Black hole solutions in $N > 4$ Gauss-Bonnet Gravity

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Abstract

Gauss-Bonnet gravity provides one of the most promising frameworks to study curvature corrections to the Einstein action in supersymmetric string theories, while avoiding ghosts and keeping second order field equations. Although Schwarzschild-type solutions for Gauss-Bonnet black holes have been known for long, the Kerr-Gauss-Bonnet metric was missing. In this paper, a five dimensional Gauss-Bonnet solution is derived for spinning black holes and the related thermodynamical properties are briefly outlined.

In any attempt to perturbatively quantize gravity as a field theory, higher-derivative interactions must be included in the action. Such terms also arise in the effective low-energy action of string theories. Furthermore, higher-derivative gravity theories are intrinsically attractive as in many cases they display features of renormalizability and asymptotic freedom. Among such approaches, Lovelock gravity [1] is especially interesting as the resulting equations of motion contain no more than second derivatives of the metric, include the self interaction of gravitation, and are free of ghosts when expanding around flat space. The four-derivative Gauss-Bonnet term is most probably the dominant correction to the Einstein-Hilbert action [2] when considering the dimensionally extended Euler densities used in the Lovelock Lagrangian. The action therefore reads as:

$$S_{GB} = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\Big[R + \alpha(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2)\Big], \tag{1}$$

where α is a coupling constant of dimension (length)^2, and G the $D-$dimensional Newton's constant defined as $G = 1/M_*^{D-2}$ in terms of the fundamental Planck scale M_*. Gauss-Bonnet gravity was shown to exhibit a very rich phenomenology in cosmology (see, *e.g.*, [3] and references therein), high-energy physics (see, *e.g.*, [4] and references therein) and black hole theory (see, *e.g.*, [5] and references therein). It also provides interesting solutions to the dark energy problem [6], offers a promising framework for inflation [7], allows useful modification of the Randall-Sundrum model [8] and, of course, solves most divergences associated with the endpoint of the Hawking evaporation process [9].

Either in D-dimensions or in 4-dimensions with a dilatonic coupling (required to make the Gauss-Bonnet term dynamical), Gauss-Bonnet black holes and their rich thermodynamical properties [10] have only been studied in the non-spinning (*i.e.* Schwarzschild-like) case.

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Although some general features can be derived in this framework, it remains mostly unrealistic as both astrophysical black holes and microscopic black holes possibly formed at colliders [11, 12, 13] are expected to be rotating (*i.e.* Kerr-like). Of course, the latter –which should be copiously produced at the *Large Hardon Collider* if the Planck scale is in the TeV range as predicted by some large extra-dimension models [14]– are especially interesting for Gauss-Bonnet gravity as they could be observed in the high-curvature region of General Relativity and allow a direct measurement of the related coupling constant [4]. The range of impact parameters corresponding to the formation of a non-rotating black hole being of zero measure, the Schwarzschild or Schwarzschild-Gauss-Bonnet solutions are mostly irrelevant. This is also of experimental importance as only a few quanta should be emitted by those light black holes, evading the Gibbons [15] and Page [16] arguments usually used to neglect the angular momentum of primordial black holes. To investigate the detailed properties of black holes in Lovelock gravity, it is therefore mandatory to derive the general solution, *i.e* the metric for the spinning case. Unlike the numerical attempts that were presented in [17] for degenerated angular momenta, the present paper focuses on the exact solution in 5 dimensions. The asymptotic behaviors are checked and the main thermodynamical properties are then briefly outlined.

Einstein equations in Gauss-Bonnet gravity with a cosmological constant Λ read as

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= \Lambda g_{\mu\nu} + \alpha \Bigg(\frac{1}{2} g_{\mu\nu} \Bigg(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \\ -4R_{\alpha\beta}R^{\alpha\beta} + R^2 \Bigg) &- 2RR_{\mu\nu} + 4R_{\mu\gamma}R^{\gamma}_{\nu} + 4R_{\gamma\delta}R^{\gamma\delta}_{\mu\nu} \\ -2R_{\mu\gamma\delta\lambda}R^{\gamma\delta\lambda}_{\nu} \Bigg), & \end{aligned} \qquad (2)$$

and the 5-dimensional metric in the spherically-symmertic Kerr-Schild type can be written as

$$\begin{aligned} ds^2 = dt^2 - dr^2 - (r^2 + a^2) \sin^2 \theta d\phi_1^2 - (r^2 + b^2) \cos^2 \theta d\phi_2^2 \\ +b^2) \cos^2 \theta d\phi_2^2 - \rho^2 d\theta^2 - 2dr \Bigg(a \sin^2 \theta d\phi_1 + b \cos^2 \theta d\phi_2 \Bigg) \\ -\beta \Bigg(dt - dr - a \sin^2 \theta d\phi_1 - b \cos^2 \theta d\phi_2 \Bigg)^2 , \end{aligned} \qquad (3)$$

where $\rho^2 = r^2 + a^2 \cos^\theta + b^2 \sin^2 \theta$ and $g_{tt} = 1 - \beta$.
The $\theta\theta$ component of Einstein equations reads :

$$A\beta'' + B\beta'^2 + C\beta' + D\beta + E = 0$$

where

$$\begin{aligned} A &= r\rho^2(4\alpha\beta - \rho^2) \\ B &= 4\alpha r \rho^2 \\ C &= 2 \Bigg[4\alpha\beta(\rho^2 - r^2) - \rho^2(\rho^2 + r^2) \Bigg] \\ D &= 2r(2r^2 - 3\rho^2) \\ E &= 2r\Lambda\rho^4. \end{aligned} \qquad (4)$$

This equation can be splitted into 2 relations depending respectively only upon β and $z \equiv \beta\beta'$ as independent unknown functions. It it then possible to introduce a few function $f(r,c)$ where $c = a^2 \cos^\theta + b^2 \sin^2\theta$ so that the equations are equivalent to to the following system :

$$\beta'' + 2(\frac{\rho^2 + r^2}{r\rho^2}\beta' - \frac{2r^2 - 3\rho^2}{\rho^4}\beta - \Lambda) - \frac{f(r,c)}{r\rho^4} = 0 \tag{5}$$

$$z' + 2\frac{\rho^2 - r^2}{r\rho^2}z - \frac{1}{2}\frac{f(r,c)}{\alpha r\rho^2} = 0 \tag{6}$$

Introducing the new function $p(r,c)$ via the transformation

$$f(r,c) = \frac{\rho^4}{r}\frac{\partial p(r,c)}{\partial r}, \tag{7}$$

the second equation can easily be solved (with $p_r \equiv \partial p(r,c)/\partial r$), leading to :

$$z = \frac{1}{2}\frac{(\int p_r dr + 2C_{21}\alpha)(r^2 + c^2)}{\alpha r^2} = (\beta\beta'), \tag{8}$$

where C_i are constants of integration. This equation can integrated to obtain :

$$\beta^2 = \frac{1}{\alpha}\int\left(p\frac{r^2 + c^2}{r^2}\right)dr + 2C_{21}\frac{r^2 - c^2}{r} + C_{20}. \tag{9}$$

The first equation can also be easily solved, which results in

$$\begin{aligned} \beta &= (C_{12}r - C_{11}(r^2 - c^2) - r\int\frac{(p_r - 2r^2\Lambda)(r^2 - c^2)}{r}dr \\ &+ (r^2 + c^2)\int(p_r + 2\Lambda r^2)dr)\frac{1}{r(r^2 + c^2)} \end{aligned} \tag{10}$$

where a simple integration by parts

$$\int p_r\frac{r^2 - c^2}{r}dr = p\frac{r^2 - c^2}{r} - \int p\frac{r^2 + c^2}{r^2}dr \tag{11}$$

leads to :

$$\begin{aligned} \beta r(r^2 + c^2) &= c_{12}r + C_{11}(r^2 - c^2) + r\int\left(p\frac{r^2 + c^2}{r^2}\right)dr \\ &+ \frac{\Lambda r^3}{6}(r^2 + c^2). \end{aligned} \tag{12}$$

As the same integral combination

$$Q = \int\left(p\frac{r^2 + c^2}{r^2}\right)dr \tag{13}$$

is involved, this can be transformed to 2 algebraic equations :

$$\begin{aligned} \beta r(r^2 + c^2) &= c_{12}r + C_{11}(r^2 - c^2) + rQ \\ &+ \frac{\Lambda r^3}{6}(r^2 + c^2) \end{aligned} \tag{14}$$

$$\alpha\beta^2 r = rQ + 2C_{21}(r^2 - c^2) + C_{20}r. \tag{15}$$

This system leads to the quadratic equation

$$\alpha\beta^2 - (r^2 + c^2)\beta + \left(C_{32} + C_{31}\frac{r^2 - c^2}{r} + \frac{\Lambda r^2}{6}(r^2 + 2c^2) \right) = 0 \tag{16}$$

where C_{3i} are new integration constants obtained from combination of C_{2i} and C_{1i}.

Taking into account the asymptotes at infinity (and therefore finding the values of the integration constants, M being the ADM mass), this leads to

$$\beta = \frac{\rho^2 \pm \sqrt{\rho^4 - 4\alpha M - \frac{2}{3}\alpha\Lambda r^2(\rho^2 - r^2)}}{2\alpha} \tag{17}$$

where the "-" branch should be chosen so as to recover the usual Kerr solution in the limit $\alpha \to 0$.

If, as suggested by geometrical arguments and by low-energy effective superstring theories, Gauss-Bonnet gravity is a realistic path toward a full quantum theory of gravity, then Kerr-Gauss-Bonnet black holes are probably among the most important objects to understand the physical basis of our World. This article has established the solution of Einstein equations in 5-dimensional Gauss-Bonnet. This allows to investigate into the details the physics of "realistic" spinning black holes, both from a pure theoretical and from a phenomenological (in the framework of low Planck-scale models) viewpoint.

Some improvements and developments can be foreseen. First, it should be very welcome to obtain the same kind of solutions for any number of dimensions. Unfortunately the method introduced in this article is not easy to generalize and a specific study should be made for each case. Then, it would be interesting to compute the greybody factors for those black holes. Following the techniques of [19], it is possible (although not straitforward) to obtain a numerical solution as soon as the metric is known, at least in the $\Lambda = 0$ case. The Kerr-Gauss-Bonnet-(Anti)-de-Sitter situation is more intricate as the metric is nowhere flat, requiring a more detailed investigation, as suggested in [20].

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