Anomalous dimensions of twist-2 operators and Pomeron in the $\mathcal{N}=4$ supersymmetric gauge theory

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Abstract

We present results for the universal anomalous dimension $\gamma_{uni}(j)$ of Wilson twist-2 operators in the $\mathcal{N}=4$ Supersymmetric Yang-Mills theory in the first three orders of perturbation theory. These expressions are obtained by extracting the most complicated contributions from the corresponding anomalous dimensions in QCD. This result is in an agreement with the hypothesis of the integrability of $\mathcal{N}=4$ Supersymmetric Yang-Mills theory in the context of AdS/CFT-correspondence.

1 Introduction

The anomalous dimensions of the twist-2 Wilson operators govern the Bjorken scaling violation for parton distributions in a framework of Quantum Chromodynamics (QCD). These quantities are given by the Mellin transformation (the simbol ˜ is used for spin-dependent case and $a_s = \alpha_s/(4\pi)$)

$$
\begin{aligned}
\gamma_{ab}(j) &= \int_0^1 dx\; x^{j-1} W_{b\to a}(x) \;=\; \gamma_{ab}^{(0)}(j)a_s + \gamma_{ab}^{(1)}(j)a_s^2 + \gamma_{ab}^{(2)}(j)a_s^3 + O(a_s^4),\\
\tilde{\gamma}_{ab}(j) &= \int_0^1 dx\; x^{j-1} \tilde{W}_{b\to a}(x) \;=\; \tilde{\gamma}_{ab}^{(0)}(j)a_s + \tilde{\gamma}_{ab}^{(1)}(j)a_s^2 + \tilde{\gamma}_{ab}^{(2)}(j)a_s^3 + O(a_s^4)
\end{aligned} \tag{1}
$$

of the splitting kernels $W_{b\to a}(x)$ and $\tilde{W}_{b\to a}(x)$ for the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation [1] which evolves the parton densities $f_a(x,Q^2)$ and $\tilde{f}_a(x,Q^2)$ (hereafter $a=\lambda,\, g,\, \phi$ for the spinor, vector and scalar particles, respectively [1]) as follows

$$
\begin{aligned}
\frac{d}{d\ln Q^2} f_a(x,Q^2) &= \int_x^1 \frac{dy}{y} \sum_b W_{b\to a}(x/y)\, f_b(y,Q^2)\,,\\
\frac{d}{d\ln Q^2} \tilde{f}_a(x,Q^2) &= \int_x^1 \frac{dy}{y} \sum_b \tilde{W}_{b\to a}(x/y)\, \tilde{f}_b(y,Q^2)\,.
\end{aligned} \tag{2}
$$

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[1] In the spin-dependent case $a=\lambda,\, g$.

The anomalous dimensions and splitting kernels in QCD are known up to the next-to-next-to-leading order (NNLO) of the perturbation theory [2, 3].

The QCD expressions for anomalous dimensions can be transformed to the case of the $\mathcal{N}$-extended Supersymmetric Yang-Mills theories (SYM) if one will use for the Casimir operators C_A, C_F, T_f the following values $C_A = C_F = N_c$, $T_f n_f = \mathcal{N} N_c/2$. For $\mathcal{N}$=2 and $\mathcal{N}$=4-extended SYM the anomalous dimensions of the Wilson operators get also additional contributions coming from scalar particles [4]. These anomalous dimensions were calculated in the next-to-leading order (NLO) [5] for the $\mathcal{N} = 4$ SYM.

However, it turns out, that the expressions for eigenvalues of the anomalous dimension matrix in the $\mathcal{N} = 4$ SYM can be derived directly from the QCD anomalous dimensions without tedious calculations by using a number of plausible arguments. The method elaborated in Ref. [4] for this purpose (it can be called as *maximal transcendentality principe*) is based on special properties of the integral kernel for the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [6]-[8] in this model and a new relation between the BFKL and DGLAP equations (see [4]). In the NLO approximation this method gives the correct results [4] for anomalous dimensions eigenvalues, which were checked by later *direct calculations* in Ref. [5]. Its properties will be reviewed below only shortly and a more extended discussion can be found in [4]. Using the results for the NNLO corrections to anomalous dimensions in QCD [3] and the method of Ref. [4] we derived the eigenvalues of the anomalous dimension matrix for the $\mathcal{N} = 4$ SYM in the NNLO approximation [9].

The obtained result is very important for the verification of the various assumptions [10]–[14] coming from the investigations of the properties of a conformal operators in the context of AdS/CFT correspondence [15].

2 Evolution equation in $\mathcal{N} = 4$ SYM

The reason to investigate the BFKL and DGLAP equations in the case of supersymmetric theories is related to a common belief, that the high symmetry may significantly simplify their structure. Indeed, it was found in the leading oredr (LO) [16], that the twist-2 operators in $\mathcal{N} = 1$ SYM are unified in supermultiplets with anomalous dimensions obtained from the universal anomalous dimension $\gamma_{uni}(j)$ by shifting its argument by an integer number. Further, the anomalous dimension matrices for twist-2 operators are fixed by the superconformal invariance [16]. Calculations in the maximally extended $\mathcal{N} = 4$ SYM, where the coupling constant is not renormalized, give even more remarkable results. Namely, it turns out, that here all twist-2 operators enter in the same multiplet, their anomalous dimension matrix is fixed completely by the super-conformal invariance and its universal anomalous dimension in leading order (LO) is proportional to $\Psi(j-1) - \Psi(1)$ (see the following section), which means, that the evolution equations for the matrix elements of twist-2 operators in the multicolour limit $N_c \to \infty$ are equivalent to the Schrödinger equation for an integrable Heisenberg spin model [17, 18]. In QCD the integrability remains only in a small sector of these operators [19]. In the case of $\mathcal{N} = 4$ SYM the equations for other sets of operators are also integrable [20, 21, 23, 22].

Similar results related to the integrability of the multi-colour QCD were obtained earlier in the Regge limit [24]. Moreover, it was shown [8], that in the $\mathcal{N} = 4$ SYM there is a deep relation between the BFKL and DGLAP evolution equations. Namely, the j-plane singularities of LO anomalous dimensions of the Wilson twist-2 operators in this case can be obtained from the eigenvalues of the BFKL kernel by their analytic continuation. The NLO calculations in $\mathcal{N} = 4$ SYM demonstrated [4], that some of these relations are valid also in higher orders of perturbation theory. In particular, the BFKL equation has the property of the hermitian separability, the linear combinations of the multiplicatively renormalized operators do not depend on the coupling constant, the eigenvalues of the anomalous dimension matrix are expressed in terms of the universal function $\gamma_{uni}(j)$ which can be obtained also from the BFKL equation [4]. The results for $\gamma_{uni}(j)$ were checked by *direct calculations* in Ref. [5]

3 LO anomalous dimension matrix in $\mathcal{N}=4$ SUSY

In the $\mathcal{N}=4$ SYM theory [25] one can introduce the following colour and $SU(4)$ singlet local Wilson twist-2 operators [4, 5]:

$$
\begin{aligned}
\mathcal{O}^g_{\mu_1,\dots,\mu_j} &= \hat{S} G^a_{\rho\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}\dots\mathcal{D}_{\mu_{j-1}}G^a_{\rho\mu_j}\,, &(3)\\
\tilde{\mathcal{O}}^g_{\mu_1,\dots,\mu_j} &= \hat{S} G^a_{\rho\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}\dots\mathcal{D}_{\mu_{j-1}}\tilde{G}^a_{\rho\mu_j}\,, &(4)\\
\mathcal{O}^\lambda_{\mu_1,\dots,\mu_j} &= \hat{S}\bar{\lambda}^a_i\gamma_{\mu_1}\mathcal{D}_{\mu_2}\dots\mathcal{D}_{\mu_j}\lambda^{a\,i}\,, &(5)\\
\tilde{\mathcal{O}}^\lambda_{\mu_1,\dots,\mu_j} &= \hat{S}\bar{\lambda}^a_i\gamma_5\gamma_{\mu_1}\mathcal{D}_{\mu_2}\dots\mathcal{D}_{\mu_j}\lambda^{a\,i}\,, &(6)\\
\mathcal{O}^\phi_{\mu_1,\dots,\mu_j} &= \hat{S}\bar{\phi}^a_r\mathcal{D}_{\mu_1}\mathcal{D}_{\mu_2}\dots\mathcal{D}_{\mu_j}\phi^a_r\,, &(7)
\end{aligned}
$$

where $\mathcal{D}_\mu$ are covariant derivatives. The spinors λ_i and field tensor $G_{\rho\mu}$ describe gluinos and gluons, respectively, and ϕ_r are the complex scalar fields. For all operators in Eqs. (3)-(7) the symmetrization of the tensors in the Lorentz indices $\mu_1,\dots,\mu_j$ and a subtraction of their traces is assumed. Due to the fact that all twist-2 operators belong to the same supermultiplet the eigenvalues of anomalous dimensions matrix can be expressed through one universal anomalous dimension $\gamma_{uni}(j)$ with shifted argument[2].

The elements of the LO anomalous dimension matrix in the $\mathcal{N}=4$ SUSY have the following form (see [18]):

for tensor twist-2 operators

$$
\begin{aligned}
\gamma^{(0)}_{gg}(j) &= 4\left(\Psi(1)-\Psi(j-1)-\frac{2}{j}+\frac{1}{j+1}-\frac{1}{j+2}\right),\\
\gamma^{(0)}_{\lambda g}(j) &= 8\left(\frac{1}{j}-\frac{2}{j+1}+\frac{2}{j+2}\right), & \gamma^{(0)}_{\varphi g}(j) &= 12\left(\frac{1}{j+1}-\frac{1}{j+2}\right),\\
\gamma^{(0)}_{g\lambda}(j) &= 2\left(\frac{2}{j-1}-\frac{2}{j}+\frac{1}{j+1}\right), & \gamma^{(0)}_{q\varphi}(j) &= \frac{8}{j},\\
\gamma^{(0)}_{\lambda\lambda}(j) &= 4\left(\Psi(1)-\Psi(j)+\frac{1}{j}-\frac{2}{j+1}\right), & \gamma^{(0)}_{\varphi\lambda}(j) &= \frac{6}{j+1},\\
\gamma^{(0)}_{\varphi\varphi}(j) &= 4\left(\Psi(1)-\Psi(j+1)\right), & \gamma^{(0)}_{g\varphi}(j) &= 4\left(\frac{1}{j-1}-\frac{1}{j}\right),
\end{aligned}
\tag{8}
$$

for the pseudo-tensor operators:

$$
\begin{aligned}
\tilde{\gamma}^{(0)}_{gg}(j) &= 4\left(\Psi(1)-\Psi(j+1)-\frac{2}{j+1}+\frac{2}{j}\right),\\
\tilde{\gamma}^{a,(0)}_{\lambda g}(j) &= 8\left(-\frac{1}{j}+\frac{2}{j+1}\right), \qquad \tilde{\gamma}^{(0)}_{g\lambda}(j) = 2\left(\frac{2}{j}-\frac{1}{j+1}\right),\\
\tilde{\gamma}^{(0)}_{\lambda\lambda}(j) &= 4\left(\Psi(1)-\Psi(j+1)+\frac{1}{j+1}-\frac{1}{j}\right).
\end{aligned}
\tag{9}
$$

Note, that in the $\mathcal{N}=4$ SUSY multiplet there are also twist-2 operators with fermion quantum numbers but their anomalous dimensions coincide up to an integer shift of the argument with the above expressions for the bosonic components (cf. ref. [16]).

[2]Non-diagonal elements of the anomalous dimensions matrix are related with non-forward anomalous dimensions by means of superconformal Ward identities [26] and can be expressed also through *non-forward* universal anomalous dimension [27].

3.1 Anomalous dimensions and twist-2 operators with a multiplicative renormalization

It is possible to construct 5 independent twist-two operators with a multiplicative renormalization. The corresponding Mellin momenta of parton distributions and their LO anomalous dimensions have the form (see [18]):

$$
\begin{aligned}
f_I(j) &= f_g(j) + f_\lambda(j) + f_\varphi(j) \sim f^{+}_{\lambda,g,\varphi}(j), \\
\gamma_I^{(0)}(j) &= 4\left(\Psi(1) - \Psi(j-1)\right) \equiv -4S_1(j-2) \equiv \gamma_+^{(0)}(j),
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
f_{II}(j) &= -2(j-1)f_g(j) + f_\lambda(j) + \frac{2}{3}(j+1)f_\varphi(j) \sim f^{0}_{\lambda,g,\varphi}(j), \\
\gamma_{II}^{(0)}(j) &= 4\left(\Psi(1) - \Psi(j+1)\right) \equiv -4S_1(j) \equiv \gamma_0^{(0)}(j),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
f_{III}(j) &= -\frac{j-1}{j+2}f_g(j) + f_\lambda(j) - \frac{j+1}{j}f_\varphi(j) \sim f^{-}_{\lambda,g,\varphi}(j), \\
\gamma_{III}^{(0)}(j) &= 4\left(\Psi(1) - \Psi(j+3)\right) \equiv -4S_1(j+2) \equiv \gamma_-^{(0)}(j),
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
f_{IV}(j) &= 2\tilde{f}_g(j) + \tilde{f}_\lambda(j) \sim \tilde{f}^{+}_{\lambda,g}(j), \\
\gamma_{IV}^{(0)}(j) &= 4\left(\Psi(1) - \Psi(j)\right) \equiv -4S_1(j-1) \equiv \tilde{\gamma}_+^{(0)}(j),
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
f_V(j) &= -(j-1)\tilde{f}_g(j) + \frac{j+2}{2}\tilde{f}_\lambda(j) \sim \tilde{f}^{-}_{\lambda,g}(j), \\
\gamma_V^{(0)}(j) &= 4\left(\Psi(1) - \Psi(j+2)\right) \equiv -4S_1(j+1) \equiv \tilde{\gamma}_-^{(0)}(j),
\end{aligned}
\tag{14}
$$

Thus, we have one supermultiplet of operators with the same anomalous dimension $\gamma_{uni}(j)$ proportional to $\Psi(1) - \Psi(j-1)$. The momenta of the corresponding linear combinations of parton distributions can be obtained from the above expressions $f_k(j)$ by an appropriate shift of their argument j in accordance with the corresponding shift of the argument of $\gamma_k(j)$. Moreover, the coefficients in these linear combinations for $N = 4$ SUSY can be found from the super-conformal invariance (cf. Ref [16]) and should be the same for all orders of the perturbation theory in an appropriate renormalization scheme.

The momenta of three multiplicatively renormalizable twist-2 operators for the unpolarized case are

$$
f_N(j) = a_g f_g(j) + a_\lambda f_\lambda(j) + a_\varphi f_\varphi(j)
$$

where the coefficients a_i can be extracted from above expressions (10)-(14). If we insert this anzatz in the DGLAP equations (2) the following representations for the corresponding anomalous dimensions

$$
\begin{aligned}
\gamma_N^{(0)}(j) &= \gamma_{gg}^{(0)}(j) + \frac{a_\lambda}{a_g}\gamma_{\lambda g}^{(0)}(j) + \frac{a_\varphi}{a_g}\gamma_{\varphi g}^{(0)}(j) \\
&= \gamma_{\lambda\lambda}^{(0)}(j) + \frac{a_g}{a_\lambda}\gamma_{g\lambda}^{(0)}(j) + \frac{a_\varphi}{a_\lambda}\gamma_{\varphi\lambda}^{(0)}(j) \\
&= \gamma_{\varphi\varphi}^{(0)}(j) + \frac{a_g}{a_\varphi}\gamma_{g\varphi}^{(0)}(j) + \frac{a_\lambda}{a_\varphi}\gamma_{\lambda\varphi}^{(0)}(j)
\end{aligned}
\tag{15}
$$

can be obtained. Eqs.(15) lead to relations among the anomalous dimension matrix $\gamma_{ab}^{(0)}(j)$ $(a, b = g, \lambda, \varphi)$ which should be valid also in the NLO approximation up to effects of breaking the superconformal invariance (see [5] and references therein).

Analogously for the set of two multiplicatively renormalizable operators in the polarized case

$$\tilde{f}_N(j) \;=\; \widetilde{a}_g \tilde{f}_g(j) + \widetilde{a}_\lambda \tilde{f}_\lambda(j)$$

we can derive the following relations

$$\widetilde{\gamma}_N^{(0)}(j) \;=\; \widetilde{\gamma}_{gg}^{(0)}(j) + \frac{\widetilde{a}_\lambda}{\widetilde{a}_g}\widetilde{\gamma}_{\lambda g}^{(0)}(j) \;=\; \widetilde{\gamma}_{\lambda\lambda}^{(0)}(j) + \frac{\widetilde{a}_g}{\widetilde{a}_\lambda}\widetilde{\gamma}_{g\lambda}^{(0)}(j)\,. \qquad (16)$$

So, we have 9 equations for the matrix elements in the case of the usual partonic distributions and 4 equations for the polarized distributions, which determines completely the anomalous dimension matrices $\gamma_{ab}^{(0)}(j)$ $(a,b = g,\lambda,\varphi)$ and $\widetilde{\gamma}_{ab}^{(0)}(j)$ $(a,b = g,\lambda)$ in terms of their eigenvalues in LO

$$\gamma_{\pm}^{(0)}(j) = -4S_1(j \mp 2), \qquad \gamma_0^{(0)}(j) = -4S_1(j), \qquad \widetilde{\gamma}_{\pm}^{(0)}(j) = -4S_1(j \mp 1)\,.$$

This procedure is considered in details in [4].

4 Method of obtaining the eigenvalues of the anomalous dimension matrix in $\mathcal{N} = 4$ SYM

As it was already pointed out in the Introduction, the universal anomalous dimension can be extracted directly from the QCD results without finding the scalar particle contribution. This possibility is based on the deep relation between the DGLAP and BFKL dynamics in the $\mathcal{N} = 4$ SYM [8, 4].

To begin with, the eigenvalues of the BFKL kernel turn out to be analytic functions of the conformal spin $|n|$ at least in two first orders of perturbation theory [4]. Further, in the framework of the $\overline{\text{DR}}$-scheme [28] one can obtain from the BFKL equation (see [8]), that there is no mixing among the special functions of different transcendentality levels i [3], i.e. all special functions at the NLO correction contain only sums of the terms $\sim 1/\gamma^i$ $(i = 3)$. More precisely, if we introduce the transcendentality level i for the eigenvalues $\omega(\gamma)$ of integral kernels of the BFKL equations in an accordance with the complexity of the terms in the corresponding sums

$$\Psi \sim 1/\gamma, \quad \Psi' \sim \beta' \sim \zeta(2) \sim 1/\gamma^2, \quad \Psi'' \sim \beta'' \sim \zeta(3) \sim 1/\gamma^3,$$

then for the BFKL kernel in the leading order (LO) and in NLO the corresponding levels are $i = 1$ and $i = 3$, respectively.

Because in $\mathcal{N} = 4$ SYM there is a relation between the BFKL and DGLAP equations (see [8, 4]), the similar properties should be valid for the anomalous dimensions themselves, i.e. the basic functions $\gamma_{uni}^{(0)}(j)$, $\gamma_{uni}^{(1)}(j)$ and $\gamma_{uni}^{(2)}(j)$ are assumed to be of the types $\sim 1/j^i$ with the levels $i = 1$, $i = 3$ and $i = 5$, respectively. An exception could be for the terms appearing at a given order from previous orders of the perturbation theory. Such contributions could be generated and/or removed by an approximate finite renormalization of the coupling constant. But these terms do not appear in the $\overline{\text{DR}}$-scheme.

It is known, that at the LO and NLO approximations (with the SUSY relation for the QCD color factors $C_F = C_A = N_c$) the most complicated contributions (with $i = 1$ and $i = 3$, respectively) are the same for all LO and NLO anomalous dimensions in QCD [2] and for the LO and NLO scalar-scalar anomalous dimensions [5] [4]. This property allows one to find the universal anomalous dimensions $\gamma_{uni}^{(0)}(j)$ and $\gamma_{uni}^{(1)}(j)$ without knowing all elements of the anomalous dimensions matrix [4], which was verified by the exact calculations in [5].

[3] Note that similar arguments were used also in [29] to obtain analytic results for contributions of some complicated massive Feynman diagrams without direct calculations.

[4] This property is correct also at the NNLO level [9].

Using above arguments, we conclude, that at the NNLO level there is only one possible candidate for $\gamma^{(2)}_{uni}(j)$. Namely, it is the most complicated part of the QCD anomalous dimensions matrix (with the SUSY relation for the QCD color factors $C_F = C_A = N_c$). Indeed, after the diagonalization of the anomalous dimensions matrix its eigenvalues should have this most complicated part as a common contribution because they differ each from others only by a shift of the argument and their differences are constructed from less complicated terms. The non-diagonal matrix elements of the anomalous dimensions matrix contain also only less complicated terms (see, for example, anomalous dimensions exact expressions at LO and NLO approximations in Refs. [2] for QCD and [5] for $\mathcal{N}=4$ SYM) and therefore they cannot generate the most complicated contributions to the eigenvalues of anomalous dimensions matrix.

Thus, the most complicated part of the NNLO QCD anomalous dimensions should coincide (up to color factors) with the universal anomalous dimension $\gamma^{(2)}_{uni}(j)$.

5 Universal anomalous dimension for $\mathcal{N}=4$ SYM

The final three-loop result [5] for the universal anomalous dimension $\gamma_{uni}(j)$ for $\mathcal{N}=4$ SYM is [9]

$$\gamma(j) \equiv \gamma_{uni}(j) \;=\; \hat{a}\gamma^{(0)}_{uni}(j) + \hat{a}^2\gamma^{(1)}_{uni}(j) + \hat{a}^3\gamma^{(2)}_{uni}(j) + ..., \qquad \hat{a} = \frac{\alpha N_c}{4\pi}\,, \tag{17}$$

where

$$\frac{1}{4}\,\gamma^{(0)}_{uni}(j+2) \;=\; -S_1, \tag{18}$$

$$\frac{1}{8}\,\gamma^{(1)}_{uni}(j+2) \;=\; \Big(S_3 + \overline{S}_{-3}\Big) - 2\,\overline{S}_{-2,1} + 2\,S_1\Big(S_2 + \overline{S}_{-2}\Big), \tag{19}$$

$$\begin{aligned}\frac{1}{32}\,\gamma^{(2)}_{uni}(j+2) \;=\;& 2\,\overline{S}_{-3}\,S_2 - S_5 - 2\,\overline{S}_{-2}\,S_3 - 3\,\overline{S}_{-5} + 24\,\overline{S}_{-2,1,1,1}\\ &+6\Big(\overline{S}_{-4,1} + \overline{S}_{-3,2} + \overline{S}_{-2,3}\Big) - 12\Big(\overline{S}_{-3,1,1} + \overline{S}_{-2,1,2} + \overline{S}_{-2,2,1}\Big)\\ &-\Big(S_2 + 2\,S_1^2\Big)\Big(3\,\overline{S}_{-3} + S_3 - 2\,\overline{S}_{-2,1}\Big) - S_1\Big(8\,\overline{S}_{-4} + \overline{S}^2_{-2}\\ &+4\,S_2\,\overline{S}_{-2} + 2\,S_2^2 + 3\,S_4 - 12\,\overline{S}_{-3,1} - 10\,\overline{S}_{-2,2} + 16\,\overline{S}_{-2,1,1}\Big)\end{aligned} \tag{20}$$

and $S_a \equiv S_a(j),\;\; S_{a,b} \equiv S_{a,b}(j),\;\; S_{a,b,c} \equiv S_{a,b,c}(j)$ are harmonic sums

$$S_a(j) \;=\; \sum_{m=1}^{j}\frac{1}{m^a},\quad S_{a,b,c,\cdots}(j) \;=\; \sum_{m=1}^{j}\frac{1}{m^a}\,S_{b,c,\cdots}(m), \tag{21}$$

$$S_{-a}(j) \;=\; \sum_{m=1}^{j}\frac{(-1)^m}{m^a},\quad S_{-a,b,c,\cdots}(j) \;=\; \sum_{m=1}^{j}\frac{(-1)^m}{m^a}\,S_{b,c,\cdots}(m),$$

$$\overline{S}_{-a,b,c,\cdots}(j) \;=\; (-1)^j\,S_{-a,b,c,\ldots}(j) + S_{-a,b,c,\cdots}(\infty)\Big(1-(-1)^j\Big). \tag{22}$$

The expression (22) is defined for all integer values of arguments (see [30, 4, 31]) but can be easily analytically continued to real and complex j by the method of Refs. [32, 4, 31].

[5]Note, that in an accordance with Ref. [7] our normalization of $\gamma(j)$ contains the extra factor $-1/2$ in comparison with the standard normalization (see [2]) and differs by sign in comparison with one from Ref. [3].

5.1 The limit $j \to 1$

The limit $j \to 1$ is important for the investigation of the small-x behavior of parton distributions (see review [33] and references therein). Especially it became popular recently because there are new experimental data at small x produced by the H1 and ZEUS collaborations in HERA [34].

Using asymptotic expressions for harmonic sums at $j = 1 + \omega \to 1$ (see [4, 5]) we obtain for the $\mathcal{N} = 4$ universal anomalous dimension $\gamma_{uni}(j)$ in Eq. (17)

$$\gamma_{uni}^{(0)}(1+\omega) \;=\; \frac{4}{\omega} + \mathcal{O}\Big(\omega^1\Big), \tag{23}$$

$$\gamma_{uni}^{(1)}(1+\omega) \;=\; -32\,\zeta_3 + \mathcal{O}\Big(\omega^1\Big), \tag{24}$$

$$\gamma_{uni}^{(2)}(1+\omega) \;=\; 32\zeta_3\,\frac{1}{\omega^2} - 232\,\zeta_4\,\frac{1}{\omega} - 1120\zeta_5 + 256\zeta_3\zeta_2 + \mathcal{O}\Big(\omega^1\Big) \tag{25}$$

in an agreement with the predictions for $\gamma_{uni}^{(0)}(1+\omega)$, $\gamma_{uni}^{(1)}(1+\omega)$ and also for the first term of $\gamma_{uni}^{(2)}(1+\omega)$ coming from an investigation of BFKL equation at NLO accuracy in [8] [6].

6 Integrability and the AdS/CFT-correspondence

We consider several important limits of (17)-(22), where our results can be compared with ones obtained in other approaches.

6.1 The limit $j \to 4$

The investigation of the integrability in $\mathcal{N} = 4$ SYM for a BMN-operators [35] gives a possibility to find the anomalous dimension of a Konishi operators [36, 23], which has the anomalous dimension coinciding with our expression (17) for $j = 4$

$$\gamma_{uni}(j)\Big|_{j=4} = -6\,\hat{a} + 24\,\hat{a}^2 - 168\,\hat{a}^3 = -\frac{3\,\alpha\,N_c}{2\pi} + \frac{3\,\alpha^2\,N_c^2}{2\pi^2} - \frac{21\,\alpha^3\,N_c^3}{8\pi^4}\,. \tag{26}$$

It is confirmed also by direct calculation in two [10, 5] and three-loop [12] orders.

A very interesting result comes from the consideration of the factorized S-matrix [13], which based on the investigation of the both sides of AdS/CFT-correspondence [35, 21, 23, 22, 37, 38] and gives a possibility to find three-loop anomalous dimension from the Bethe ansatz for arbitrary values of the Lorenz spin. The resulting Bethe ansatz reproduces our results for universal anomalous dimension $\gamma_{uni}(j)$ Eq. (17) and, thus confirms the hypotheses on integrability in $\mathcal{N} = 4$ SYM.

6.2 The limit $j \to \infty$

In the limit $j \to \infty$ the results (18)-(20) are simplified significantly. Note, that this limit is related to the study of the asymptotics of structure functions and cross-sections at $x \to 1$ corresponding to the quasi-elastic kinematics of the deep-inelastic *ep* scattering.

We obtain the following asymptotics for the $\mathcal{N} = 4$ universal anomalous dimension $\gamma_{uni}(j)$ in Eq. (17) with

$$\gamma_{uni}^{(0)}(j) \;=\; -4\Big(\ln j + \gamma_e\Big) + \mathcal{O}\Big(j^{-1}\Big), \tag{27}$$

$$\gamma_{uni}^{(1)}(j) \;=\; 8\zeta_2\,\Big(\ln j + \gamma_e\Big) + 12\zeta_3 + \mathcal{O}\Big(j^{-1}\Big), \tag{28}$$

$$\gamma_{uni}^{(2)}(j) \;=\; -88\zeta_4\,\Big(\ln j + \gamma_e\Big) - 16\zeta_2\zeta_3 - 80\zeta_5 + \mathcal{O}\Big(j^{-1}\Big). \tag{29}$$

[6] Unfortunately, the results of Refs. [8, 4] contain a misprint. Namely, the coefficient in front of $\hat{a}^3$ obtained in the limit $j \to 1$ in Eq. (39) of Ref. [4] should be multiplied by a factor 4.

Recently the coefficients in the front of $\ln j$ in Eqs. (27)–(29) were confirmed by Eden and Staudacher [14], where higher order terms $\gamma^{(n)}_{uni}(j)$ $(n \geq 3)$ have been also obtained using Bethe ansatz approach.

6.3 Resummation of γ_{uni} and the AdS/CFT correspondence

Last several years there was a great progress in the investigation of the $\mathcal{N} = 4$ SYM theory in a framework of the AdS/CFT correspondence [15] where the strong-coupling limit $\alpha_s N_c \to \infty$ is described by a classical supergravity in the anti-de Sitter space $AdS_5 \times S^5$. In particular, a very interesting prediction [39] (see also [40]) was obtained for the large-j behavior of the anomalous dimension for twist-2 operators

$$\gamma(j) = a(z)\, \ln j\,, \qquad\qquad z = \frac{\alpha N_c}{\pi} = 4\hat{a} \tag{30}$$

in the strong coupling regime (see Ref. [41] for asymptotic corrections):

$$\lim_{z\to\infty} a = -z^{1/2} + \frac{3\ln 2}{8\pi} + \mathcal{O}\left(z^{-1/2}\right)\,. \tag{31}$$

On the other hand, all anomalous dimensions $\gamma_i(j)$ and $\tilde{\gamma}_i(j)$ $(i = +, 0, -)$ coincide at large j and our results for $\gamma_{uni}(j)$ in Eq. (17) allow one to find three first terms of the small-z expansion of the coefficient $a(z)$ (see also Eqs. ([?])-([?]))

$$\lim_{z\to 0} a = -z + \frac{\pi^2}{12}\, z^2 - \frac{11}{720}\pi^4 z^3 + \dots\,. \tag{32}$$

For resummation of this series we suggest the following equation for $\tilde{a}$ [5]

$$z = -\tilde{a} + \frac{\pi^2}{12}\,\tilde{a}^2 \tag{33}$$

interpolating between its weak-coupling expansion up to NNLO

$$\tilde{a} = -z + \frac{\pi^2}{12}\, z^2 - \frac{1}{72}\pi^4 z^3 + \mathcal{O}(z^4) \tag{34}$$

and strong-coupling asymptotics

$$\tilde{a} = -\frac{2\sqrt{3}}{\pi}\, z^{1/2} + \frac{6}{\pi^2} + \mathcal{O}\left(z^{-1/2}\right) \approx -1.1026\; z^{1/2} + 0.6079 + \mathcal{O}\left(z^{-1/2}\right). \tag{35}$$

It is remarkable, that the prediction for NNLO based on the above simple equation is valid with the accuracy $\sim 10\%$. It means, that this extrapolation seems to be good for all values of z.

7 Relation between pomeron and graviton in the framework of the AdS/CFT correspondense

Further, for $j \to 2$ let us take into account, that according to the BFKL equation [7] the anomalous dimension of twist-2 operators is quantized in the Regge kinematics:

$$\gamma = 1/2 + i\nu + (j-1)/2 = 1 + (j-2)/2 + i\nu \tag{36}$$

for the principal series of unitary representations of the Möbius group. On the other hand, in the diffusion approximation valid near the leading singularity of the t-channel partial wave the eigenvalue of the BFKL kernel is

$$j - 1 = \omega_0 - D\nu^2\,, \tag{37}$$

where ω_0 and D are the Pomeron intercept and diffusion coefficient, respectively. These quantities are functions of the coupling constant. We assume, that for the large coupling constant in $N = 4$ SUSY the Pomeron intercept approaches the graviton intercept in the $AdS_5 \times S_5$ space [42], which means, that

$$j_0 = 1 + \omega_0 = 2 - \Delta\,, \tag{38}$$

where Δ is a small number. Further, due to the energy-momentum conservation ($\gamma = 0$ for $j = 2$) the parameters Δ and D are equal and $\gamma(j)$ can be expressed near $j = 2$ only in terms of one parameter

$$\gamma(j) = (j-2)\,\left[\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right]. \tag{39}$$

The derivative $\gamma'(2)$ can be calculated from our results in three first orders of the perturbation theory:

$$\gamma'(2) = \frac{1}{2} - \frac{1}{2\Delta} = -\frac{\pi^2}{6}\,z + \frac{\pi^4}{72}\,z^2 - \frac{\pi^6}{540}\,z^3 + \ldots. \tag{40}$$

Similar to the case of large j for a resummation of this series we used the following equation for $\widetilde{\widetilde{a}} = \gamma'(2)$ (see [5])

$$\frac{\pi^2}{6}\,z = -\,\widetilde{\widetilde{a}} + \frac{1}{2}\,\widetilde{\widetilde{a}}^2\,. \tag{41}$$

Its solution at small z is

$$\widetilde{\widetilde{a}} = -\frac{\pi^2}{6}\,z + \frac{\pi^4}{72}\,z^2 - \frac{\pi^6}{432}\,z^3 + \ldots. \tag{42}$$

One can verify from eqs. (40) and (42), that the prediction for NNLO based on the simple equation (41) is valid with the accuracy $\sim 20\%$. Therefore we can hope, that this method of resummation gives us a good estimate also for the behavior of a at large z

$$\gamma'(2) = 1 - \sqrt{\frac{\pi^2}{3}\,z + 1} \approx -\frac{\pi}{\sqrt{3}}\,z^{1/2} + 1 + \mathcal{O}\left(z^{-1/2}\right)\,. \tag{43}$$

Thus, one obtains for the intercept of the Pomeron in $N = 4$ SUSY from the resummation (41) at large z the result

$$j = 2 - \frac{\sqrt{3}}{2\pi}\,z^{-1/2} - \frac{3}{4\pi^2}\,z^{-1} - \mathcal{O}\left(z^{-2}\right). \tag{44}$$

On the other hand, from eqs. (36) and (37), using also the following relation valid in ADS/CFT correspondence for the string energy at j close to 2 [15, 39]

$$E^2 = (j+\Gamma)^2 - 4, \quad \Gamma = -2\gamma, \tag{45}$$

we obtain, that the BFKL equation in the diffusion approximation (36) is equivalent to the equation for the leading Regge trajectory in the superstring theory

$$j = 2 + \frac{\alpha'}{2}t, \quad t = E^2/R^2, \quad \alpha' = \frac{R^2}{2}\Delta, \tag{46}$$

where R is the radius of the anti-de-Sitter space.

It is naturally to expect that this Regge trajectory remains approximately linear (up to corrections to diffusion approximation of the BFKL equation) for all values of t and j. We can attempt to use expression (39) also for large z

$$\gamma(j)|_{z\to\infty} = -\sqrt{j-2}\Delta^{-1/2}. \tag{47}$$

This relation is in an agreement with the prediction of A. Polyakov and other authors [15, 39] [7].

$$\gamma(j)|_{(z,j)\to\infty} = -\frac{1}{2}E = -\sqrt{\pi j}z^{1/4} - \frac{3\sqrt{\pi}}{4}\frac{j^{3/2}}{z^{1/4}} + ..., \tag{48}$$

providing that

$$\Delta = \frac{1}{\pi}z^{-1/2}. \tag{49}$$

This number coincides up to 15% with the estimate $\Delta = [\sqrt{3}/(2\pi)]z^{-1/2}$ obtained in (44) from the resummation procedure (41). We can expect, that expression (47) with parameter Δ calculated in (49) gives the anomalous dimension of twist-2 operators for $z \to \infty$ and all j (neglecting the nonlinearity effects).

Recently, the results (47)–(49) (see also [9, 43]) have been confirmed in the paper [44].

8 Conclusion

Thus, in this review we presented the results for anomalous dimension $\gamma_{uni}(j)$ of twist-2 Wilson operators in the $\mathcal{N} = 4$ supersymmetric gauge theory up to the next-to-next-to-leading approximation and verified its self-consistency in the Regge ($j \to 1$) and quasi-elastic ($j \to \infty$) regimes. Our result for universal anomalous dimension at $j = 4$ could be used to determine the anomalous dimension of Konishi operator [36] up to 3-loops. It is remarkable, that our results coincide [8] with corresponding predictions from dilatation operator approach and integrability [23, 37]. The method, developed for this construction, can be applied also to less symmetric cases of $\mathcal{N} = 1,\ 2$ SYM and QCD, which are very important for phenomenological applications. For the verification of the AdS/CFT correspondence the calculations of the various physical quantities in $\mathcal{N} = 4$ SYM attract a great interest due to a possibility to develop non-perturbative approaches to QCD.

We demonstrated above that the expressions interpolating between the week and strong regime work remarkably well both in limit $j \to \infty$ and $j \to 2$. The integrability of the evolution equations for the twist-2 operators in LO [17, 18] is an interesting property of $\mathcal{N} = 4$ SYM which should be verified on NLO and NNLO level. We hope to discuss these problems in our future publications.

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