# Yang-Mills on the light cone and beyond

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**Abstract**

We review a recently constructed (non-local) change of variables which brings the light-cone gauge Yang-Mills action to the MHV-form.

## 1 Introduction

Recently, a new approach to the perturbative calculations in Yang-Mills (YM) theory has been suggested by Cachazo, Svrček and Witten (CSW) [1]. In this new formalism, the vertices are obtained from the so-called MHV-amplitudes (i.e. the amplitudes maximally violating the helicity) by a suitable continuation off shell. This technique was shown to reproduce all known gluon tree amplitudes and predicts a number of new results [2]. The successful generalization for the one-loop amplitudes has been also developed [3] although a new additional vertex has to be added at one-loop level in YM theory without supersymmetry. The MHV-like diagrams for the gravity case have been formulated as well [4]. A complete list of references can be found in [5].

In this short review we discuss the question of equivalence between the MHV diagrams and the conventional YM perturbation theory expansion. The MHV diagram rules can of course be described with help of an action functional, which we call the CSW action. It turns out that there exists a change of variables transforming the standard YM action to the CSW action. The formula for such a change of variables is obtained as follows [6] (see also ref. [7] for a different approach). First, we recall a certain solution to the self-duality equation which serves for a swift derivation of the MHV-amplitudes [8, 9]. This self-dual gauge field can be continued off shell in the spirit of ref. [1] and provides very explicit change of variables which brings YM Lagrangian in the light-cone gauge into the form of CSW Lagrangian. At present, we can check this by a brute-force calculation only and feel that a better, more conceptual understanding of our result is needed. This is despite the fact that the formula for the change of variables is perfectly explicit and the geometrical origin of the self-dual solution behind it seems to be well understood.

The paper is organized as follows. First, we remind the MHV diagram rules (Section 2) and present the YM action in the light-cone gauge (Section 3). Then, in Section 4 we describe a solution to the self-duality equation which is relevant to the MHV-amplitudes. A change of variables in the light-cone YM action, which renders it to the CSW action, is introduced in Section 5. Some open questions are mentioned in the concluding Section.

## 2 MHV amplitudes

The helicity of gluons is not in general preserved by the scattering processes in the Yang-Mills theory. A maximal violation of helicity happens in the scattering where one has two gluons of negative helicity and arbitrary number of positive helicity gluons (we treat all the scattering particles as incoming). The corresponding scattering amplitude is given by the following simple rational function of the on-shell momenta of external massless particles [10, 11][1]:

$$A(1^-, 2^-, 3^+ \ldots, n^+) = g_{YM}^{n-2} \frac{\langle 1,2\rangle^4}{\langle 1,2\rangle\langle 2,3\rangle \ldots \langle n,1\rangle} \tag{1}$$

where the on-shell momentum of a massless particle in the standard spinor notations reads as $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$, $\lambda_\alpha$ and $\tilde{\lambda}_{\dot{\alpha}}$ are positive and negative helicity spinors. The inner products in spinor notations read as $\langle \lambda_1, \lambda_2 \rangle = \epsilon_{\alpha\beta}\lambda_1^\alpha \lambda_2^\beta = \langle 1,2\rangle$ and $[\tilde{\lambda}_1, \tilde{\lambda}_2] = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_1^{\dot{\alpha}}\tilde{\lambda}_2^{\dot{\beta}}$. These amplitudes were interpreted as correlators in the auxiliary two-dimensional theory in [12] and in terms of topological string on twistor target space in [13]. There are no amplitudes with zero or one negative helicity gluons at the tree level. However, these amplitudes emerge at one-loop level in the YM theory without supersymmetry [14]. For instance, a one-loop all-plus amplitude reads as

$$A^{one-loop}(+, \ldots, +) = g_{YM}^n \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \frac{\langle i_1, i_2\rangle [i_2, i_3] \langle i_3, i_4\rangle [i_4, i_1]}{\langle 1,2\rangle\langle 2,3\rangle \ldots \langle n,1\rangle} \tag{2}$$

It was suggested in [1] that the conventional YM Feynman diagrams (in both supersymmetric and non-supersymmetric gauge theories) can be reorganized in a different way as the so-called MHV (or CSW) diagrams. The building blocks of this diagrammatics are MHV vertices extended off-shell and the canonical propagator $\frac{1}{P^2}$ involving $(+-)$ degrees of freedom and connecting two MHV vertices. The continuation off-shell suggested in [1] for $\lambda$ in any internal line reads as

$$\lambda_\alpha = p_{\alpha,\dot{\alpha}} \eta^{\dot{\alpha}} \tag{3}$$

where $\eta$ is arbitrary spinor fixed for off-shell lines in all diagrams relevant for a given amplitude.

[1]Here, we omit the colour factor and write down only a portion of the whole amplitude corresponding to a particular colour-ordering of the external gluons.

At higher loops the situation turns out to be more subtle at least in the theory without supersymmetry. The non-vanishing all-plus one-loop amplitude cannot be derived from the MHV vertices only. That is why it was suggested in [3] that the one-loop all-plus diagram has to be added to the CSW Lagrangian as a new vertex. It was also argued that there is no need to add one-loop vertex $(-,+\ldots+)$ to new Lagrangian.

In spite of the considerable success of this approach its conceptual origin remained obscure and it was unclear how these effective degrees of freedom involved into the CSW Lagrangian are related with the conventional YM gauge fields. It is the goal of this paper to argue that these effective degrees of freedom emerge from the standard YM variables in the light-cone gauge upon the particular "dressing " procedure.

## 3 Light cone gauge formulation

In this Section we briefly discuss YM theory in the light-cone gauge which involves only two physical degrees of freedom. The Lagrangian of YM theory in the light-cone variables has been found in N=4 SUSY case [15, 16]. In what follows we shall exploit Mandelstam two-field formulation [15] which has been successfully used recently in one-loop calculations in YM theories with various amount of supersymmetry [17]. Two fields $\Phi_+$ and $\Phi_-$ are related to the physical transverse degrees of freedom of the gluon as follows

$$\Phi_-(x) = \partial_+^{-1} A(x), \quad \Phi_+(x) = \partial_+ \bar{A}(x) \tag{4}$$

We shall be interested in the non-supersymmetric theory with the action in $A_+ = 0$ gauge[2]

$$S = \int d^4x [\Phi_+^a \Box \Phi_-^a + 2g f^{abc} \partial_+ \Phi_-^a \bar{\partial} \Phi_-^b \Phi_+^c + 2g f^{abc} \partial_+^2 \Phi_-^a \partial_+^{-2} \partial \Phi_+^b \partial_+^{-1} \Phi_+^c$$
$$-2g^2 f^{abc} f^{ade} \partial_+^{-2} (\partial_+ \Phi_-^b \Phi_+^c)(\partial_+^{-1} \Phi_+^d \partial_+^2 \Phi_-^e)] \tag{5}$$

where $\partial = \frac{1}{\sqrt{2}}(\partial_{x_1} + i\partial_{x_2})$ is expressed in terms of the derivatives with respect to the transverse coordinates $x_1, x_2$ and $\bar{\partial} = \partial^*$. The action contains local and non-local triple vertices as well as non-local quartic vertex.

Let us make a few comments on the form of the action (5). First note that it involves two fields of dimensions 0 and 2 hence positive and negative helicity fields enter Lagrangian asymmetrically. In particular, vertex $(-++)$ is local in the coordinate space while $(--+)$ is not. There are two classes of solutions to the equations of motion which correspond to the

[2] We shall understand everywhere the fields $\Phi_+, \Phi_-$ as matrix valued fields except the formula for the action below, which is written in terms of components $\Phi_+^a, \Phi_-^a$ with respect to a basis of matrices.

self-duality and anti-self-duality equations written in a little bit unusual form, namely

$$\Phi_{-}=0 \qquad \Box\Phi_{+}+(\Phi_{+},\Phi_{+})=0 \tag{6}$$

and

$$\Phi_{+}=0 \qquad \Box\Phi_{-}+\{\Phi_{-},\Phi_{-}\}=0 \tag{7}$$

where the schematically written r.h.s. are obtained by the variations of the cubic terms in the action (5).

## 4 Self-dual solutions and tree level MHV amplitudes

Let us discuss the formula (1) for the MHV-amplitude in more detail. A generating functional for the tree amplitudes can in general be described with help of a certain solution to the classical field equations (we call it the perturbiner). In case we are going to consider mostly positive helicity gluons on the external lines, we can reduce the general YM field equations to the self-dual ones. The latter are of course much easier to solve. This fact has been recognized some time ago by Bardeen [8] and has been elaborated further in [18, 19, 20, 9]. The shape of the relevant self-dual solution will help us in finding a needed field transformation in the next section.

Let us briefly remind the derivation of the perturbiner solution to the self-duality equation following [9]. The self-dual perturbiner yields the form-factor of the one off-shell gluon between the vacuum and arbitrary number of gluons of the same helicity, momenta $p_j$ and color orientations $t_j$. The starting point is the transition to the twistor representation with the additional spinor homogeneous coordinate $\rho^{\alpha}$ on the auxiliary $CP^1$. The self-duality equation in the twistor representation is equivalent to the zero-curvature condition

$$[\nabla_{\dot{\alpha}}\nabla_{\dot{\beta}}]=0 \tag{8}$$

where $\nabla_{\dot{\alpha}}=\rho^{\alpha}\nabla_{\dot{\alpha},\alpha}$. Hence the solution to the self-duality equation can be represented in the following form

$$A_{\dot{\alpha}}=g^{-1}\partial_{\dot{\alpha}}g \tag{9}$$

where $\partial_{\dot{\alpha}}=\rho^{\alpha}\partial_{\dot{\alpha},\alpha}$ , $A_{\dot{\alpha}}=\rho^{\alpha}A_{\dot{\alpha},\alpha}$ and $g$ is group valued function depending on $\rho$ and $x$, as well as on the quantum numbers $p_j$ and $t_j$ of the external particles. Since both, $A_{\dot{\alpha}}$ and $\partial_{\dot{\alpha}}$, are linear in $\rho$, the function $g$ has to be homogeneous (of zero degree) meromorphic function of $\rho$. However, the singularities of $g$ are constrained by the requirement that $A_{\dot{\alpha}}$ is regular.

The perturbiner is defined as a solution to the self-duality equation of the shape of a formal expansion in the (non-commuting) variables $E_j=t_je^{ip_jx}$, which are essentially the plane waves of the external gluons of the same, say positive, helicity. Thus, we look for the group valued function providing such a solution to the zero curvature equation in the form of an expansion in plane waves:

$$g_{ptb}(\rho) = 1 + \sum_j g_j(\rho) E_j + \ldots + \sum_{j_1 \ldots j_L} g_{j_1 \ldots j_L}(\rho) E_{j_1} \ldots E_{j_L} + \ldots \ , \tag{10}$$

where different terms with $L$ of $E$'s correspond to different color orderings in the form-factors with $L$ external particles. The above constraints on the singular behavior of $g_{ptb}(\rho)$ leads us immediately to a unique solution, where

$$g_{j_1 \ldots j_L}(\rho) = \frac{\langle \rho, q \rangle}{\langle \rho, j_1 \rangle \langle j_1, j_2 \rangle \langle j_2, j_3 \rangle \ldots \langle j_{L-1}, j_L \rangle \langle j_L, q \rangle} \tag{11}$$

where the so-called reference spinor $q_\alpha$ is the one which enters into the polarization vectors $\epsilon^j_{\dot\alpha,\alpha} = q_\alpha \tilde\lambda^j_{\dot\alpha}$.

The corresponding self-dual gauge field $A_{ptb}$ can be found upon the substitution of the solution into (9). Let us note that perturbiner solution itself is localized on the line in the twistor space if one performs half-Fourier transforms for all massless particles involved in the form-factor similar to [13].

The perturbiner solution describes form-factor or off-shell current of the form $\langle A_{\dot\alpha,\alpha}(k) \rangle_{k_1,\ldots,k_n}$ where the gluon with momentum $k$ is off-shell while all other gluons are on-shell and have the same helicity. Using the explicit form of the solution one can verify that this form-factor has no pole in $k^2$ and, hence, gives zero upon the application of the reduction formula. This corresponds to the vanishing of the amplitude with all but one gluons of the same helicity.

To get the MHV amplitudes from the perturbiner solution one has to consider the linearized YM equation in the background of the perturbiner. The most compact form of the generating function for the tree level MHV amplitudes has the following structure [9]

$$M(k_1, k_2) = \langle 1, 2 \rangle^2 \int d^4x \, \mathrm{tr}[E_1 g_{ptb}^{-1} E_2 g_{ptb}] \, , \tag{12}$$

where $E_1, k_1, E_2, k_2$ refer to the negative helicity gluons while the plane waves of the positive helicity gluons are substituted into $g_{ptb}$. Recall that the group element $g_{ptb}$ given in eq. (11) depends on the twistor spinor variable $\rho$ and the reference spinor $q$. In eq. (12), it is assumed that one sets $\rho = \lambda_1$ and $q = \lambda_2$, where $\lambda_1, \lambda_2$ are "spinor momenta" corresponding to $k_1, k_2$ respectively. Then, upon a direct calculation of the left hand side of eq. (12), one recovers the MHV-formula of eq. (1).

The group elements which depend on the twistor variable have to be taken at points $\rho_i$ corresponding to the momenta of negative helicity gluons.

## 5 MHV variables

In the last section we described the generating function for certain tree amplitudes as a solution of the self-duality equation. It means that we have a transformation $F$ which takes a self-dual solution of the linear wave equation (i.e. a sum of self-dual plane waves) as an input and produces from it a solution of the non-linear self-duality equation. If we now continue the definition of $F$ from the plane wave solutions to arbitrary (i.e. off-shell) fields such a transformation must link the linear free-field wave operator $\Box\phi_+$ and the non-linear one appearing in the left hand side of the second of eqs. (6). In more concrete terms it means that we are able to define a field transformation

$$\Phi_+ = F(\phi_+) \tag{13}$$

which obeys

$$\Box\Phi_+ + (\Phi_+, \Phi_+) = \partial_+ F'(\phi_+, \partial_+^{-1}\Box\phi_+)\,. \tag{14}$$

Here, $F'$ is the derivative of $F$ (that is $\delta F = F'(\phi, \delta\phi)$).

A formula for such a field transformation $F$ is obtained if we perform an off-shell continuation of the formulas of the last section in a way analogous to the one of ref. [1]. Explicitly, we define

$$\Phi_+ = F(\phi_+) = \partial_+^2 \sum_{n\geq 1} \frac{1}{\partial_{+,1}\langle\bar\partial_1\,,\bar\partial_2\rangle\ldots\partial_{+,n}} \underbrace{\phi_+ \ldots \phi_+}_{n}\,, \tag{15}$$

where $\bar\partial = (\eta^{\dot\alpha}\partial_{\alpha\dot\alpha})$ and $\bar\partial_k$ acts only on the k-th factor $\phi_+$ in the product. This expression for the transformation $F(\phi)$ can also be written in terms of a group-valued function $g_{ptb}(\phi)$, which is an analogously done off-shell continuation of $g_{ptb}$ of the last section, as follows:

$$F(\phi_+) = \partial_+(g_{ptb}^{-1}(\phi_+)\partial g_{ptb}(\phi_+))\,. \tag{16}$$

An important observation to make here is that the transformation (15) obeys the following property:

$$\int d^4x\, \mathrm{tr}[F'(\phi, u)\partial_+^{-3}F'(\phi, v)] = \int d^4x\, \mathrm{tr}[u\partial_+^{-3}v]\,, \tag{17}$$

where $u$ and $v$ are arbitrary (matrix-valued) fields. This can be phrased as that $F$ preserves a symplectic form on the field space. However, the only proof of eq. (17) we know is achieved by a straightforward algebra.

According to eq. (14), the shape of the transformation $F$ is designed to kill the (++-) vertex in the Lagrangian (5). On the other hand, the property (17) allows us to find also an appropriate field redefinition in the "minus sector", that is for the field $\Phi_-$. Indeed, if we want to express $\Phi_-$ in terms of the new variables $\phi_+$ and $\phi_-$ we should require that the kinetic term retains its canonical form. Put together, the above means that we want the following equality:

$$\int d^4x\,\mathrm{tr}\Phi_-(\Box\Phi_+ + (\Phi_+,\Phi_+)) = \int d^4x\,\mathrm{tr}\phi_-\Box\phi_+\,, \tag{18}$$

Let us define the remaining field transformation as

$$\Phi_- = \partial_+^{-4}F'(\phi_+,\partial_+^4\phi_-) \tag{19}$$

Then, eqs. (14,17) show us immediately that the equality (18) does indeed hold. To prove that the change of variables (15,19) brings the YM Lagrangian to the CSW form it remains to check only the effect of our field redefinition on the interaction part of the Lagrangian. We have indeed verified that the $(+--)$ and $(++--)$ vertices in the light-cone YM Lagrangian (5) get combined together to the correct interaction terms in the CSW Lagrangian. That is we have proved that the change of variables yields at the tree level both the correct propagator and the CSW vertices. The technical details concerning the change of variables shall be presented elsewhere.

To conclude, let us comment on the one-loop extension of the CSW Lagrangian. As we have already mentioned in the non-supersymmetric case it has to be extended by all-plus one-loop amplitude. The possible origin of such correction is clear in our approach — this is the Jacobian of the change of variables. We have not proved that the Jacobian reproduces the desired answer but there are certain arguments favoring this possibility.

The work of A.G. was supported in part by grants CRDF RUP2-261-MO-04 and RFBR-06-02-17382 and work of A.R. by grants RFBR-03-02-17554, NSch-1999.2003.2 and INTAS-03-51-6346.

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