Modified gravity alternative to dark energy

K. Koyama[a*]

[a] *Institute of Cosmology and Gravitation, University of Portsmouth*
Portsmouth PO1 2EG, UK

Abstract

We study the DGP model as a modified gravity model alternative to dark energy. It is shown that due to the modification of the gravity at large distance, a new scalar degree of freedom appears. This provides a possibility to discriminate the model from dark energy models in general relativity, but at the same time, poses a question of theoretical consistency of the model. This paper is based on Refs. [1] - [4] which are collaborations with D. Sergei, K. Izumi, R. Maartens, S. Sibiryakov and T. Tanaka

1 Introduction

The acceleration of the late-time universe, as implied by observations of Supernovae redshifts, cosmic microwave background anisotropies and the large-scale structure, poses one of the deepest theoretical problems facing cosmology. Within the framework of general relativity, the acceleration must originate from a dark energy field with effectively negative pressure, such as vacuum energy or a slow-rolling scalar field ("quintessence"). So far, none of the available models has a natural explanation. For example, in the simplest option of vacuum energy, leading to the "standard" LCDM model, the incredibly small,

$$\rho_{\Lambda,\mathrm{obs}} = \frac{\Lambda}{8\pi G} \sim H_0^2 M_P^2 \ll \rho_{\Lambda,\mathrm{theory}}\,, \tag{1}$$

and incredibly fine-tuned,

$$\Omega_\Lambda \sim \Omega_m\,, \tag{2}$$

value of the cosmological constant cannot be explained by current particle physics.

An alternative to dark energy plus general relativity is provided by models where the acceleration is due to modifications of gravity on very large scales, $r \gtrsim H_0^{-1}$. One of the simplest covariant models is based on the Dvali-Gabadadze-Porrati (DGP) brane-world model [5], in which gravity leaks off the 4-dimensional Minkowski brane into the 5-dimensional "bulk" Minkowski spacetime at large scales. The 5D action describing the DGP model is given by

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}R + \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-\gamma}\ {}^{(4)}R - \int d^4x\sqrt{-\gamma}\mathcal{L}_m. \tag{3}$$

On small scales, gravity is effectively bound to the brane and 4-dimensional Newtonian dynamics is recovered to a good approximation. The transition from 4- to 5-dimensional behaviour is governed by a crossover scale r_c; the weak-field gravitational potential behaves as

$$\Psi \sim \begin{cases} r^{-1} & \text{for} \quad r < r_c \\ r^{-2} & \text{for} \quad r > r_c \end{cases} \tag{4}$$

*e-mail: Kazuya.Koyama@port.ac.uk

The DGP model was generalized by Deffayet to a Friedman-Robertson-Walker brane in a Minkowski bulk [6]; the gravity leakage at late times initiates acceleration – not due to any negative pressure field, but due to the weakening of gravity on the brane. The energy conservation equation remains the same as in general relativity, but the Friedman equation is modified:

$$\dot{\rho} + 3H(\rho + p) = 0\,, \tag{5}$$

$$H^2 - \frac{H}{r_c} = \frac{8\pi G}{3}\rho\,. \tag{6}$$

It is important to stress that the modification to the Friedman equation is derived from a covariant 5-dimensional action and junction conditions across the brane [6].

The modified Friedman equation (6) shows that at late times in a CDM universe, with $\rho \propto a^{-3} \to 0$, we have

$$H \to H_\infty = \frac{1}{r_c}\,. \tag{7}$$

Since $H_0 > H_\infty$, in order to achieve acceleration at late times, we require $r_c \gtrsim H_0^{-1}$, and this is confirmed by fitting SN observations [7]. Like the LCDM model, the DGP model is simple, with a single parameter r_c to control the late-time acceleration although the DGP model does not provide a natural solution to the late-acceleration problem; similarly to the LCDM model, where Λ must be fine-tuned, the DGP parameter r_c must be fine-tuned to match observation.

The most interesting aspect of the DGP model is that there is a possibility to distinguish the model from dark energy models in general relativity. This is because the recovery of the general relativity is very subtle. Although the weak-field gravitational potential behaves as 4D on scales smaller than r_c, the linearized gravity is not described by general relativity. This is because the modification of the gravity introduces a new scalar degrees of freedom. Due to this scalar degrees of freedom, the linearized gravity is described by Brans-Dicke gravity. However, this scalar mode becomes non-linear on larger scales than expected. Let us consider the static source with mass M. Gravity becomes non-linear near the Schwarzshild radius $r_g = 2GM$. However, the scalar mode becomes non-linear at $r_* = (r_g r_c^2)^{1/3}$ which is much larger than r_g if $r_c \sim H_0^{-1}$. In fact, for the Sun r_* is much larger than the size of the solar system. A remarkable finding is the once the brane bending becomes non-linear, general relativity is recovered. This non-linear shielding of the scalar mode is crucial to escape from the tight solar system constraints Fig. 1 summarize the behaviour of gravity in the DGP model.

This complicated behaviour of gravity gives us an opportunity to distinguish the DGP model from dark energy models in general relativity. On largest scales, gravity becomes 5D. Then the large scale Integrated Sachs Wolfe (ISW) effect is sensitive to the 5D gravity. Even on small scales compared with r_c, the scalar mode of gravity gives a distinct features to large scale structure. If perturbations become non-linear, the theory approaches to general relativity. This transition from linear theory described by Brans-Dicke to non-linear physics described by general relativity can be probed by gravitational lensing. Fig. 1 also shows possible cosmological probes of the DGP gravity.

Due to this possibility of discrimination from general relativity models, the DGP model is a very popular model for modified gravity alternative to dark energy. However, from a theoretical point of view, these interesting features could be, at the same time, signatures of inconsistencies of the model. For example, the non-linear interaction of the scalar mode becomes important on very large scales. If we consider a Planck mass particle, this scale becomes $r_* = (\ell_{pl} r_c^2)^{1/3} \sim 1000$km where ℓ_{pl} is a Planck length and $r_c \sim H_0^{-1}$. This implies that quantum corrections cannot be neglected below 1000km. We will come back to this problem in conclusion. Even if we focus on the linearized behaviour of the scale mode, there appears a problem of a ghost instability. In the case where the brane is described by de Sitter spacetime, it is proved that the scalar mode becomes a ghost.

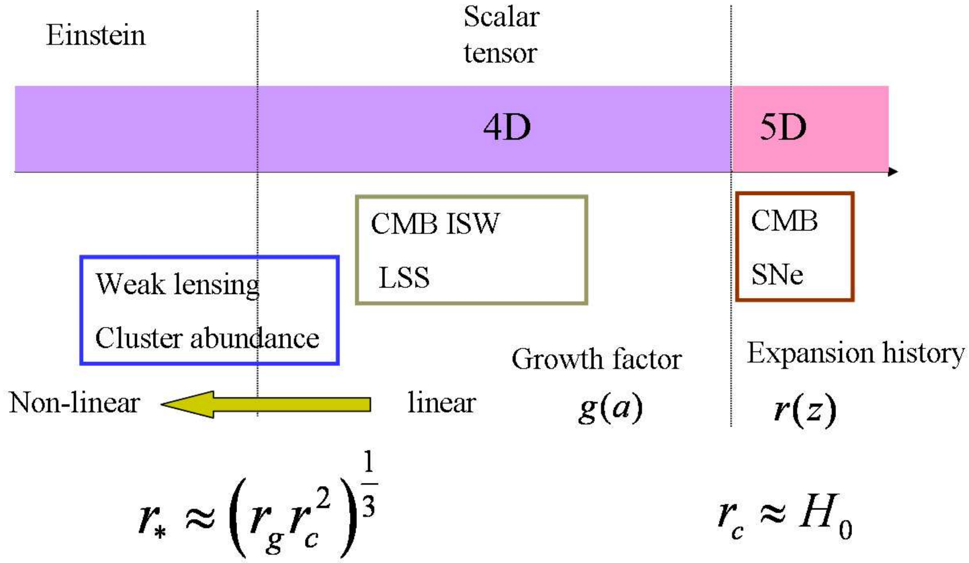

Figure 1: Summary of the behaviour of gravity in the DGP model.

In this paper, we focus on the behavior of linear cosmological perturbations in cosmological background. In section 2, we discuss the solutions for linearized metric perturbations under horizon and explain a possibility to discriminate the DGP model from general relativity models of dark energy. Then we point out that the behaviour of the metric perturbations show a signature of the inconsistency of the theory, namely the existence of the ghost-like excitation. In section 3, we confirm the existence of the ghost in de Sitter background. It is argued that the ghost is associated with the difficulty of massive gravity theory in de Sitter background. In section 4, the scalar mode is identified as a brane bending mode and the effective theory of this mode is discussed. Section 5 is devoted to conclusion.

2 Linear growth rate

In Ref. [3], solutions for metric perturbations under horizon are obtained by consistently solving 5D perturbations. Scalar metric perturbations are given in longitudinal gauge by

$$ds^2 = -(1+2\Psi)dt^2 + a^2(1+2\Phi)d\vec{x}^2\,, \tag{8}$$

and the perturbed energy-momentum tensor for matter is given by

$$\delta T^\mu_\nu = \begin{pmatrix} -\delta\rho & a\delta q_{,i} \\ -a^{-1}\delta q^{,i} & \delta p\, \delta^i{}_j \end{pmatrix}. \tag{9}$$

The solutions for the brane metric perturbations are

$$\frac{k^2}{a^2}\Phi = 4\pi G\left(1-\frac{1}{3\beta}\right)\rho\triangle, \tag{10}$$

$$\frac{k^2}{a^2}\Psi = -4\pi G\left(1+\frac{1}{3\beta}\right)\rho\triangle, \tag{11}$$

where

$$\beta = 1 - 2r_c H\left(1+\frac{\dot{H}}{3H^2}\right), \tag{12}$$

and

$$\Delta = \delta\rho - 3H\delta q. \tag{13}$$

This agrees with the results obtained by Lue, Scoccimarro and Starkman [11]. They find spherical symmetric solutions by closing the 4D equations using an anzatz for the metric and checked in retrospect that the obtained solutions satisfy the regularity in the bulk. Here we have shown that the solutions (10) and (11) are uniquely determined by the regularity condition in the bulk within our approximations.

The modified Poisson equation (10) shows the suppression of growth. The rate of growth is determined by Δ, and for CDM,

$$\ddot{\triangle} + 2H\dot{\triangle} = -\frac{k^2}{a^2}\Psi\,. \tag{14}$$

which leads to

$$\ddot{\triangle} + 2H\dot{\triangle} = 4\pi G\left(1 + \frac{1}{3\beta}\right)\rho\,\triangle\ . \tag{15}$$

Thus the growth rate receives an additional modification from the time variation of Newton's constant through β.

In Fig. 2, we show the linear growth factor Δ/a for the DGP model, and compare it with LCDM and with the general relativity dark energy model whose background evolution matches that of the DGP model. We also showed the incorrect DGP result in which the inconsistent assumption is effectively adopted, which has been frequently adopted in literatures. The correct equations for subhorizon density perturbations are crucial for meaningful tests of DGP predictions against structure formation observations. And such tests are essential for breaking the degeneracy with LCDM that arises with SN redshift observations [10, 12]. The distance-based SN observations draw only upon the background 4D Friedman equation (6) in DGP models, and therefore there are quintessence models in general relativity that can produce precisely the same SN redshifts as DGP. By contrast, structure formation observations require the 5D perturbations in DGP, and one cannot find equivalent general relativity models.

While the linear grwoth rate provides us the opportunity to distinguish the DGP model from general relativity models, it also shows a signature of the inconsistency of the model. The modification of metric perturbations can be described by a linearized scalar-tensor gravity with Brans-Dicke parameter [11]

$$\omega = \frac{3}{2}(\beta - 1), \tag{16}$$

where the gravitational scalar corresponds to the bending of the brane. If we take $r_c \sim H_0$, ω is always smaller than $-3/2$. In BD theory, if $\omega < -3/2$, the BD scalar becomes a ghost. This implies that the scalar mode behaves as a ghost. In fact, the growth rate shows the suppression of the gravitational collapse. It is understood as the effect of the ghost that mediates the repulsive force. In the next section, we prove the existence of the ghost in a more rigorous way restricting spacetime on a brane to de Sitter spacetime. Note that the condition for the ghost in de Sitter spacetime reduced from $\omega < -3/2$ becomes

$$Hr_c > 1/2, \tag{17}$$

where $Hr_c > 1$ for the positive tension brane and $Hr_c = 1$ for self-accelerating universe.

3 Ghost in de Sitter spacetime

The 5D solution for the metric with the 4D de Sitter brane can be obtained as

$$ds^2 = dy^2 + N(y)^2\gamma_{\mu\nu}dx^\mu dx^\nu, \quad N(y) = 1 + Hy, \tag{18}$$

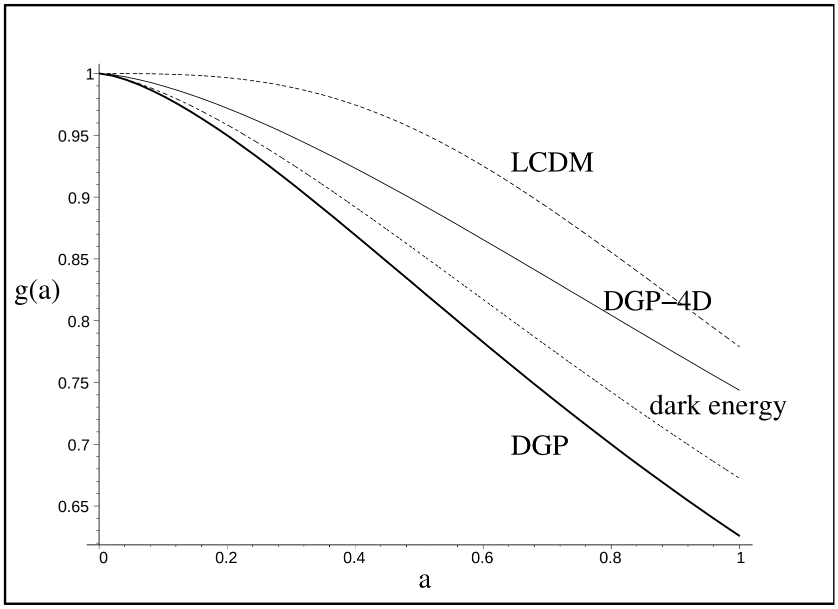

Figure 2: The growth history $g(a) = \triangle(a)/a$ is shown for LCDM (long dashed) and DGP (solid, thick). The growth history for a dark energy model (short dashed) with the identical expansion histories with DGP. Due to the time variation of Newton's constant through β in Eq. (15), the growth factor $g(a)$ receives an additional suppression compared with the dark energy model. DGP-4D (solid, thin) shows the incorrect result in which the inconsistent assumption is adopted [10]. We set the desity parameter for matter today as $\Omega_{m0} = 0.3$.

where $\gamma_{\mu\nu}$ is the metric for the de Sitter spacetime and the brane is located at $y = 0$. There is a solution for the de Sitter spacetime without σ,

$$H = \frac{1}{r_c}. \tag{19}$$

We call this solution the self-accelerating universe.

Let us investigate the perturbations $N(y)^2\gamma_{\mu\nu} + h_{\mu\nu}$ about the background de Sitter spacetime. In the following, we assume $Hr_c \neq 1$ and treat the case $Hr_c = 1$ separately. In addition to the gravitational perturbations $h_{\mu\nu}$, we must take into account a perturbation of the position of the brane $y = \varphi(x)$ [14]. Using the transverse-traceless gauge $\nabla^\mu h_{\mu\nu} = h = 0$, the perturbed junction condition is given by

$$\begin{aligned} k_{\mu\nu} &- \mathcal{H}h_{\mu\nu} - r_c\left[X_{\mu\nu}(h) - \kappa_4^2\left(T_{\mu\nu} - \frac{1}{3}\gamma_{\mu\nu}T\right)\right] \\ &= -(1-2Hr_c)\left(\nabla_\mu\nabla_\nu + H^2\gamma_{\mu\nu}\right)\varphi, \end{aligned} \tag{20}$$

where $k_{\mu\nu} = (1/2)\partial_y h_{\mu\nu}$ on the brane and $X_{\mu\nu}$ is given by

$$\begin{aligned} X_{\mu\nu} &= \delta^{(4)}G_{\mu\nu} + 3H^2 h_{\mu\nu} \\ &= -\frac{1}{2}\left(\Box_4 h_{\mu\nu} - \nabla_\mu\nabla_\alpha h^\alpha_\nu - \nabla_\nu\nabla_\alpha h^\alpha_\mu + \nabla_\mu\nabla_\nu h\right) \\ &- \frac{1}{2}\gamma_{\mu\nu}(\nabla_\alpha\nabla_\beta h^{\alpha\beta} - \Box_4 h) + H^2\left(h_{\mu\nu} + \frac{1}{2}\gamma_{\mu\nu}h\right). \end{aligned} \tag{21}$$

The equation of motion for φ is obtained from the traceless condition $h = 0$;

$$(1-2Hr_c)(\Box_4 + 4H^2)\varphi = \frac{\kappa^2 T}{6}.$$

Let us find solutions for the vacuum brane $T_{\mu\nu} = 0$. Using the separation of variables $h_{\mu\nu} = \int dm\ e_{\mu\nu}(x) F_m(y)$, the equation of motion in the bulk is written as

$$F_m'' + \frac{1}{N^2}(m^2 - 2H^2)F_m = 0, \tag{22}$$

where prime denotes a derivative with respect to y. There are two types of solutions. One type of solution is an inhomogeneous solution sourced by the scalar mode φ. We call this solution the spin-0 perturbation. The other solution is a homogeneous solution with $\varphi = 0$, which is called the spin-2 perturbation. The spin-2 perturbations $\chi_{\mu\nu}$ satisfy the junction condition without φ

$$\chi_{\mu\nu}' - 2\mathcal{H}\chi_{\mu\nu} = -m^2 r_c \chi_{\mu\nu}. \tag{23}$$

We find a tower of continuous Kaluza-Klein (KK) modes starting from $m^2 = (9/4)H^2$ as well as a normalizable discrete mode

$$\frac{m_d^2}{H^2} = \frac{1}{(Hr_c)^2}(3Hr_c - 1), \tag{24}$$

for $Hr_c > 2/3$ [13]. For $Hr_c > 1$, the mass is in the range $0 < m_d^2 \leq 2H^2$ where $m_d^2 = 2H^2$ for the self-accelerating universe $Hr_c = 1$ and $m_d^2 \to 0$ for $Hr_c \to \infty$.

For the spin-0 perturbations, there is a normalizable solution given by

$$h_{\mu\nu} = \frac{1 - 2Hr_c}{H(1 - Hr_c)}(\nabla_\mu \nabla_\nu + H^2 \gamma_{\mu\nu})\varphi. \tag{25}$$

This is a solution with $m^2 = 2H^2$.

We can construct the 2nd order action for $h_{\mu\nu}$ and φ from the 5D action. The result is given by

$$\delta_2 S = -\frac{1}{4\kappa^2}\int d^5x \sqrt{-g} N^{-4} h^{\mu\nu} \delta^{(5)} G_{\mu\nu} + \frac{1}{\kappa^2}\int d^4x \sqrt{-\gamma}\mathcal{L}_B, \tag{26}$$

where $\delta^{(5)} G_{\mu\nu}$ is the 5D perturbed Einstein tensor and

$$\begin{aligned} \mathcal{L}_B &= k^{\mu\nu} h_{\mu\nu} - kh + \frac{1}{2}\mathcal{H}(h^2 - h^{\mu\nu} h_{\mu\nu}) \\ &+ (1 - 2\mathcal{H}r_c)\left(h_{\mu\nu}\nabla^\mu \nabla^\nu \varphi - h\nabla^\rho \nabla_\rho \varphi - 3H^2 h\varphi\right) \\ &- 3\mathcal{H}\left(-(1 - 2\mathcal{H}r_c)\varphi(\Box_4 + 4H^2)\varphi + \frac{\kappa^2}{3}T\varphi\right) \\ &+ \frac{1}{2}\kappa^2 h^{\mu\nu} T_{\mu\nu} - \frac{r_c}{2} h^{\mu\nu} X_{\mu\nu}(h). \end{aligned} \tag{27}$$

This action gives the correct equation of motion and the junction condition for $h_{\mu\nu}$ and the equation of motion for φ.

We can derive an effective action for the brane fluctuation φ by substituting the 5D solution for $h_{\mu\nu}$ given by φ (25) into the 5D action and get the off-shell action for φ by integrating out only with respect to the extra coordinate y [16]. This yields the action for φ as

$$S_\varphi = \frac{3H}{2\kappa^2}\left(\frac{1 - 2Hr_c}{1 - Hr_c}\right)\int d^4x \sqrt{-\gamma}\varphi(\Box_4 + 4H^2)\varphi. \tag{28}$$

The 4D effective action for the spin-2 perturbations is also obtained in a similar way. For the discrete mode with m_d^2, we get

$$S_\chi = \frac{r_c(3Hr_c - 1)}{4\kappa^2(3Hr_c - 2)}\int d^4x \sqrt{-\gamma}\chi^{\mu\nu}(\Box_4 - 2H^2 - m_d^2)\chi_{\mu\nu}, \tag{29}$$

where transverse-traceless gauge fixing conditions $\nabla^\mu \chi_{\mu\nu} = \chi^\mu_\nu = 0$ are imposed . This is exactly the same action for the spin-2 perturbations in the 4D massive gravity theory where the Pauli-Fierz (PF) mass term is added to the Einstein-Hilbert action by hand [17]

$$S_M = -\frac{M^2}{8\kappa_4^2} \int d^4x \sqrt{-\gamma} (h^{\mu\nu} h_{\mu\nu} - h^2). \tag{30}$$

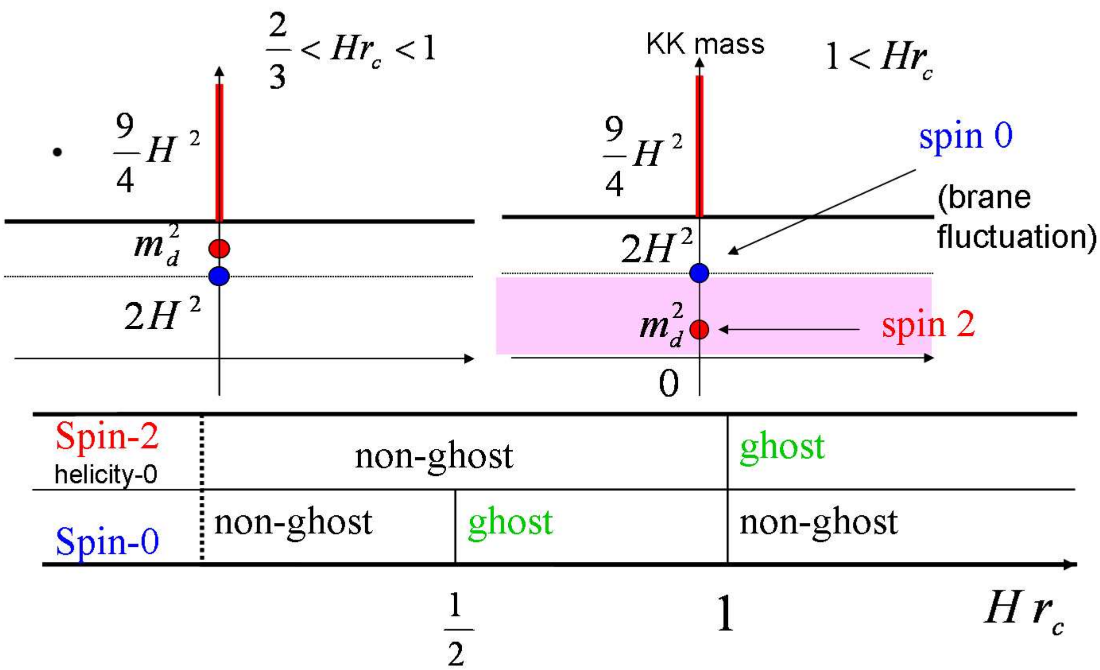

Figure 3: Summary of the mass spectrum of the scalar perturbations in + branch. Spin-2 perturbation has continuous modes with $m^2 \geq (9/4)H^2$ and a discrete mode with $m^2 = m_d^2$ while spin-0 perturbation has $m^2 = 2H^2$. In the limit $Hr_c \to 1$, both the helicity-0 excitation of spin-2 perturbation and the spin-0 perturbation have mass $m^2 = 2H^2$ and there is a resonance.

Ref. [1] studied the existence of the ghost based on the above effective action. Let us focus on + branch. The result is summarized in Fig 3. For $Hr_c > 1$, the discrete mode of spin-2 perturbations has mass given by $0 < m^2 < 2H^2$. It is well known that the spin-2 perturbations contain a helicity-0 excitation that is a ghost in this mass region. For $Hr_c < 1$, the spin-2 perturbations become healthy but the spin-0 perturbations becomes a ghost.

On a self-accelerating universe $Hr_c = 1$, the situation is complicated because the spin-0 perturbation and the discrete mode of the spin-2 perturbations has the same mass and there occur a mixing. Ref KK2 studied this case very carefully. The solution for the spin-2 perturbations become

$$h_{\mu\nu} = A_{\mu\nu}(x) + \frac{1}{H}(\nabla_\mu \nabla_\nu + H^2 \gamma_{\mu\nu})\varphi(x) \log(1 + Hy) \ . \tag{31}$$

where

$$\Box A_{\mu\nu} - 4H^2 A_{\mu\nu} = H \left(\nabla_\mu \nabla_\nu + H^2 \gamma_{\mu\nu} \right) \varphi \ . \tag{32}$$

Substituting this solution to the 5D action, the effective action is obtained as

$$\begin{aligned} S_{eff} = \frac{1}{\varkappa^2 H} \int d^4x \sqrt{-\gamma} \Big\{ & - A^{\mu\nu} X_{\mu\nu}(A) - H^2 A^{\mu\nu} A_{\mu\nu} + H^2 A^2 \\ & - H (A^{\mu\nu} \nabla_\mu \nabla_\nu \varphi - A \Box \varphi - 3H^2 A \varphi) - \frac{9H^2}{4} \varphi (\Box + 4H^2) \varphi \Big\} \ , \end{aligned} \tag{33}$$

where we introduced the notation $A \equiv A^\mu_\mu$.

In order to study the existence of the ghost, it is needed to write the effective action only in terms of phyical degrees of freedom. Ref. [3] performs the Hamiltonian analysis to derive the reduced Hamiltonian written only in terms of physial degrees of freedom. There are two dynamical degrees of freedom, namely the spin-0 mode and the helicity-0 excitation of spin-2 perturbations. It is found that, in general, the Hamiltonian cannot be diagonalized. However, on small scales under horizon, it is possible to diagonalize the Hamiltonian and we find a ghost from a mixing between a spin-0 perturbation and a helicity-0 excitation of spin-2 perturbations.

4 Spin-2 and spin-0 connection -amplitude analysis

The existence of the ghost is summarized in Fig 3. An interesting issue is that the spin-0 ghost appears as soon as the spin-2 ghost disappears. This comes from the fact that at $m^2 = 2H^2$, the spin-0 and spin-2 perturbations degenerate as is shown in Ref. [4]. In fact we can make the spin-2 perturbations from a scalar

$$h_{\mu\nu} = (\nabla_\mu \nabla_\nu - H^2 \gamma_{\mu\nu}) X, \tag{34}$$

if X satisfies

$$(\Box + 4H^2) X = 0. \tag{35}$$

This is a scalar mode with mass squared $-4H^2$. At the same time $h_{\mu\nu}$ is a transverse-tarceless perturbations and from the identity,

$$(\Box - 4H^2) \left(\nabla_\mu \nabla_\nu - H^2 \gamma_{\mu\nu}\right) X = \left(\nabla_\mu \nabla_\nu - H^2 \gamma_{\mu\nu}\right) (\Box + 4H^2) X = 0,$$

it is identified to have a mass given by $m^2 = 2H^2$. In fact, the massive gravity with $m^2 = 2H^2$ has an 'symmetry' where Eq. (34) is a gauge mode. Then this symmetry can be used to eliminate the helicity-0 mode and hence a ghost. However, in the self-accelerating universe, there is a spin-0 perturbations which breaks this symmetry. In this section, we show that this existence of the spin-0 mode is inevitable to have a consistent theory with matter perturbations with $T \neq 0$.

In order to see this fact, it is convenient to calculate an amplitude $\mathcal{A} = (1/2) h_{\mu\nu} T^{\mu\nu}$. In the self-accelerating branch we have massive spin-2 perturbations (one discrete mode and an infinite tower of massive modes) and the spin-0 perturbations. Then the amplitude can be separated as

$$\mathcal{A} \equiv \frac{1}{2} h_{\mu\nu}(0) T^{\mu\nu} = \mathcal{A}_{s=2} + \mathcal{A}_{s=0}. \tag{36}$$

Using the result for the spectrum, the spin-2 contribution is calculated as

$$\begin{aligned} \mathcal{A}_{s=2} &= -\kappa \sum_i u_i(0)^2 \Big[T_{\mu\nu} (\Box^{(4)} - 2H^2 - m_i^2)^{-1} T^{\mu\nu} \\ &\quad - \frac{1}{3} \frac{m_i^2 - 3H^2}{m_i^2 - 2H^2} T (\Box^{(4)} + 6H^2 - m_i^2)^{-1} T \Big], \end{aligned} \tag{37}$$

where the solution for the mode functions $u_i(0)^2$ are given by

$$\begin{aligned} u_d^2(0) &= \frac{1}{2r_c} \frac{3Hr_c - 2}{3Hr_c - 1}, \quad m_d^2 = \frac{3Hr_c - 1}{r_c^2}, \\ u_m^2(0) &= \frac{H}{\pi} \frac{k^2}{\left(\frac{m^2 r_c}{H} - \frac{3}{2}\right)^2 + k^2}, \quad m^2 > \frac{9H^2}{4}, \end{aligned} \tag{38}$$

and

$$k=\sqrt{\frac{m^2}{H^2}-\frac{9}{4}}. \tag{39}$$

Here we used the fact that each massive state is described by Pauli-Fierz massive gravity.

The effective action for the spin-0 was calculated as

$$S_{s=0}=\frac{3H}{2\kappa}\left(\frac{1-2Hr_c}{1-Hr_c}\right)\int d^4x\sqrt{-\gamma}\xi^y(\Box^{(4)}+4H^2)\xi^y. \tag{40}$$

From the equation of motion for ξ^y

$$\xi^y=\frac{1}{1-2Hr_c}(\Box^{(4)}+4H^2)^{-1}\frac{\kappa T}{6}, \tag{41}$$

the effective interaction between ξ^y and T is given by

$$\mathcal{A}_{s=0}=-\frac{1}{2}H(1-Hr_c)^{-1}\xi^y T. \tag{42}$$

Then we get the spin-0 contribution

$$\mathcal{A}_{s=0}=-\frac{\kappa}{12}\frac{H}{(1-Hr_c)(1-2Hr_c)}T(\Box^{(4)}+4H^2)^{-1}T. \tag{43}$$

The amplitude $\mathcal{A}_{s=0}$ and $\mathcal{A}_{s=2}$ look singular at $Hr_c=1$ where $m_d^2=2H^2$. However, we show that the total amplitude is finite at this point and the origin of the singularity comes from the degeneracy between the spin-2 and spin-0 at $m^2=2H^2$, that is, the separation between the spin-2 and spin-0 becomes singular.

In order to see this fact, we rewrite the amplitude as follows. Using the solutions for $u_i(0)^2$, we can show the identity

$$\begin{aligned}\frac{\kappa}{3}\sum_i\frac{H^2u_i(0)^2}{m_i^2-2} &= \frac{H^2u_d^2}{m_d^2-2}+\frac{H}{\pi(Hr_c)^2}\int_0^\infty\frac{k^2}{\left(k^2-\frac{9}{4}\right)\left(k^2+\frac{1}{4}\right)\left(k^2+\frac{9}{4}-\frac{m_d^2}{H^2}\right)} \\ &= \frac{\kappa}{12}\frac{H}{1-Hr_c}.\end{aligned} \tag{44}$$

Then we can rearrange the amplitude as

$$\begin{aligned}\mathcal{A} &= -\kappa\sum_i u_i(0)^2\left[T_{\mu\nu}\frac{1}{\Box^{(4)}-2H^2-m_i^2}T^{\mu\nu}-\frac{1}{3}T\frac{1}{\Box^{(4)}+6H^2-m_i^2}T\right] \\ &\quad -\frac{\kappa}{3}\sum_i u_i(0)^2\frac{1}{m_i^2-2H^2}T\left(\frac{1}{\Box^{(4)}+6H^2-m_i^2}-\frac{1}{\Box^{(4)}+4H^2}\right)T \\ &\quad -\frac{\kappa}{6}\frac{H}{1-2Hr_c}T\frac{1}{\Box^{(4)}+4H^2}T.\end{aligned} \tag{45}$$

The second line shows the connection between spin-2 and spin-0 around $m_i^2=2H^2$. It seems that $m_i^2=2H^2$ is a singular point but this is not true. In fact the second line can be written as

$$\begin{aligned}\mathcal{A}_{0-2} &\equiv -\frac{\kappa}{3}\sum_i u_i(0)^2\frac{1}{m_i^2-2H^2}T\left(\frac{1}{\Box^{(4)}+6H^2-m_i^2}-\frac{1}{\Box^{(4)}+4H^2}\right)T, \\ &= -\frac{\kappa}{3}\sum_i u_i(0)^2T\frac{1}{(\Box^{(4)}+6H^2-m_i^2)(\Box^{(4)}+4H^2)}T.\end{aligned} \tag{46}$$

Therefore there is no divergence at $m_i^2=2H^2$. The singularity comes from the fact that the spin-0 and the spin-2 are degenerate and we cannot separate the two at $m_i^2=2H^2$.

Around $m_i^2 = 2H^2$, the second term determines the existence of the ghost. It is clear that it is impossible to avoid the spin-2 ghost and spin-0 ghost at the same time because the spin-2 interaction and the spin-0 interaction act in an opposite way. For $m_i^2 < 2H^2$, the spin-2 pole gives the ghost. In this case, the spin-0 pole behaves normal. For $m_i^2 > 2H^2$, the spin-2 becomes normal but the spin-0 pole gives the ghost. This connection between the spin-0 and spin-2 is an essential difference between the DGP and 4D massive gravity theory. Unfortunately, this fact makes it very difficult to remove the ghost - once we remove the spin-2 ghost, the spin-0 ghost appears!

5 Effective theory on small scales

On small scales, the degeneracy between spin-2 and spin-0 become unimportant and the theory becomes quite simplified. Let us consider the limit

$$\Box^{(4)} \gg H^2, m_i^2. \tag{47}$$

Then the amplitude is approximated as

$$\mathcal{A} = -\kappa \sum_i u_i(0)^2 \left[T_{\mu\nu} \frac{1}{\Box^{(4)}} T^{\mu\nu} - \frac{1}{3} T \frac{1}{\Box^{(4)}} T \right] - \frac{\kappa}{6} \frac{H}{1 - 2Hr_c} T \frac{1}{\Box^{(4)}} T. \tag{48}$$

Here we neglected $\mathcal{A}_{0-2}$ using the assumption (47) because

$$\mathcal{A}_{0-2} \to -\frac{\kappa}{3} \sum_i u_i(0)^2 T \frac{1}{(\Box^{(4)})^2} T, \tag{49}$$

hence this part does not give any contributions within the approximations (47). Then on small scales it is possible to diagonalise spin-0 and spin-2.

We can see $Hr_c = 1$ is not a special point anymore. This is the reason why the boundary effective action approach, which uses the small scales limit does not see any pathology at $Hr_c = 1$.

Now using the solutions for the mode functions we get

$$\begin{aligned} \sum_i u_i(0)^2 &= u_d(0)^2 + \sum u_m(0)^2 \\ &= \frac{1}{2r_c} \frac{3Hr_c - 2}{3Hr_c - 1} + \frac{H}{\pi (Hr_c)^2} \int_0^\infty dk \frac{k^2}{\left(k^2 + \frac{9}{4}\right)\left(k^2 + \frac{9}{4} - m_d^2\right)} \\ &= \frac{1}{2r_c}. \end{aligned} \tag{50}$$

This is an expected result. The effective gravitational coupling is read as

$$\kappa \sum_i u_i(0)^2 = \kappa_4, \tag{51}$$

so we see that the 4D gravity is recovered by the summation of massive states. Then the amplitude is calculated as

$$\mathcal{A} = -\kappa_4 \left[T_{\mu\nu} \frac{1}{\Box^{(4)}} T^{\mu\nu} - \frac{1}{3} T \frac{1}{\Box^{(4)}} T \right] - \frac{\kappa_4}{3} \frac{Hr_c}{1 - 2Hr_c} T \frac{1}{\Box^{(4)}} T. \tag{52}$$

The first terms is exactly the same as the amplitude in Minkowski brane in the normal branch. Due to the scalar polarisation, the coefficient in front of $T \Box^{(4)\,-1} T$ is 1/3 not 1/2. The last term represents the effect of the curvature of the brane. Then finally we get

$$\mathcal{A} = -\kappa_4 \left[T_{\mu\nu} \frac{1}{\Box^{(4)}} T^{\mu\nu} - \frac{1}{3} \frac{1 - 3Hr_c}{1 - 2Hr_c} T \frac{1}{\Box^{(4)}} T \right]. \tag{53}$$

This result can be compared with the 4D Brans-Dicke (BD) theory. In the BD theory with BD parameter ω, the amplitude is given by

$$\mathcal{A} = -\kappa_4 \left[T_{\mu\nu} \frac{1}{\Box^{(4)}} T^{\mu\nu} - \frac{1}{3} \frac{1+\omega}{1+\frac{2\omega}{3}} T \frac{1}{\Box^{(4)}} T \right]. \tag{54}$$

Then the BD parameter is given by

$$\omega = -3Hr_c. \tag{55}$$

It is known that the BD theory contains a ghost if

$$\omega < -\frac{3}{2}. \tag{56}$$

This means that there is a ghost if $Hr_c > 1/2$ which agrees with the spectrum analysis.

The origin of the ghost is the last term in (52). In order to identify this origin, let us derive the amplitude in a different way. The amplitude is also written as

$$\mathcal{A} = \frac{1}{2} h_{\mu\nu}(0) T^{\mu\nu} = \frac{1}{2} h^{(TT)}_{\mu\nu} T^{\mu\nu} - H \xi^y T. \tag{57}$$

Here the brane bending mode satisfies

$$(\Box^{(4)} + 4H^2)\xi^y = \frac{1}{1-2Hr_c} \frac{\kappa}{6} T. \tag{58}$$

The junction condition for the transverse-traceless modes is given by

$$(\partial_y - 2H)\, h^{(TT)}_{\mu\nu} = -\kappa \Sigma_{\mu\nu} - r_c(\Box^{(4)} - 2H^2) h^{(TT)}_{\mu\nu}, \tag{59}$$

where

$$\Sigma_{\mu\nu} = T_{\mu\nu} - \frac{1}{3} H^2 \gamma_{\mu\nu} T + \frac{1}{3} (\nabla_\mu \nabla_\nu + H^2 \gamma_{\mu\nu})(\Box^{(4)} + 4H^2)^{-1} T. \tag{60}$$

The solution for transverse-traceless perturbations can be obtained by the Green's function method as

$$h^{(TT)}_{\mu\nu} = -2\kappa \sum_i \frac{u_i(0)^2}{\Box^{(4)} - 2H^2 - m_i^2} \Sigma_{\mu\nu}. \tag{61}$$

Then $\mathcal{A}$ is written as

$$\begin{aligned} \mathcal{A} \;=\; & -\kappa \sum_i u_i(0)^2 \Bigg[T_{\mu\nu} \frac{1}{\Box^{(4)} - 2H^2 - m_i^2} T^{\mu\nu} - \frac{1}{3} T \frac{1}{\Box^{(4)} + 6H^2 - m_i^2} T \\ & + \frac{1}{3} H^2 T \frac{1}{\Box^{(4)} + 6H^2 - m_i^2} \frac{1}{\Box^{(4)} + 4H^2} T \Bigg] \\ & - \frac{H}{1-2Hr_c} \frac{\kappa}{6} T \frac{1}{\Box^{(4)} + 4H^2} T. \end{aligned} \tag{62}$$

It is easy to see that this is equivalent to (45) using (46). Then we see that the last term in (52) is noting but the contribution from ξ^y

This indicates that on small scales we only need to look at the contribution of the brane bending mode and we do not need to solve the 5D perturbations explicitly. This is the method adopted in Ref. NR Let us again consider the limit

$$\Box^{(4)} \gg r_c^{-2}. \tag{63}$$

Using this approximations, we can only keep the 4D terms in the 5D second order action. Then the 4D boundary effective action is obtained as

$$
\begin{aligned}
S_B &= \frac{1}{\kappa} \int d^4x \sqrt{-\gamma} \Big[(1-2\mathcal{H}r_c) \left(h_{\mu\nu} \nabla^\mu \nabla^\nu \xi^y - h \nabla^\rho \nabla_\rho \xi^y - 3H^2 h \xi^y \right) \\
&\quad -3\mathcal{H} \left(-(1-2\mathcal{H}r_c) \xi^y (\Box_4 + 4H^2) \xi^y + \frac{\kappa^2}{3} T \xi^y \right) \\
&\quad + \frac{1}{2} \kappa^2 h^{\mu\nu} T_{\mu\nu} - \frac{r_c}{2} h^{\mu\nu} X_{\mu\nu}(h) \Big], \qquad (64)
\end{aligned}
$$

where $\mathcal{H} = H$ in the self-accelerating branch. As this is 4D theory, it is easy to calculate the amplitude. We get

$$
\begin{aligned}
\mathcal{A} &= -\kappa_4 \Big[T_{\mu\nu} \frac{1}{\Box^{(4)}} T^{\mu\nu} - \frac{1}{3} T \frac{1}{\Box^{(4)}} T + \frac{1}{3} H^2 T \frac{1}{(\Box^{(4)})^2} T \Big] \\
&\quad - \frac{H}{1-2Hr_c} \frac{\kappa}{6} T \frac{1}{\Box^{(4)}} T \\
&= -\kappa_4 \Big[T_{\mu\nu} \frac{1}{\Box^{(4)}} T^{\mu\nu} - \frac{1}{3} \frac{1-3Hr_c}{1-2Hr_c} T \frac{1}{\Box^{(4)}} T \Big]. \qquad (65)
\end{aligned}
$$

This is consistent with the result obtained in the previous section.

We can diagonalise the action by defining

$$
h_{\mu\nu} = \chi_{\mu\nu} - r_c^{-1} (1-2Hr_c) \gamma_{\mu\nu} \xi^y. \qquad (66)
$$

The resultant action is

$$
\begin{aligned}
S_B &= \frac{1}{2\kappa_4} \int d^4x \sqrt{-\gamma} \Big[-\chi_{\mu\nu} X^{\mu\nu}(\chi) + \kappa_4 \chi_{\mu\nu} T^{\mu\nu} \\
&\quad + \frac{3}{r_c} (1-2Hr_c) \xi^y (\Box^{(4)} + 4H^2) \xi^y - \frac{\kappa_4}{r_c} \xi^y T \Big]. \qquad (67)
\end{aligned}
$$

It is clear that ξ^y becomes a ghost for $Hr_c > 1/2$.

6 Conclusion

In this paper, we studied the DGP model as an alternative to dark energy models. We concentrated on the self-accelerating universe where the accelerating universe is realized without introducing a cosmological constant. The cross over scale is fine-tuned $r_c = H_0^{-1}$. Due to a new scalar mode introduced by the modification of gravity, the linearized gravity is described by the Brans-Dicke (BD) gravity under horizon. This gives a possibility to distinguish the model from dark energy models in general relativity. However, the BD parameter is shown to be smaller than $-3/2$, which indicates the existence of ghost like excitations. We confirm the existence of the ghost in de Sitter in several ways.

We come back to the strong coupling problem. As we explained in the introduction, we need to take into account the non-linearity of the brane bending mode at $r_* = (r_c^2 r_g)^{1/3}$. This scale would be different in de Sitter spacetime. The effective action including the non-linear interaction of ξ^y looks like

$$
S = \frac{1}{2\kappa_4} \int d^4x \sqrt{-\gamma} \left[\frac{3}{r_c} (1-2Hr_c) \xi^y (\Box^{(4)} + 4H^2) \xi^y + (\partial \xi^y)^2 \Box^{(4)} \xi^y \right]. \qquad (68)
$$

in de Sitter spacetime where the non-linear term is the same as Minkowski case [9]. The point is that the quadratic part of the action is modified in the de Sitter spacetime, the non-linear interaction is insensitive to the curvature of the background spacetime. Then the strong coupling scale becomes

$$r_* = \left(r_c^2(1-2Hr_c)^{-2}r_g\right)^{1/3}. \tag{69}$$

Then for $Hr_c \gg 1$, r_* becomes small. This confirms the suggestion made by Ref. [9] where the kinetic term for the brane bending receives a correction from the extrinsic curvature of the brane. This opens up a possibility to avoid the strong coupling problem.

The DGP model provided us the first concrete model for the modified gravity alternative to dark energy models where we can study the behaviour of gravity from a single covariant action. It is revealed that the model can be distinguished from dark energy models in general relativity from future observations. However, it also reveals the difficulties to construct the consistent theory once we modified general relativity on large scales. More efforts have to be made to develop consistent models and test again ever improving cosmological observations.

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