \quad (54)$$

Next Q_σ is the part of operator \tilde{Q} (51) independent of the ghost fields η_G , \mathcal{P}_G with the substitution $h \rightarrow -\sigma$. Analogously, operator K_σ is operator (50) where substitution $h \rightarrow -\sigma$ be done.

The gauge symmetry induced by nilpotency of the operator Q_σ will be reducible with the first stage of reducibility

$$\delta|\chi\rangle_n = Q_\sigma|\Lambda\rangle_n \quad gh(|\Lambda\rangle_n) = -1, \quad (55)$$

$$\delta|\Lambda\rangle_n = Q_\sigma|\Omega\rangle_n, \quad gh(|\Omega\rangle_n) = -2. \quad (56)$$

6.3 Unified description of all massive integer spin fields

It is evident, the fields with different spins $s = n$ may have different masses which we denote m_n . First of all we introduce the state vectors with definite spin and mass as follows

$$|\chi, m\rangle_{n, m_n} = |\chi\rangle_n \delta_{m, m_n}, \quad (57)$$

with $|\chi\rangle_n$ being defined in (53) and m in (57) is now a new variable of the states $|\chi, m\rangle_{n, m_n}$. Second, we introduce the mass operator M acting on the variable m so that the states $|\chi, m\rangle_{n, m_n}$ are eigenvectors of the operator M with the eigenvalues m_n

$$M|\chi, m\rangle_{n, m_n} = m_n|\chi, m\rangle_{n, m_n} = m|\chi, m\rangle_{n, m_n}. \quad (58)$$

Construction of the Lagrangian describing unified dynamics of fields with all spins is realized in terms of a single state $|\chi\rangle$ containing the fields of all spins (57)

$$|\chi\rangle = \sum_{n=0}^{\infty} |\chi, m\rangle_{n, m_n}. \quad (59)$$

This Lagrangian describing a propagation of all integer spin fields with different masses simultaneously looks like [3]

$$\mathcal{L} = \int d\eta_0 \, \langle\chi|K_\sigma Q_{\sigma M}|\chi\rangle. \quad (60)$$

Let us turn to the gauge transformations. Analogously to (57) we introduce the gauge parameters for the fields with given spin and mass

$$|\Lambda, m\rangle_{n, m_n} = |\Lambda\rangle_n \delta_{m, m_n}, \quad |\Omega, m\rangle_{n, m_n} = |\Omega\rangle_n \delta_{m, m_n} \quad (61)$$

and analogously to (59) we denote

$$|\Lambda\rangle = \sum_{n=0}^{\infty} |\Lambda, m\rangle_{n, m_n}, \quad |\Omega\rangle = \sum_{n=0}^{\infty} |\Omega, m\rangle_{n, m_n}. \quad (62)$$

Summing up (55), (56) over all n we find gauge transformation for the field $|\chi\rangle$ (59) and transformation for the gauge parameter $|\Lambda\rangle$

$$\delta|\chi\rangle = Q_{\sigma M}|\Lambda\rangle, \quad \delta|\Lambda\rangle = Q_{\sigma M}|\Omega\rangle. \quad (63)$$

7 Summary

We have developed the BRST approach to derivation of gauge invariant Lagrangians both for massless fermionic and massive bosonic higher spin fields. We investigated the (super)algebras generated by the constraints which are necessary to define these irreducible representations of the Poincare group and found that the algebras have an identical structure. In particular, the algebras contain operators which are not constraints neither in the space of bra-vectors nor in the space of ket-vectors. For the operators which are not constraints to be made harmless this method includes construction of a new representation of the algebra, after which the BRST operator can be obtained as if all the operators were the first class constraints.

The main obtained results are

- The Lagrangians for free arbitrary spin fields are constructed in terms of completely symmetric tensor(-spinor) fields (see eq. (39) for massless fermionic fields and eq. (52) for massive bosonic fields) in concise form. No off-shell constraints (including tracelessness) on the fields and the gauge parameters are used. All the equations which define an irreducible representation of the Poincare group (including tracelessness of the fields) are consequences of the Lagrangian equations of the motion and the gauge fixing.
- The models under consideration are reducible gauge theories. In the bosonic case the models have the first order of reducibility and in the fermionic case the order of reducibility grows with the value of spin.
- Lagrangian describing propagation of all massless fermionic fields simultaneously is constructed (43). Lagrangian describing propagation of all bosonic massive fields (with different masses) simultaneously is constructed (60).

There are several possibilities for extending our results. This approach can be applied to Lagrangian construction of fermionic massive fields [4], for higher spin fields in AdS space [5] and to Lagrangian construction of higher spin fields (both massive and massless) with mixed symmetry of Lorentz indices (see [9] for corresponding bosonic massless case).

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